TOWARDS A REGULATION PROCEDURE FOR INSTANTANEOUS REACTIVE POWER IN NONLINEAR ELECTRICAL CIRCUITS

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Abstract: In a recent paper the authors proposed a classification of nonlinear RLC circuits into "dominantly inductive" or "dominantly capacitive" depending on an order relation between stored magnetic and electric energies that, in the linear case, exactly leads to the classical definitions based on reactive power. Associated to each of the classes is a suitably defined passive map with corresponding supply rate functions, that we interpret as generalized reactive power. In this paper we further investigate the properties of these functions deriving a very simple, and physically interpretable, expression for their rate of change and a procedure for its regulation with external sources. *Copyright ©2005 IFAC*

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1. PRELIMINARIES

1.1 Class of RLC circuits

In this note we consider RLC circuits consisting of interconnections of (possibly nonlinear) lumped dynamic (inductors, capacitors) and static (resistors and voltage and current sources) elements, whose behavior is described as follows. An n_L–port inductor is defined by a vector function relating flux and current $\mathbf{p}_L = \hat{\mathbf{p}}_L(\mathbf{i}_L)$, with $\hat{\mathbf{p}}_L : \mathbb{R}^{n_L} \to \mathbb{R}^{n_L}$, and Faraday's law

$$\mathbf{v}_L = \dot{\mathbf{p}}_L = \mathbf{L}(\mathbf{i}_L)\frac{d\mathbf{i}_L}{dt}, \tag{1}$$

where we defined the inductance matrix $\mathbf{L}(\mathbf{i}_L) := \nabla\hat{\mathbf{p}}_L$.[1] Analogously, for n_C–port capacitors we have that the charges are related to the voltages as $\mathbf{q}_C = \hat{\mathbf{q}}_C(\mathbf{v}_C)$, with $\hat{\mathbf{q}}_C : \mathbb{R}^{n_C} \to \mathbb{R}^{n_C}$, and

$$\mathbf{i}_C = \dot{\mathbf{q}}_C = \mathbf{C}(\mathbf{v}_C)\frac{d\mathbf{v}_C}{dt}, \tag{2}$$

where $\mathbf{C}(\mathbf{v}_C) := \nabla\hat{\mathbf{q}}_C$. We also have the following relationships for the energy functions $\mathcal{E}_L(\mathbf{p}_L),\ \mathcal{E}_C(\mathbf{q}_C)$, where $\mathcal{E}_L : \mathbb{R}^{n_L} \to \mathbb{R},\ \mathcal{E}_C : \mathbb{R}^{n_C} \to \mathbb{R}$,

$$\mathbf{i}_L = \nabla\mathcal{E}_L, \qquad \mathbf{v}_C = \nabla\mathcal{E}_C. \tag{3}$$

The circuit has n_R resistors, which are 1–ports characterized by the functions $v_{kR} = \hat{v}_{kR}(i_{kR}),\ k = 1, \ldots, n_R$, if they are current–controlled or by the

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[1] We use $\nabla_x(\cdot) := \frac{\partial}{\partial x}$, when clear from the context the argument will be omitted.

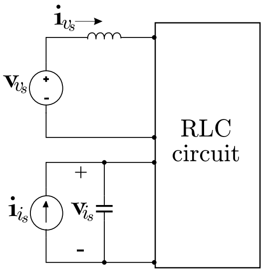

Fig. 1. RLC network with port variables the regulated current or voltage sources.

functions $i_{kR} = \hat{i}_{kR}(v_{kR})$ if they are voltage–controlled, where $\hat{v}_{kR}, \hat{i}_{kR} : \mathbb{R} \to \mathbb{R}$, (see Fact 1 below). It is clear that *constant* voltage and current sources can be easily added as particular instances of resistors. The circuit is interconnected with the environment through n_{vS} regulated voltage sources (in series with inductors) or n_{iS} regulated current sources (in parallel with capacitors). We denote their voltages and currents as $\mathbf{v}_{vS}, \mathbf{i}_{vS} \in \mathbb{R}^{n_{vS}}$, and $\mathbf{v}_{iS}, \mathbf{i}_{iS} \in \mathbb{R}^{n_{iS}}$, respectively. See Fig. 1.

To simplify the notation, we will group all capacitors of the circuit into one n_C–port and all inductors into one n_L–port with corresponding energies the sum of the energies of all multi–port capacitors and inductors, respectively. Also, we will sometimes group all port variables into vectors denoted by $\mathbf{v} := \mathrm{col}(\mathbf{v}_C, \mathbf{v}_L, \mathbf{v}_R, \mathbf{v}_S), \mathbf{i} := \mathrm{col}(\mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_R, -\mathbf{i}_S)$, where we have adopted the standard sign convention for the sources currents.

We make the following assumptions:

A.1 The energy functions, $\mathcal{E}_L(\mathbf{p}_L)$, $\mathcal{E}_C(\mathbf{q}_C)$, are twice differentiable and strictly convex—which implies that inductors and capacitors are passive and, furthermore, $\mathbf{L}(\mathbf{i}_L) > 0$ and $\mathbf{C}(\mathbf{v}_C) > 0$.

A.2 The characteristic functions of all resistors, $v_{kR} = \hat{v}_{kR}(i_{kR})$, $i_{kR} = \hat{i}_{kR}(v_{kR})$, live in the first–third quadrant, which is tantamount to saying that the resistors are passive.

A.3 The circuit is *complete*, which means that the currents in the inductors and the voltages in the capacitors, via Kirchhoff's laws and the laws of the resistors characteristics, determine the voltages and currents in all the branches.

Fact 1. (Brayton and Moser, 1964) Complete RLC circuits can be split into two subnetworks Σ_L, Σ_C that, respectively, contain all the inductors and capacitors. According to this partition, we can split the resistors into two sets,

- n_{vR} voltage–controlled resistors belonging to Σ_C, whose port variables will be denoted by $(\mathbf{i}_{R_C}, \mathbf{v}_{R_C})$, and have characteristic functions $i_{kR_C} = \hat{i}_{kR_C}(v_{kR_C})$; and
- n_{iR} current–controlled resistors belonging to Σ_L, with port variables $(\mathbf{i}_{R_L}, \mathbf{v}_{R_L})$ and characteristic functions $v_{kR_L} = \hat{v}_{kR_L}(i_{kR_L})$. ◁

As shown in (Brayton and Moser, 1964), see also (Ortega *et al.*, 2003) for an alternative derivation including the sources, the dynamics of the circuit is described by

$$\begin{aligned} \mathbf{L}(\mathbf{i}_L)\,\dot{\mathbf{i}}_L &= -\nabla_{\mathbf{i}_L} P + \mathbf{B}_{vS}\mathbf{v}_{vS} \\ \mathbf{C}(\mathbf{v}_C)\dot{\mathbf{v}}_C &= \nabla_{\mathbf{v}_C} P + \mathbf{B}_{iS}\mathbf{i}_{iS} \end{aligned} \tag{4}$$

where

$$P(\mathbf{i}_L, \mathbf{v}_C) := \mathbf{i}_L^\top \mathbf{\Gamma} \mathbf{v}_C + G(\mathbf{\Gamma}_L \mathbf{i}_L) - F(\mathbf{\Gamma}_C \mathbf{v}_C) \tag{5}$$

is the mixed potential function, [2]

$$F(\mathbf{v}_{R_C}) := \sum_{k=1}^{n_{vR}} \int_0^{v_{kR_C}} \hat{i}_{kR_C}(v'_{kR_C})dv'_{kR_C}, \ \mathbf{v}_{R_C} := \mathbf{\Gamma}_C \mathbf{v}_C$$

$$G(\mathbf{i}_{R_L}) := \sum_{k=1}^{n_{iR}} \int_0^{i_{kR_L}} \hat{v}_{kR_L}(i'_{kR_L})di'_{kR_L}, \ \mathbf{i}_{R_L} := \mathbf{\Gamma}_L \mathbf{i}_L \tag{6}$$

are the co–content and the content of the voltage–controlled and the current–controlled resistors, respectively, $\mathbf{B}_{vS} \in \mathbb{R}^{n_L \times n_{vS}}$, $\mathbf{B}_{iS} \in \mathbb{R}^{n_C \times n_{iS}}$ are input (full rank) matrices, and $\mathbf{\Gamma} \in \mathbb{R}^{n_L \times n_C}$, $\mathbf{\Gamma}_C \in \mathbb{R}^{n_{R_C} \times n_C}$, $\mathbf{\Gamma}_L \in \mathbb{R}^{n_{R_L} \times n_L}$ are constant matrices determined by the circuit topology.

Before closing this subsection we recall the classical definition of passivity (Willems, 1992; Van der Schaft, 2000) and a well–known consequence of it.

Definition 1. (*Passivity*). We say that an m–port system with state $\mathbf{x} = \mathrm{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$ and port variables $(\mathbf{u}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^m$, is *passive* if there exists a non–negative function $\mathcal{E} : \mathbb{R}^n \to \mathbb{R}_+$, called the storage function, such that

$$\mathcal{E}[\mathbf{x}(t)] - \mathcal{E}[\mathbf{x}(0)] \leq \int_0^t \mathbf{u}^\top(s)\mathbf{y}(s)ds, \tag{7}$$

along all trajectories of the system. [3] The function $w : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, defined as $w(\mathbf{u}, \mathbf{y}) := \mathbf{u}^\top \mathbf{y}$, is called the supply rate.

The following fact will be important to justify our definition of generalized reactive power introduced in Section 2, see also (Willems, 1992; Van der Schaft, 2000).

Fact 2. In physical systems the port variables, $(\mathbf{u}, \mathbf{y})$, are conjugated—in the sense that their product has units of power—and the function $\mathcal{E}(\mathbf{x})$ is the total stored energy. On the other hand, since $\mathcal{E}(\mathbf{x})$ is non–negative, the passivity inequality (7) implies

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[2] For a description of the class of nonlinear RLC networks that enable an explicit expression for the mixed potential function (5), we refer the reader to (Weiss *et al.*, 1998)

[3] In the seminal paper (Willems, 1992) a system satisfying condition (7) is said to be dissipative with respect to the supply rate $w(\mathbf{u}, \mathbf{y})$. The use of the word "dissipative" in this context may generate some confusion, we therefore prefer to avoid its utilization here.

$$-\int_0^t \mathbf{u}^\top(s)\mathbf{y}(s)ds \leq \mathcal{E}[\mathbf{x}(0)]$$

where we underscore the negative sign. This inequality indicates that from a passive system you can only extract a finite amount of energy, that cannot exceed the energy initially stored in the system.

1.2 A passivity–based classification

The basic postulate of (Garcia–Canseco *et al.*, 2004) is that the (inductive or capacitive) nature of a nonlinear circuit should be determined, as it is in the linear case, by the order relationship between the stored electric and magnetic energies. Indeed, for single port LTI RLC circuits with sinusoidal voltage source, $v_S = \sqrt{2}V_S\cos(\omega t)$ and current $i_S = \sqrt{2}I_S\cos(\omega t + \phi)$ we have established in (Garcia–Canseco and Ortega, 2004) that

$$Q_\omega = 2\omega[\mathcal{E}_{L_{av}}(\omega) - \mathcal{E}_{C_{av}}(\omega)], \tag{8}$$

where Q_ω is the classical reactive power [4] defined as $Q_\omega = V_S I_S \sin\phi$, and $\mathcal{E}_{L_{av}}(\omega), \mathcal{E}_{C_{av}}(\omega)$ are the averaged energies. (Equation (8) indicates that the circuit is inductive (resp., capacitive) if and only if the average magnetic energy dominates (resp., is dominated by) the average electric energy—this, in its turn, is equivalent to the reactive power being non–positive (resp., non–negative) for all $\omega \in \mathbb{R}$.)

The result above is easily proven with simple frequency response arguments. To extend our postulate to the nonlinear case it is clear that we cannot rely on sinusoidal steady state reasoning and we should look for a more general framework, which turns out to be the provided by the passivity formalism. For the LTI case we have shown in (Garcia–Canseco and Ortega, 2004) that there exists a one–to–one correspondence between passivity of some (suitably defined) maps and reactive power, namely

$$\begin{aligned} (\dot{v}_S, i_S) \text{ is passive } &\Leftrightarrow\ Q_w < 0 \\ (\dot{\widehat{i}}_S, v_S) \text{ is passive } &\Leftrightarrow\ Q_w > 0. \end{aligned} \tag{9}$$

An extension to the nonlinear case was given in (Garcia–Canseco *et al.*, 2004) where, for the general nonlinear circuit of Fig. 1 with constant current sources $\mathbf{i}_{iS}$, it was proven that

$$\mathcal{E}_L(\mathbf{p}_L) >> \mathcal{E}_C(\mathbf{q}_C) \Rightarrow (\mathbf{v}_{vS}, \dot{\widehat{\mathbf{i}}}_{vS}) \quad \text{is passive}$$

Dually, if the voltage sources $\mathbf{v}_{vS}$ are constant, we have

$$\mathcal{E}_C(\mathbf{q}_C) >> \mathcal{E}_L(\mathbf{p}_L) \Rightarrow (\mathbf{i}_{iS}, \dot{\mathbf{v}}_{iS}) \quad \text{is passive.}$$

Comparing with (8), (9) we notice that the sharp inequality and the equivalence have been replaced by $>>$ (resp. $<<$) and sufficiency only.

[4] We use the subindex ω to underscore the dependence of the reactive power on the frequency of operation.

2. RATE OF CHANGE OF GENERALIZED REACTIVE POWER

The discussion above highlights the importance of the functions

$$-\int_0^t \mathbf{v}_{vS}^\top(\tau)\,\dot{\widehat{\mathbf{i}}}_{vS}(\tau)d\tau, \quad -\int_0^t \mathbf{i}_{iS}^\top(\tau)\dot{\mathbf{v}}_{iS}(\tau)d\tau,$$

that, as indicated in Fact 2, have the interpretation of extracted "generalized energies", while the supply rates, $\dot{\mathbf{v}}_{iS}^\top\mathbf{i}_{iS}$ and $\mathbf{v}_{vS}^\top\,\dot{\widehat{\mathbf{i}}}_{vS}$, are "generalized powers".

In this section we the study the behavior of the sum of the supply rates of the energy storing elements of the circuit that—with an obvious abuse of notation—we will call total *generalized reactive power* in the sequel. [5] We will derive a simple expression for the time evolution of the total generalized reactive power—that highlights the role of dissipation and suggests a procedure to regulate it with the inclusion of controlled sources. For ease of presentation we will assume first that the external sources are constant, and leave for a remark the case of regulated sources. In Section 3 we present two examples, with a general time–varying source, and with a regulated source for reactive power control.

Proposition 1. Consider the RLC circuit of Fig. 1, described by (4), (5), and satisfying Assumption A.1–A.3. Define the scalar signal

$$q(t) := \mathbf{v}_L^\top(t)\,\dot{\widehat{\mathbf{i}}}_L(t) + \mathbf{i}_C^\top(t)\dot{\mathbf{v}}_C(t), \tag{10}$$

that we call total generalized reactive power, and assume:

A.4 The regulated current, $\mathbf{i}_{iS}$, and voltage, $\mathbf{v}_{vS}$, sources are constant.

Then, along the trajectories of the circuit we have

$$\dot{q}(t) = -\dot{\widehat{\mathbf{i}}}_L^\top(t)\nabla^2 G\,\dot{\widehat{\mathbf{i}}}_L(t) - \dot{\mathbf{v}}_C^\top(t)\nabla^2 F\dot{\mathbf{v}}_C(t) + g(t) \tag{11}$$

where

$$\begin{aligned} g(t) :=& -\dot{\widehat{\mathbf{i}}}_L^\top(t)\nabla_{\mathbf{i}_L}\Big[(\nabla_{\mathbf{i}_L}P)^\top\mathbf{L}^{-1}(\mathbf{i}_L(t))\Big]\mathbf{L}(\mathbf{i}_L(t))\,\dot{\widehat{\mathbf{i}}}_L(t) \\ &+\dot{\mathbf{v}}_C^\top(t)\nabla_{\mathbf{v}_C}\Big[(\nabla_{\mathbf{v}_C}P)^\top\,\mathbf{C}^{-1}(\mathbf{v}_C(t))\Big]\,\mathbf{C}(\mathbf{v}_C(t))\dot{\mathbf{v}}_C(t) \end{aligned}$$

and $P(\mathbf{i}_L, \mathbf{v}_C)$ is given in (5). In particular, if the inductors and capacitors are linear, the term $g(t)$ simplifies yielding

$$\dot{q}(t) = -2\,\dot{\widehat{\mathbf{i}}}_L^\top(t)\nabla^2 G\,\dot{\widehat{\mathbf{i}}}_L(t) - 2\dot{\mathbf{v}}_C^\top(t)\nabla^2 F\dot{\mathbf{v}}_C(t)$$

[5] In (Waytt and Ilic, 1990) the difference of the supply rates, called $p_{\text{react},2}$, is proposed as a definition of instantaneous reactive power. We refer the reader to this paper for many interesting discussions.

Proof. First, we write the system (4), (5) in the form

$$\mathbf{M}(\mathbf{i}_L,\mathbf{v}_C)\begin{bmatrix}\dot{\widehat{\mathbf{i}}}_L\\ \dot{\mathbf{v}}_C\end{bmatrix}=\nabla P_A \tag{12}$$

where we defined $\mathbf{M}(\mathbf{i}_L,\mathbf{v}_C):=\begin{bmatrix}-\mathbf{L}(\mathbf{i}_L) & 0\\ 0 & \mathbf{C}(\mathbf{v}_C)\end{bmatrix}$, $P_A(\mathbf{i}_L,\mathbf{v}_C):=P(\mathbf{i}_L,\mathbf{v}_C)-\mathbf{i}_L^\top\mathbf{B}_{vS}\mathbf{v}_{vS}+\mathbf{v}_C^\top\mathbf{B}_{iS}\mathbf{i}_{iS}$, and we note that $\mathbf{M}(\mathbf{i}_L,\mathbf{v}_C)$ is full rank.

The proof uses, again, Proposition 5 of (Ortega *et al.*, 2003) to generate an alternative description of the system dynamics. Specifically, we will prove now that the system can be written as

$$\tilde{\mathbf{M}}(\mathbf{i}_L,\mathbf{v}_C)\begin{bmatrix}\dot{\widehat{\mathbf{i}}}_L\\ \dot{\mathbf{v}}_C\end{bmatrix}=\nabla\tilde{P}_A, \tag{13}$$

where

$$\tilde{\mathbf{M}}(\mathbf{i}_L,\mathbf{v}_C):=\begin{bmatrix}\tilde{\mathbf{M}}_{11}(\mathbf{i}_L,\mathbf{v}_C) & \tilde{\mathbf{M}}_{12}(\mathbf{i}_L,\mathbf{v}_C)\\ \tilde{\mathbf{M}}_{21}(\mathbf{i}_L,\mathbf{v}_C) & \tilde{\mathbf{M}}_{22}(\mathbf{i}_L,\mathbf{v}_C)\end{bmatrix}$$

with

$$\tilde{\mathbf{M}}_{11}(\mathbf{i}_L,\mathbf{v}_C)=-\nabla^2 G-\nabla_{\mathbf{i}_L}\Big[(\nabla_{\mathbf{i}_L}P)^\top\mathbf{L}^{-1}(\mathbf{i}_L)\Big]\mathbf{L}(\mathbf{i}_L),$$

$$\tilde{\mathbf{M}}_{12}(\mathbf{i}_L,\mathbf{v}_C)=2\mathbf{\Gamma},$$

$$\tilde{\mathbf{M}}_{21}(\mathbf{i}_L,\mathbf{v}_C)=-2\mathbf{\Gamma}^\top,$$

$$\tilde{\mathbf{M}}_{22}(\mathbf{i}_L,\mathbf{v}_C)=-\nabla^2 F+\nabla_{\mathbf{v}_C}\Big[(\nabla_{\mathbf{v}_C}P)^\top\mathbf{C}^{-1}(\mathbf{v}_C)\Big]\mathbf{C}(\mathbf{v}_C)$$

and

$$\tilde{P}_A(\mathbf{i}_L,\mathbf{v}_C):=\nabla^\top P_A\begin{bmatrix}\mathbf{L}^{-1}(\mathbf{i}_L) & 0\\ 0 & \mathbf{C}^{-1}(\mathbf{v}_C)\end{bmatrix}\nabla P_A. \tag{14}$$

From (12) and (13) it is obvious that to establish the claim it suffices to prove that

$$\tilde{\mathbf{M}}(\mathbf{i}_L,\mathbf{v}_C)\mathbf{M}^{-1}(\mathbf{i}_L,\mathbf{v}_C)\nabla P_A=\nabla\tilde{P}_A,$$

that can be easily verified via direct calculations.

The key observation now is that, replacing (12) in (14), we get

$$\tilde{P}_A=\begin{bmatrix}(\dot{\widehat{\mathbf{i}}}_L)^\top & \dot{\mathbf{v}}_C^\top\end{bmatrix}\begin{bmatrix}\mathbf{L}(\mathbf{i}_L) & 0\\ 0 & \mathbf{C}(\mathbf{v}_C)\end{bmatrix}\begin{bmatrix}\dot{\widehat{\mathbf{i}}}_L\\ \dot{\mathbf{v}}_C\end{bmatrix}$$
$$=(\dot{\widehat{\mathbf{i}}}_L)^\top\mathbf{v}_L+\dot{\mathbf{v}}_C^\top\mathbf{i}_C,$$

where we have used (1), (2) to get the last identity. Comparing with (10), we have $\tilde{P}_A(\mathbf{i}_L(t),\mathbf{v}_C(t))=q(t)$.

The expression of $\dot{q}(t)$ is obtained pre-multiplying (13) by $\begin{bmatrix}(\dot{\widehat{\mathbf{i}}}_L)^\top & \dot{\mathbf{v}}_C^\top\end{bmatrix}$ and invoking Assumption A.4. ◁

Remark 1. The previous analysis remains unaffected if we include current–dependent voltage sources in series with the inductors and/or voltage–dependent current sources in parallel with the capacitors. Indeed, the expressions (10) and (11) remain valid with the new co–content and content functions

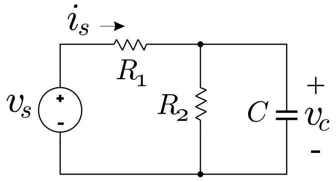

Fig. 2. A simple voltage–driven linear RC circuit example.

$$\tilde{F}(\mathbf{v}_C)=F(\mathbf{v}_C)+\sum_{k=1}^{n_{iu}}\int_0^{v_{kC}}\hat{u}_{ki}(v'_{kC})dv'_{kC}$$

$$\tilde{G}(\mathbf{i}_L)=G(\mathbf{i}_L)+\sum_{k=1}^{n_{vu}}\int_0^{i_{kL}}\hat{u}_{kv}(i'_{kL})di'_{kL},$$

where $n_{iu}\le n_C,\ n_{vu}\le n_L$ are the number of added current and voltage sources, respectively, and $\hat{u}_{ki},\hat{u}_{kv}:\mathbb{R}\to\mathbb{R}$ are their characteristic functions, that are chosen by the designer.[6] As indicated in (11), and illustrated in Example B in the next subsection, these control actions enter through the Hessians $\nabla^2\tilde{F},\ \nabla^2\tilde{G}$. Henceforth, $q(t)$ can be regulated via a suitable selection of the "slopes" of the characteristic functions of the sources.

3. ILLUSTRATIVE EXAMPLES

A. A Forced LTI Circuit Consider the simple LTI RC circuit of Fig. 2 driven by a time–varying voltage source $v_S(t)$. The dynamics are described by (12) with

$$P_A(v_C,t)=-\frac{1}{2R}v_C^2+\frac{1}{R_1}v_Cv_S(t),\quad M=C,$$

where $\frac{1}{R}:=\frac{1}{R_1}+\frac{1}{R_2}$. The transformed model (13) has the parameters

$$\tilde{P}_A(v_C,t)=\frac{1}{C}\left[-\frac{1}{R}v_C+\frac{1}{R_1}v_S(t)\right]^2,\quad \tilde{M}=-\frac{2}{R}.$$

It is clear that

$$\tilde{P}_A(v_C(t),t)=\dot{v}_C(t)i_C(t)=q(t).$$

Computing its time derivative we get

$$\dot{q}(t)=-\frac{2}{R}\dot{v}_C^2(t)$$
$$+\frac{2}{C}\left[-\frac{1}{R}v_C(t)+\frac{1}{R_1}v_S(t)\right]\frac{1}{R_1}\dot{v}_S(t) \tag{15}$$

where the first right hand term is the one given in Proposition 1, with $G=g(t)=0$, while the remaining ones appear due to the time–variations of $v_S(t)$.

In this simple example we can actually compute an explicit expression of $q(t)$. Indeed, (15) can be written in the form

$$\dot{q}(t)=-\frac{2}{RC}q(t)+2\sqrt{\frac{q(t)}{C}}\frac{1}{R_1}\dot{v}_S(t)$$

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[6] For ease of presentation, but without loss of generality, we have made the sources dependent on the first few elements of the vectors $\mathbf{v}_C$ and $\mathbf{i}_L$.

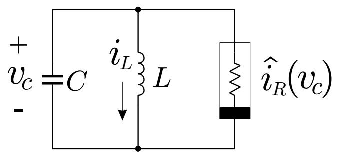

Fig. 3. A van der Pol oscillator.

Consider the case when $v_s(t) = \sqrt{2}V\cos(\omega t)$, then the solution of the differential equation above is easily obtained as

$$q(t) = \frac{2V^2R^2C\omega^2}{R_1^2(1+R^2C^2\omega^2)^2}\left[RC\omega\cos(\omega t) - \sin(\omega t)\right]^2 + \epsilon_t$$

where ϵ_t are exponentially decaying terms due to initial conditions. The average value of this function coincides with the classical reactive power of the circuit, as computed for instance in Example 11.7 of (De Carlo and Lin, 2001), that is

$$\frac{1}{T}\int_0^T q(t)dt = -\omega Q_\omega.$$

B. Reactive Power Control of an Oscillator Let us consider now the example of the van der Pol oscillator depicted in Fig. 3 that is described by

$$\begin{aligned} L\,\dot{\hat{i}}_L &= v_C \\ C\dot{v}_C &= -i_L - \hat{i}_R(v_C) \\ \hat{i}_R(v_C) &:= \alpha v_C(v_C^2 - 1) \end{aligned}$$

where $L, C, \alpha > 0$. The dynamics can be written in the form (12) with

$$P_A(i_L, v_C) = -i_L v_C - F(v_C), \quad M = \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}$$

where $F(v_C) = \frac{-\alpha}{4}v_C^2(v_C^2 - 2)$ is the circuits co–content. The transformed model (13) has the parameters

$$\tilde{P}_A(i_L, v_C) = \frac{1}{L}v_C^2 + \frac{1}{C}\left[-i_L + \alpha v_C(v_C^2 - 1)\right]^2,$$

$$\tilde{M}(v_C) = \begin{bmatrix} 0 & -1 \\ 1 & \alpha(3v_C^2 - 1) \end{bmatrix}.$$

Once again, we have that

$$\tilde{P}_A(i_L(t), v_C(t)) = \dot{\hat{i}}_L(t)v_L(t) + \dot{v}_C(t)i_C(t) = q(t).$$

The time derivative of the generalized reactive power is, according with (11), given by

$$\dot{q}(t) = -2\nabla^2 F\dot{v}_C^2(t) = 2\alpha(3v_C^2 - 1)\dot{v}_C^2(t)$$

As indicated in Remark 1 the generalized reactive power of the circuit can be "controlled" adding regulated sources. For instance, let us add a voltage–dependent current source in parallel with the capacitor, as shown in Fig. 4, where the control action is given as $u = \hat{u}_i(v_C)$, with $\hat{u}_i : \mathbb{R} \to \mathbb{R}$ a function to

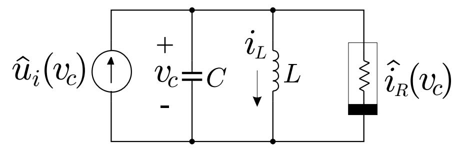

Fig. 4. van der Pol oscillator with a regulated current source.

be defined. One can easily verify that the previous analysis remains unaffected, and only the circuit co–content has to be changed to

$$\tilde{F}(v_C) = \frac{-\alpha}{4}v_C^2(v_C^2 - 2) + \int_0^{v_C}\hat{u}_i(v_C')dv_C'$$

The rate of change of the reactive power becomes now

$$\dot{q}(t) = -2\nabla^2\tilde{F}\dot{v}_C^2(t) = 2[\alpha(3v_C^2(t) - 1) - \nabla\hat{u}_i]\dot{v}_C^2(t).$$

The previous expression shows how we can modify $q(t)$ via a suitable selection of the "slope" of the function $\hat{u}_i(v_C)$. A similar effect is obtained, but now modulated by $(\dot{\hat{i}}_L)^2$, placing a current–dependent voltage source in series with the inductor.

4. OUTLOOK AND OPEN PROBLEMS

The results reported in this work are part of a long term research program whose final objective is the development of model–based compensator design methods for electric energy processing systems with nonlinear loads. A review of the literature reveals that the vast majority of the authors adopt a signal–processing viewpoint of the problem. At its most basic level, that prevails for instance in single phase active filters design, a desired waveform for the current is defined and compared with the actual signal to generate an error that a compensator (usually a set of nested PI's) tries to minimize. The intrinsic limitations of this approach, that makes no attempt to model the loads and essentially boils down to estimating derivatives of (highly noisy) signals, are quite evident. More "advanced" schemes try to capture on a (possibly vector) signal the effect of the loads. More precisely, looking at the voltage and current on the terminals of the load, a signal—axiomatically called reactive power or some variation of this name—is computed. It is then claimed that this signal measures the inactive power, therefore reducing its magnitude yields efficient compensation schemes.

A paradigmatic example of this approach is the highly popular DQ Reactive Power Compensation scheme of (Akagi *et al.*, 1984) that can be briefly described as follows, see also (Akagi *et al.*, 1999) for a modern tutorial account, and (Willems, 1972) for a particularly illuminating exposition of the main ideas that we briefly recall here. Consider a polyphase system with voltages and currents $(\mathbf{v}, \mathbf{i})$ providing a constant voltage to a load. Define

$$\mathbf{i}_p = \frac{\mathbf{i}^T\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}, \quad \mathbf{i}_q = \mathbf{i} - \mathbf{i}_p, \quad q_A = |\mathbf{v}||\mathbf{i}_q|.$$

With some simple geometric arguments we conclude that

$$|\mathbf{i}|^2 = \frac{p^2 + q_A^2}{|\mathbf{v}|^2}.$$

where $p = \mathbf{i}^T\mathbf{v}$ is the instantaneous active power. Now, using compensators without energy storage, p cannot be changed, therefore for a given voltage energy, the expression above shows that the current losses are minimized reducing q_A. This basic practical approach, introduced 21 years ago, was easily understood by practitioners and universally adopted as the *de facto* standard for power electronic compensators—probably because of the lack of a better option. It is only recently, see e.g. (Lev–Ari and Stankovic, 2003), that it is recognized that a compensator based on this notion will not only fail to eliminate the inactive power, but will certainly introduce additional undesired harmonics into the current waveform—putting a serious question mark into the effectiveness of this approach. (See (Pérez *et al.*, 2004) for a rectifier application where the DQ Reactive Power Compensation control is compared with a passivity–based LTI PI control.)

Substantiated by the arguments advanced in Willems in (Willems, 1991), it is our contention that this kind of signal–processing viewpoint is inadequate for the solution of the problem of interest in this paper. Therefore, following (Ortega *et al.*, 2001), we advocate an alternative control–by–interconnection viewpoint and adopt an operator–theoretic framework for our program. It is natural that the first steps in this program concern modelling aspects, as well as the exploration of new properties of these models—the results may be found in (Jeltsema and Scherpen, 2002; Ortega and Shi, 2002; Ortega *et al.*, 2003; Jeltsema *et al.*, 2003; Garcia–Canseco and Ortega, 2004), with some preliminary investigation on stabilization included in (Ortega *et al.*, 2003).

The present work constitutes the next, modest, step and its main contribution are some exploratory ideas for compensator design, based on the behavior of the operators supply rate, that we claim provides a measure of the circuits reactive power.

Many problems and questions remain open, among them we might cite: in a typical control configuration, the compensator is a multiport which is placed between the generator and the load. Therefore, the generator "sees" now a new load consisting of the cascade of the compensator and the original load. Following the philosophy of this paper and that of (Garcia–Canseco *et al.*, 2004), where loads are classified in terms of passivity of some suitably defined multiports, the aim of the compensator is then to modify the passivity properties of the new multiport associated to this new load. How to formalize this conceptual scheme, and propose a practical implementation for it, are open questions to be investigated.

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