RELATIONSHIPS BETWEEN AFFINE FEEDBACK POLICIES FOR ROBUST CONTROL WITH CONSTRAINTS

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Abstract: This paper is concerned with the analysis of a recently-proposed robust control policy for linear discrete-time systems subject to bounded state disturbances with mixed constraints on the states and inputs, which parameterizes the input as an affine function of the past disturbance sequence. The paper shows that this disturbance feedback policy is equivalent to the class of affine state feedback policies with memory of prior states, and thus subsumes the well-known classes of open-loop and pre-stabilising control policies. Furthermore, the parameterization transforms the non-convex problem of finding an admissible state feedback policy to an equivalent and tractable convex problem.

Keywords: Robust Control, Constraint Satisfaction, Predictive Control, Optimal Control

1. INTRODUCTION

The problem of finding a nonlinear state feedback control law, which guarantees that a set of state and input constraints are satisfied for all time, despite the presence of a persistent state disturbance, has been the subject of study for many authors (Bemporad *et al.*, 2003; Glover and Schweppe, 1971; Bertsekas and Rhodes, 1971; Blanchini, 1999; Mayne *et al.*, 2000; Mayne, 2001; Mayne and Schroeder, 1996; Scokaert and Mayne, 1998; Diehl and Björnberg, 2004). However, the problem is that the solutions offered to date are exponentially complex or intractable for online implementation. As a consequence, many researchers have proposed compromise solutions, which, though not able to guarantee the same level of performance, are computationally tractable (Bemporad, 1998; Chisci *et al.*, 2001; Lee and Kouvaritakis, 1999; Langson *et al.*, 2004; Smith, 2004).

We propose a nonlinear control scheme that is implemented by selecting from amongst the set of constraint-admissible affine state feedback policies at each stage. Such a scheme subsumes the well known classes of "pre-stabilizing" (Lee and Kouvaritakis, 1999; Chisci *et al.*, 2001) and "open-loop" (Mayne *et al.*, 2000, Sect. 4.5) policies for robust control of constrained systems. We demonstrate via a simple example that, if implemented directly, this scheme is problematic due to non-convexity in the set of constraint-admissible state-feedback policies.

We therefore exploit a recently-proposed method for solving so-called *robust optimization* problems with hard constraints (Ben-Tal *et al.*, 2002; Guslitser, 2002). The authors proposed that, instead of solving for a general, nonlinear function that guarantees that the constraints are met for all values of the uncertainty, one could aim to formulate a control policy that is an affine function of the uncertainty.

This type of parameterization appears to have originally been suggested some time ago within the context of stochastic programs with recourse (Gartska and Wets, 1974). More recently, it has also been revisited as a way of finding solutions to robust model predictive control problems (Löfberg, 2003; van Hessem and Bosgra, 2002; van Hessem, 2004).

We prove that this affine uncertainty parameterization is *equivalent* to an affine state feedback parameterization, and that the proposed scheme enables a *convex* reformulation of the *non-convex* problem of finding a constraint-admissible affine state feedback control policy. A feasible robust control policy can thus be calculated using convex optimization techniques.

Notation: For matrices A and B, $A \otimes B$ is the Kronecker product of A and B, $A^\dagger$ is the one-sided or pseudo-inverse of A, and $A \leq B$ denotes element-wise inequality and $\text{abs}(A)$ is the element-wise absolute value of A. A matrix, not necessarily square, is referred to as *(strictly) lower triangular* if the (i,j) entry is zero for all $i < j$ ($i \leq j$). A block partitioned matrix is referred to as *(strictly) block lower triangular* if the (i,j) block is zero when $i < j$ ($i \leq j$); note that a *block* lower triangular matrix is not necessarily lower triangular. $\mathbb{Z}_{[k,l]}$ represents the set of integers $\{k, k+1, \ldots, l\}$. $\mathbf{1}$ is a column vector of ones. For vectors x and y, $\text{vec}(x,y) := [x^T\ y^T]^T$.

2. DEFINITIONS AND STANDING ASSUMPTIONS

Consider the following discrete-time LTI system:

$$x^+ = Ax + Bu + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input and $w \in \mathbb{R}^n$ is the disturbance[1]. It is assumed that (A, B) is stabilizable and that at each sample instant a measurement of the state is available. The current and future values of the disturbance are unknown and may change unpredictably from one time instant to the next, but are contained in a convex and compact (closed and bounded) set W.

The system is subject to mixed constraints on the state and input:

$$\mathcal{Z} := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{2}$$

where the matrices $C \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$; s is the number of affine inequality constraints that define $\mathcal{Z}$. A design goal is to guarantee that the state and input of the closed-loop system remain in $\mathcal{Z}$ for all time and for all allowable disturbance sequences.

In addition to $\mathcal{Z}$, a target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{3}$$

where the matrix $Y \in \mathbb{R}^{r\times n}$ and the vector $z \in \mathbb{R}^r$; r is the number of affine inequality constraints that define X_f. It is also assumed that X_f is bounded and contains the origin in its interior. The set X_f can be used as a target set in time-optimal control or to define a receding horizon controller with guaranteed invariance and stability properties (Goulart *et al.*, 2005).

Before proceeding, we define some additional notation. In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{nN}$, respectively, as

$$\mathbf{x} := \text{vec}(x_0, \ldots, x_N), \tag{4a}$$
$$\mathbf{u} := \text{vec}(u_0, \ldots, u_{N-1}), \tag{4b}$$
$$\mathbf{w} := \text{vec}(w_0, \ldots, w_{N-1}), \tag{4c}$$

where $x_0 = x$ denotes the current measured value of the state and $x_{i+1} := Ax_i + Bu_i + w_i$, $i \in \{0, \ldots, N-1\}$ denote the prediction of the state after i time instants into the future. Finally, let the set $\mathcal{W} := W^N := W \times \cdots \times W$, so that $\mathbf{w} \in \mathcal{W}$.

3. AFFINE STATE FEEDBACK PARAMETERIZATION

One natural approach to controlling the system in (1), while ensuring the satisfaction of the constraints, is to search over the set of affine state feedback control policies with memory of prior states:

$$u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \tag{5}$$

where each $L_{i,j} \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN\times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} := \begin{bmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{bmatrix}, \tag{6a}$$

[1] This assumption on the disturbance is without loss of generality; the results in this paper are easily generalized to the case where $x^+ = Ax + Bu + Ew$.

and

$$\mathbf{g} := \mathrm{vec}(g_0, \ldots, g_{N-1}), \tag{6b}$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}. \tag{7}$$

For a given initial state x, we say that the pair $(\mathbf{L}, \mathbf{g})$ is admissible if the control policy (5) guarantees that for all allowable disturbance sequences of length N, the constraints (2) are satisfied over the horizon $i = 0, \ldots, N-1$ and that the state is in the target set (3) at the end of the horizon. More precisely, the set of admissible $(\mathbf{L}, \mathbf{g})$ is defined as

$$\Pi_N^{sf}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \;\middle|\; \begin{array}{c} (\mathbf{L}, \mathbf{g}) \text{ satisfies (6)}, \ x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i} L_{i,j} x_j + g_i \\ (x_i, u_i) \in \mathcal{Z}, \ x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]}, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{8}$$

The set of initial states x for which an admissible control policy of the form (5) exists is defined as

$$X_N^{sf} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{sf}(x) \neq \emptyset \right\}. \tag{9}$$

It is critical to note that it is generally *not possible* to select a single $(\mathbf{L}, \mathbf{g})$ such that it is admissible for all $x \in X_N^{sf}$. Indeed, it is possible that for some pair $(x, \tilde{x}) \in X_N^{sf} \times X_N^{sf}$, $\Pi_N^{sf}(x) \bigcap \Pi_N^{sf}(\tilde{x}) = \emptyset$. For problems of non-trivial size, it is therefore necessary to calculate an admissible pair $(\mathbf{L}, \mathbf{g})$ on-line, given a measurement of the current state.

Once an admissible control policy is computed for the current state, it can then be implemented either in a time-varying, time-optimal or receding-horizon fashion. In general, the implemented control policy will be a *nonlinear* function with respect to the initial state, even though it may have been defined in terms of the class of affine state feedback policies (5).

Remark 1. Note that the state feedback policy (5) subsumes the well-known class of "pre-stabilizing" control policies (Lee and Kouvaritakis, 1999; Chisci *et al.*, 2001), in which the control policy takes the form $u_i = Kx_i + c_i$, where K is computed off-line and only c_i is computed on-line.

Computing an admissible pair $(\mathbf{L}, \mathbf{g})$, given the current state x, is seemingly a very difficult problem, due to the following property:

Proposition 2. (Non-convexity). For a given state $x \in X_N^{sf}$, the set of admissible affine state feedback control parameters $\Pi_N^{sf}(x)$ is non-convex, in general.

This is easily shown by the following example:

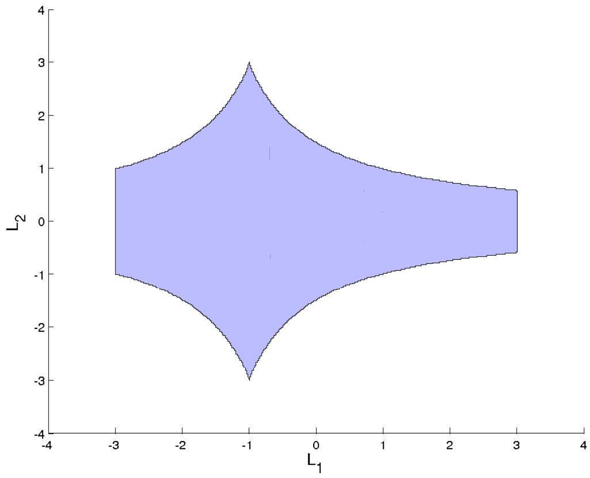

Fig. 1. Non-Convexity of $\Pi_N^{sf}(0)$ in Example 3

Example 3. Consider the SISO system

$$x^+ = x + u + w$$

with initial state $x_0 = 0$, input constraint $|u| \leq 3$, bounded disturbances $|w| \leq 1$ and a planning horizon of $N = 3$. Consider a control policy of the form (5) with $\mathbf{g} = 0$ and $L_{2,1} = 0$, so that $u_0 = 0$ and

$$u_1 = L_{1,1} w_0$$
$$u_2 = [L_{2,2}(1 + L_{1,1})]\, w_0 + L_{2,2} w_1$$

In order to satisfy the input constraints for all allowable disturbance sequences, the controls u_i must satisfy

$$|u_i| \leq 3, \ i = 1, 2, \ \forall \mathbf{w} \in \mathcal{W}$$

or, equivalently,

$$\max_{\mathbf{w} \in \mathcal{W}} |u_i| \leq 3, \ i = 1, 2.$$

Since the constraints on the components of $\mathbf{w}$ are independent, it is easy to show that the input constraints are satisfied for all $\mathbf{w} \in \mathcal{W}$ if and only if

$$|L_{1,1}| \leq 3$$
$$|L_{2,2}(1 + L_{1,1})| + |L_{2,2}| \leq 3.$$

It is straightforward to verify that the set of gains $\mathbf{L}$, which satisfy these constraints, is non-convex for this problem; the set of admissible values for $(L_{1,1}, L_{2,2})$ is shown in Figure 1.

Despite the fact that the set of admissible parameters $\Pi_N^{sf}(x)$ may be non-convex, we will proceed to show that one can actually find an admissible $(\mathbf{L}, \mathbf{g})$ by solving a single, *tractable* and *convex* programming problem using an appropriate reparameterization. We introduce this parameterization in the next section.

4. AFFINE DISTURBANCE FEEDBACK PARAMETERIZATION

An alternative to (5) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}, \qquad (10)$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and $v_i \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]}. \qquad (11)$$

For notational convenience, we define the vector $\mathbf{v} \in \mathbb{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix} \qquad (12a)$$

and

$$\mathbf{v} := \mathrm{vec}(v_0, \ldots, v_{N-1}), \qquad (12b)$$

so that the control input sequence can be written as

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}. \qquad (13)$$

For a given initial state x, we say that the pair $(\mathbf{M}, \mathbf{v})$ is admissible if the control policy (10) guarantees that for all allowable disturbance sequences of length N, the constraints (2) are satisfied over the horizon $i = 0, \ldots, N-1$ and that the state is in the target set (3) at the end of the horizon. More precisely, the set of admissible $(\mathbf{M}, \mathbf{v})$ is defined as

$$\Pi_N^{df}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (12)}, \ x = x_0 \\ x_{i+1} = Ax_i + Bu_i + w_i \\ u_i = \sum_{j=0}^{i-1} M_{i,j} w_j + v_i \\ (x_i, u_i) \in \mathcal{Z}, \ x_N \in X_f \\ \forall i \in \mathbb{Z}_{[0,N-1]}, \ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \qquad (14)$$

The set of initial states x for which an admissible control policy of the form (10) exists is defined as

$$X_N^{df} := \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_N^{df}(x) \neq \emptyset \right\}. \qquad (15)$$

Before proceeding, we note that $\Pi_N^{df}(x)$ can be expressed more compactly by eliminating the state variables and defining matrices $\mathbf{E} \in \mathbb{R}^{n(N+1) \times nN}$ and $\mathbf{A} \in \mathbb{R}^{n(N+1) \times n}$ as

$$\mathbf{E} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I \end{bmatrix}, \ \mathbf{A} := \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \qquad (16)$$

and $\mathbf{B} \in \mathbb{R}^{n(N+1) \times mN}$, $\mathbf{C} \in \mathbb{R}^{(qN+r) \times n(N+1)}$ and $\mathbf{D} \in \mathbb{R}^{(qN+r) \times mN}$ as $\mathbf{B} := \mathbf{E}(I \otimes B)$,

$$\mathbf{C} := \begin{bmatrix} (I \otimes C) & 0 \\ 0 & Y \end{bmatrix}, \ \mathbf{D} := \begin{bmatrix} (I \otimes D) \\ 0 \end{bmatrix}.$$

By further defining $F := \mathbf{CB} + \mathbf{D}$, $G := \mathbf{CE}$, $T := -\mathbf{CA}$, and $c := \mathrm{vec}(\mathbf{1} \otimes b, z)$, the expression for $\Pi_N^{df}(x)$ becomes

$$\Pi_N^{df}(x) = \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (12)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Tx \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \qquad (17)$$

4.1 Convexity of $\Pi_N^{df}(x)$

The main advantage of the disturbance feedback parameterization in (10) over the state feedback parameterization in (5) is formalized in the following statement:

Proposition 4. (Convexity). For a given state $x \in X_N^{df}$, the set of admissible affine disturbance feedback parameters $\Pi_N^{df}(x)$ is convex and closed. Furthermore, the set of states X_N^{df}, for which at least one admissible affine disturbance feedback parameter exists, is also convex and closed.

PROOF. Consider the set

$$\mathcal{C}_N := \left\{ (\mathbf{M}, \mathbf{v}, x) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (12)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Tx \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}$$

which is closed and convex, since it can be written as the intersection of closed and convex sets:

$$\mathcal{C}_N = \bigcap_{\mathbf{w} \in \mathcal{W}} \left\{ (\mathbf{M}, \mathbf{v}, x) \;\middle|\; \begin{array}{r} (\mathbf{M}, \mathbf{v}) \text{ satisfies (12)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Tx \end{array} \right\}.$$

The sets $\Pi_N^{df}(x)$ and X_N^{df} are just projections of this closed and convex set onto suitably defined subspaces, and are thus also closed and convex. □

This result is of fundamental importance. If $\mathcal{W}$ is convex and compact, then it is conceptually possible to compute a pair $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ in a computationally tractable way, given the current state x. For example, if $\mathcal{W}$ is a polytope, then an admissible policy may be found by solving a single LP in a tractable number of decision variables and constraints(Ben-Tal *et al.*, 2002). If $\mathcal{W}$ is an ellipsoid, an admissible policy may be found via a tractable SOCP. See (Goulart *et al.*, 2005) for further details and examples.

Remark 5. Note that the proof of Proposition 4 does not require $\mathcal{W}$ to be convex. However, convexity of $\mathcal{W}$ is important for the efficient computation of an admissible pair $(\mathbf{M}, \mathbf{v})$.

5. EQUIVALENCE BETWEEN STATE AND DISTURBANCE FEEDBACK PARAMETERIZATIONS

Having introduced both the non-convex state feedback and convex disturbance feedback parameterizations, we arrive at our main result.

Theorem 6. The set of admissible states $X_N^{df} = X_N^{sf}$. Additionally, for any admissible $(\mathbf{L}, \mathbf{g})$ an admissible $(\mathbf{M}, \mathbf{v})$ can be found that yields the same input and state sequence for all allowable disturbance sequences, and vice-versa.

PROOF. The set equality is established by showing both $X_N^{sf} \subseteq X_N^{df}$ and $X_N^{df} \subseteq X_N^{sf}$.

$X_N^{sf} \subseteq X_N^{df}$: By definition, for a given $x \in X_N^{sf}$, there exists a pair $(\mathbf{L}, \mathbf{g})$ that satisfies the constraints in (8). For a given disturbance sequence $\mathbf{w} \in \mathcal{W}$, the states of the system may be written as

$$\mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}.$$

Given the pair $(\mathbf{L}, \mathbf{g})$, the inputs and states can be written as:

$$\begin{aligned} \mathbf{u} &= \mathbf{L}\mathbf{x} + \mathbf{g} \\ \mathbf{x} &= \mathbf{B}(\mathbf{L}\mathbf{x} + \mathbf{g}) + \mathbf{E}\mathbf{w} + \mathbf{A}x \\ &= (I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g} + \mathbf{E}\mathbf{w} + \mathbf{A}x) \end{aligned}$$

The matrix $I - \mathbf{B}\mathbf{L}$ is always non-singular, since $\mathbf{B}\mathbf{L}$ is strictly block lower triangular. The control sequence can then be rewritten as an affine function of the disturbance sequence $\mathbf{w}$:

$$\mathbf{u} = \mathbf{L}(I-\mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g}+\mathbf{A}x)+\mathbf{L}(I-\mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w}+\mathbf{g},$$

and an admissible $(\mathbf{M}, \mathbf{v})$ constructed by choosing

$$\mathbf{M} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}\mathbf{E} \tag{18a}$$
$$\mathbf{v} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{B}\mathbf{g} + \mathbf{A}x) + \mathbf{g}. \tag{18b}$$

This choice of $(\mathbf{M}, \mathbf{v})$ gives exactly the same input sequence as the pair $(\mathbf{L}, \mathbf{g})$, so the state and input constraints in (14) are satisfied. The constraint (12) that $\mathbf{M}$ be strictly block lower triangular is satisfied because $\mathbf{M}$ is chosen in (18) as a product of the block lower triangular matrices $(I - \mathbf{B}\mathbf{L})^{-1}$ and $\mathbf{L}$ and the strictly block lower triangular matrix $\mathbf{E}$. Therefore, $(\mathbf{M}, \mathbf{v}) \in \Pi_N^{df}(x)$ and thus $x \in X_N^{sf} \Rightarrow x \in X_N^{df}$.

$X_N^{df} \subseteq X_N^{sf}$: By definition, for a given $x \in X_N^{df}$, there exists a pair $(\mathbf{M}, \mathbf{v})$ that satisfies the constraints in (14). For a given disturbance sequence $\mathbf{w} \in \mathcal{W}$, the inputs and states of the system can be written as:

$$\begin{aligned} \mathbf{u} &= \mathbf{M}\mathbf{w} + \mathbf{v} \\ \mathbf{x} &= \mathbf{B}(\mathbf{M}\mathbf{w} + \mathbf{v}) + \mathbf{E}\mathbf{w} + \mathbf{A}x \end{aligned}$$

Recall that since full state feedback is assumed, the disturbances can be determined exactly from

$$w_i = x_{i+1} - Ax_i - Bu_i, \quad \forall i \in \mathbb{Z}_{[0,N-1]},$$

which can be written in matrix form as

$$\mathbf{w} = \begin{bmatrix} 0 & I & 0 & \cdots & \cdots & 0 \\ 0 & -A & I & 0 & \ddots & \vdots \\ 0 & 0 & -A & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -A & I \end{bmatrix} \mathbf{x} - \begin{bmatrix} I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} Ax + (I \otimes B)\mathbf{u},$$

or more compactly as

$$\mathbf{w} = \mathbf{E}^\dagger \mathbf{x} - \mathcal{I}Ax + \mathbf{E}^\dagger \mathbf{B}\mathbf{u},$$

where $\mathcal{I} := [I \ 0 \ \cdots \ 0]^T$. It is easy to verify that the matrices $\mathbf{E}^\dagger$ and $\mathcal{I}^T$ are left inverses of $\mathbf{E}$ and $\mathbf{A}$ respectively, so that $\mathbf{E}^\dagger\mathbf{E} = I$ and $\mathcal{I}^T\mathbf{A} = I$.

The input sequence can then be rewritten as

$$\begin{aligned} \mathbf{u} &= \mathbf{M}(\mathbf{E}^\dagger\mathbf{x} - \mathcal{I}Ax + \mathbf{E}^\dagger\mathbf{B}\mathbf{u}) + \mathbf{v} \\ &= (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}(\mathbf{M}\mathbf{E}^\dagger\mathbf{x} - \mathbf{M}\mathcal{I}Ax + \mathbf{v}). \end{aligned}$$

The matrix $I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B}$ is non-singular because the product $\mathbf{M}\mathbf{E}^\dagger\mathbf{B} = \mathbf{M}(I \otimes B)$ is strictly block lower triangular. An admissible $(\mathbf{L}, \mathbf{g})$ can then be constructed by choosing

$$\mathbf{L} = (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^\dagger \tag{19a}$$
$$\mathbf{g} = (I - \mathbf{M}\mathbf{E}^\dagger\mathbf{B})^{-1}(\mathbf{v} - \mathbf{M}\mathcal{I}Ax). \tag{19b}$$

This choice of $(\mathbf{L}, \mathbf{g})$ gives exactly the same input sequence as the pair $(\mathbf{M}, \mathbf{v})$, so the state and input constraints in (8) are satisfied. The constraint that $\mathbf{L}$ be block lower triangular is satisfied because it is the product of block lower triangular matrices. Therefore, $(\mathbf{L}, \mathbf{g}) \in \Pi_N^{sf}(x)$ and thus $x \in X_N^{df} \Rightarrow x \in X_N^{sf}$. □

Remark 7. It is important to note that the result in Theorem 6 will general not hold if additional structural restrictions are placed on $\mathbf{M}$ or $\mathbf{L}$ (e.g. that one or both be banded and/or block-Toeplitz), because the nonlinear transformations in (18) and (19) only allow a limited number of structural constraints to be preserved.

We conclude this section by comparing Theorem 6 with Proposition 4. This leads immediately to the following result, which, in the light of Proposition 2, is rather surprising:

Corollary 8. (Convexity of X_N^{sf}). The set of states X_N^{sf}, for which an admissible affine state feedback policy of the form (5) exists, is closed and convex.

6. CONCLUSIONS

We have demonstrated that the disturbance feedback policy defined in Section 4 is a convex reparameterization of an affine state feedback policy with memory of prior states. This result allows for computation of an admissible robust control policy using standard convex optimization techniques. It would be interesting to see if it is possible to derive a similar convex reparameterization in the case where the control at each stage is an affine function of the *current state only*.

The set description in (17) may be extended to exploit any additional structure inherent in the robust finite horizon control problem for different classes of disturbance; some results along these lines are already available for a class of problems with ∞-norm bounded disturbances (Goulart and Kerrigan, 2005).

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