EQUIVALENCE OF DIFFERENT REALIZABILITY CONDITIONS FOR NONLINEAR MIMO DIFFERENTIAL EQUATIONS

Ülle Kotta [*,1] **Tanel Mullari** [*]

[*] *Institute of Cybernetics at TUT, Akadeemia tee 21, 12618 Tallinn, Estonia. Phone: +372 620 4153, Fax: +372 620 4151, Emails: kotta@cc.ioc.ee, tanel@parsek.yf.ttu.ee*

Abstract: The relationship between three state space realizability conditions for nonlinear multi-input multi-output differential equations, formulated in terms of different mathematical tools, is studied. Moreover, explicit formulas are provided for calculation of the differentials of the state coordinates which, in case the necessary and sufficient realizability conditions are satisfied, can be integrated to obtain the state coordinates.

Keywords: Nonlinear control systems, Differential input-output equations, State space realization, Algebraic approaches, Geometric approaches.

1. INTRODUCTION

The paper compares distinct realizability conditions and realization algorithms in order to systematize the knowledge and to provide the *explicit* formulas for calculation the differentials of the state coordinates which, in case the necessary and sufficient realizability conditions are satisfied, can be integrated to obtain the state coordinates. Our aim is to extend the results of (Kotta and Mullari, 2003) to the multi-input multi-output (MIMO) case. In the above paper three apparently distinct (algebraic, geometric and Lie brackets based) intrinsic necessary and sufficient realizability conditions (der Schaft, 1987; der Schaft, 1989; Delaleau and Respondek, 1995; Moog *et al.*, 2002) for input-output differential equation are proved to be equivalent. Moreover, it was shown that the sufficient algorithm-dependent realizability conditions (Crouch and Lamnabhi-Lagarrigue, 1988; Glad, 1989) are tightly related to the above intrinsic condsitions as the algorithm constructs the basis vectors for the algebraic condition. Finally, alternative explicit formulas for calculation the differentials of the state coordinates is suggested. Since in (Moog *et al.*, 2002) only single-input single-output (SISO) systems are studied, instead of the above paper, we concentrate on paper (Kotta, 1998) that gives algebraic conditions under which the derivatives of the inputs can be eliminated in the generalized state equations and so, can be viewed as realizability conditions. We also extend the algorithms for calculating the state coordinates from (Crouch and Lamnabhi-Lagarrigue, 1988; Glad, 1989) and explicit formula from (Kotta and Mullari, 2003) for calculation the differentials of state coordinates to the MIMO case. Note that generalization to the MIMO case, though technically involved, is not difficult *once the extended system* corresponding to the set of input-output equations, *is properly defined* (more about this in Section 3), and the results carry over to the MIMO case.

[1] This work was partially supported by the Estonian Science Foundation Grant nr. 5405.

Note that explicit formula for calculating the basis of $\mathcal{H}_{s+2}$ is valid only under the assumption that $\mathcal{H}_1, \ldots, \mathcal{H}_{s+1}$ are completely integrable, though this assumption was erroneously missing in our previous paper (Kotta and Mullari, 2003).

Note that it is not our purpose to compare the results which study realization in *specific* state-space form, e. g. bilinear (Sontag, 1988), polynomial (Wang and Sontag, 1992) or state affine (Liu and Moog, 1994; Crouch and Lamnabhi-Lagarrigue, 1992).

2. THE REALIZABILITY CONDITIONS

Consider a nonlinear system described by p ($i = 1, \ldots, p$) i/o differential equations where the highest derivatives of y appear linearly

$$y_i^{(n_i)} = \varphi_i(y_k, \ldots, y_k^{(n_{ik})}, u_j, \ldots, u_j^{(s_{ij})}, \quad k = 1, \ldots, p, j = 1, \ldots, m). \tag{1}$$

Assumption 1. System (1) is strictly proper, i. e. $s_{ij} < n_i$, for $i = 1, \ldots, p$, $j = 1, \ldots, m$.

Assumption 2. System (1) is in a *canonical form*, which means that, $n_i \geq 1$, $n_1 \leq n_2 \leq \ldots \leq n_p$, $n_{ik} < \min(n_i, n_k)$, and $n_1 + n_2 + \ldots + n_p = n$ is the order of the system.

The latter implies that whenever (1) admits a Kalmanian realization, the indices n_i, associated to each output y_i, $i = 1, \ldots, p$, are the observability indices of any observable state-space realization of order n. The form (1) is an extension of the echelon canonical matrix description, introduced in (Popov, 1969) for linear systems. Note that every strictly proper system can be transformed into the above form (der Schaft, 1988). Define $s := \max s_{ij}$ and note that Assumption 2 yields $s < n_p$ [2]. The realization problem studied in this paper is defined as follows. Given equations (1), with $\varphi_i(\cdot)$ analytic, find, if possible, the state coordinates $x \in I\!R^n$, $x = \psi(y_i, \ldots, y_i^{(n_i-1)}, u_j, \ldots, u_j^{(s)}), \quad i = 1, \ldots, p, \quad j = 1, \ldots, m)$ such that in these coordinates the system takes the classical state space form, called the realization of (1):

$$\dot{x} = F(x, u), \quad y = h(x, u). \tag{2}$$

The solution of the realization problem in (der Schaft, 1987; der Schaft, 1989; Delaleau and Respondek, 1995; Kotta, 1998) is formulated in terms of the extended state-space system,

$$\dot{z} = f(z) + \sum_{j=1}^{m} g_j v_j, \tag{3}$$

associated with (1), with the inputs $v_j = u_j^{(s+1)}$, the state $z = [y_1, \ldots, y_1^{(n_1-1)}, \ldots, y_p, \ldots, y_p^{(n_p-1)}, u_1, \ldots, u_m, \ldots, u_1^{(s)}, \ldots, u_m^{(s)}]^T \in I\!R^{n+m(s+1)}$ and the vector fields $f(z)$ and g_j defined respectively as $f(z) = [z_2, \ldots, z_{n_1}, \varphi_1(z), \ldots, z_{n_1+\ldots+n_{p-1}+2}, \ldots, z_n, \varphi_p(z), z_{n+2}, \ldots, z_{n+s+1}, 0, \ldots, z_{n+(m-1)(s+1)+2}, \ldots, z_{n+m(s+1)}, 0]^T$ and $g_j = [0 \ldots 010 \ldots 0]^T$ where the $(n + ms + j)$th element is the only non-zero entry of g_j. In many papers on nonlinear control, system (3) is treated as the realization of (1). The disadvantage of the *extended state space realization* is that it uses the $(s+1)$th derivative of control $u^{(s+1)} = v$ as input. For linear systems it is possible to find an extended state coordinate transformation such that the system description in the new coordinates does not involve the explicit differentiation of the input. Unfortunately, this is not always possible for nonlinear systems. Therefore, it is important to characterize the input-output models (1) which do have an observable state space representation (2) of order n and to provide the algorithm to find the state coordinates. Below we give a brief exposition of realizability conditions.

A. Algebraic realizability conditions. Applying the results of (Kotta, 1998) to a realization problem, one has to start not from the i/o differential equation (1) but from the generalized state equations

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n_1-1} &= z_{n_1} \\
\dot{z}_{n_1} &= \varphi_1(z_1, \ldots, z_n, u, \dot{u}, \ldots, u^{(s)}) \\
&\vdots \\
\dot{z}_{n_1+\ldots+n_{p-1}+1} &= z_{n_1+\ldots+n_{p-1}+2} \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= \varphi_p(z_1, \ldots, z_n, u, \dot{u}, \ldots, u^{(s)})
\end{aligned} \tag{4}$$

associated to equation (1). Equations (4) are, aside a slight difference in notations, first n equations of the extended state space description (3).

In (Kotta, 1998) the realization problem for MIMO nonlinear systems is studied using the language of differential forms. Let $\mathcal{K}$ denote the field of meromorphic functions in the variables $\{z, v\}$, associated to (3). Over the field $\mathcal{K}$ one can define a vector space $\mathcal{E}^* := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$, spanned by the differentials of the elements of $\mathcal{K}$. Consider an one-form $\omega \in \mathcal{E}^*$: $\omega = \sum_i \alpha_i \mathrm{d}\varphi_i, \quad \alpha_i, \varphi_i \in \mathcal{K}$; its derivative $\dot{\omega}$ is defined according to $\dot{\omega} = \sum_i \dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i$ where $\dot{z}$ is defined by (3). The relative degree r of an one-form $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}$

---

[2] As for differences between the geometric conditions (der Schaft, 1989) and the commutativity conditions (Delaleau and Respondek, 1995) for the case $s = n_p$, see (Delaleau and Respondek, 1995)

is defined to be the least integer such that the rth derivative of one-form $\omega^{(r)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}$. If such an integer does not exist, we set $r = \infty$. A decreasing sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}^*$ is defined by (Aranda-Bricaire *et al.*, 1995)

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\} \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\},\ k \geq 1. \end{aligned} \tag{5}$$

Note that $\mathcal{H}_k$ is the space of one-forms whose relative degree is greater than or equal to k.

Theorem 3. (Kotta, 1998) The i/o differential equations (1) are locally realizable in the observable state space form (2) iff for $1 \leq k \leq s+2$ the subspaces $\mathcal{H}_k$ defined by (5) for the extended system (3) are integrable. The state coordinates can be found by integrating the basis vectors of $\mathcal{H}_{s+2}$.

B. Geometric realizability conditions. The realization problem in (der Schaft, 1987; der Schaft, 1989) is studied using the language of vector fields. The increasing sequence of distributions $\{S_k\}$ of $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\partial/\partial y_i, \ldots, \partial/\partial y_i^{(n_i-1)}, \partial/\partial u_j, \ldots, \partial/\partial u_j^{(s+1)}, i = 1, \ldots, p, j = 1, \ldots, m\}$ is defined by [3]

$$\begin{aligned} S_1 &= \mathrm{span}_{\mathcal{K}}\left\{\partial/\partial u_j^{(s+1)},\quad j = 1, \ldots, m\right\}, \\ S_{k+1} &= \bar{S}_k + [f, \bar{S}_k \cap \ker \mathrm{d}y \cap \ker \mathrm{d}u],\quad k \geq 1 \end{aligned} \tag{6}$$

where $\bar{S}$ denotes the involutive closure of the distribution S, and $[f, S]$ denotes the distribution spanned by all Lie brackets $[f, X]$, with X a vector field belonging to S. The distribution $S^* = S_{s+2}$ is the minimal conditionally invariant distribution for the extended system (3). Using the specific structure of the extended state space system (3), it has been proved by van der Schaft (1989) that if S_k, for $k = 1, \ldots, s+2$, are all involutive, then

$$\begin{aligned} &S_k \subset \ker \mathrm{d}u \cap \ker \mathrm{d}y,\quad k = 1, \ldots, s+1 \\ &S_{s+2} \cap \ker \mathrm{d}u \cap \ker \mathrm{d}y = S_{s+1} \\ &\dim S_k = km,\quad k = 1, \ldots, s+2 \end{aligned} \tag{7}$$

Theorem 4. (der Schaft, 1989) The i/o differential equations (1) are locally realizable in the observable state space form (2) iff all the distributions $S_1, \ldots, S_{s+2}$ defined by (6) for the extended system (3) are involutive.

C. Realizability conditions in terms of commutativity of iterative Lie brackets. Delaleau and Respondek also start from equations (4). The realizability conditions in (Delaleau and Respondek, 1995) are formulated in terms of the iterative Lie brackets of vector fields $f = \sum_{i=1}^{p}\left(\dot{y}_i(\partial/\partial y_i) + \ldots + \varphi_i(\cdot)(\partial/\partial y_i^{(n_i-1)})\right) + \sum_{j=1}^{m}\left(\dot{u}_j(\partial/\partial u_j) + \ldots + u_j^{(s+1)}(\partial/\partial u_j^{(s)})\right)$ and $g_j = \partial/\partial u_j^{(s)}$, $j = 1, \ldots, m$, defined by the extended system (3). Denote for $j = 1, \ldots, m$.

$$\begin{aligned} &L_f^0(\partial/\partial u_j^{(s)}) = \partial/\partial u_j^{(s)}, \\ &L_f^k(\partial/\partial u_j^{(s)}) = \left[f, L_f^{k-1}(\partial/\partial u_j^{(s)})\right], k \geq 1. \end{aligned}$$

Theorem 5. (Delaleau and Respondek, 1995) The i/o differential equations (1) are locally realizable in the observable state space form (2) iff for $0 \leq q$, $r \leq s$, $1 \leq j, l \leq m$

$$\left[L_f^q \frac{\partial}{\partial u_j^{(s)}}, L_f^r \frac{\partial}{\partial u_l^{(s)}}\right] \equiv 0. \tag{8}$$

Note that, in order to lower the order of the input derivative in (4) by one, the condition (8) has to hold for $0 \leq q$, $r \leq 1$, $1 \leq j, l \leq m$. The latter condition is satisfied only if $\partial^2\varphi_i(\cdot)/\ (\partial u_j^{(s)})^2 \equiv 0$, or equivalently if equations (1) are linear with respect to the highest derivatives of the inputs. In the MIMO case linearly with respect to the highest derivatives of controls is not sufficient for lowering the input derivatives by one. The system $\ddot{y} = y\dot{u}_1 + \dot{y}^2\dot{u}_2$ serves as an example.

3. MAIN RESULTS

The purpose of this section is to prove the equivalence of the three different realizability conditions recalled in the previous section. Moreover, we will provide explicit formulas for calculation of the basis vectors of the subspaces of one-forms $\mathcal{H}_k$, for $k = 3, \ldots, s+2$ and extend the algorithm-based solutions from (Crouch and Lamnabhi-Lagarrigue, 1988) to the MIMO case. Finally, we will demonstrate that the latter can be understood as the method to compute the basis vectors for $\mathcal{H}_k$, $k = 3, \ldots, s+2$.

3.1 Relationship of the sequences $\{\mathcal{H}_k\}$ and $\{\mathcal{S}_k\}$

This subsection establishes the relation between the sequences $\{\mathcal{H}_k\}$ and $\{S_k\}$.

Lemma 6. Assume that the distribution S_k, for $k = 1, \ldots, s+1$, is involutive, and the subspace of one-forms $\mathcal{H}_k$ annihilates the distribution S_k. Then the subspace of one-forms $\mathcal{H}_{k+1}$ annihilates the distribution S_{k+1}, that is $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$ for $k = 1, 2, \ldots, s+1$.

[3] Note that in (der Schaft, 1987) $S_1 = \mathrm{span}_{\mathcal{K}}\left\{(\partial/\partial u_j^{(s)})\right\}$.

This technical Lemma, proved in (Kotta and Mullari, 2003) for the SISO case, can be easily extended to the MIMO case, if we assume (as done in this paper) that in (3) we take the highest derivatives of all inputs equal to $s = \max s_{ij}$ even if in equations (1) the highest derivatives of the components are different. Otherwise, $S_k \not\subset \ker \mathrm{d}u \cap \ker \mathrm{d}y$ for all k values up to $s+1$ as in (7), and therefore, starting from next k S_k. Therefore we omit the proof. Note, that the condition of involutivity of S_k is essential to the proof of Lemma 6. If we drop this assumption, $\mathcal{H}_{k+1}$ does not necessarily annihilate S_{k+1}.

Lemma 7. For the extended system (3), $\dim \mathcal{H}_k = n + (s+2-k)m$, for $k = 1, \ldots, s+2$.

Theorem 8. The subspaces $\mathcal{H}_k$, $k = 3, \ldots, s+2$, defined by (5) for the extended system (3) are integrable iff the distributions S_k, $k = 3, \ldots, s+2$ defined by (6) for the extended system (3) are involutive.

Proof. From involutivity of a constant dimensional distribution follows complete integrability of its maximal annihilator and vice versa. Therefore, to prove the theorem, we have to show that $\mathcal{H}_{k+1}$ for $k = 2, \ldots, s+1$ is the maximal annihilator of S_{k+1}, i. e. $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$ and that $\operatorname{codim} \mathcal{H}_{k+1} = \dim S_{k+1}$, given that either S_k is involutive or $\mathcal{H}_k$ is completely integrable. The codimension of $\mathcal{H}_k$ in $\mathcal{E}^*$ is defined to be the dimension of $\mathcal{E}^*/\mathcal{H}_k$. Consider the subspace $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-1)}, i = 1, \ldots, p, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s-1)}, j = 1, \ldots, m\}$ which is obviously a maximal annihilator of $S_2 = \operatorname{span}_{\mathcal{K}}\{\partial/\partial u_j^{(s)}, \partial/\partial u_j^{(s+1)}\}$, i. e. $\mathcal{H}_2(S_2) \equiv 0$, and moreover, $\operatorname{codim}\mathcal{H}_2 = \dim S_2$. The proof is now by induction on k. We will show that if S_k is involutive then $\mathcal{H}_{k+1}$ is a *maximal* annihilator of S_{k+1}. From Lemma 6, $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$. Next, since by Lemma 7 $\dim \mathcal{H}_{k+1} = \dim \mathcal{H}_k - m$, or equivalently, $\operatorname{codim} \mathcal{H}_{k+1} = \operatorname{codim} \mathcal{H}_k + m$, the proof is completed by the fact that from (7) $\dim S_{k+1} = \dim S_k + m$.

Note that $\mathcal{H}_1$ is integrable by the definition and integrability of $\mathcal{H}_2$ follows from the special structure of the extended system (3). In a similar manner S_1 is involutive by the definition and involutivity of S_2 comes from the specific structure of (3).

3.2 The relationship between the geometric conditions and conditions in terms of iterative Lie brackets

Theorem 9. Involutivity of the distributions S_1, …, S_k, for $k = 3, \ldots, s+2$ is equivalent to condition (8) for $0 \le q$, $r \le k-2$, $1 \le j, l \le m$.

This Theorem, proved in (Kotta and Mullari, 2003) for the SISO case, can again be easily extended to the MIMO case if in (3) we take the highest derivatives of all inputs equal to $s = \max s_{ij}$. We omit the proof.

3.3 The algorithms for calculating the basis vectors of $\mathcal{H}_{s+2}$

In principle, $\mathcal{H}_{s+2}$ can be found using either definition (5), or the algorithm, given in (Aranda-Bricaire *et al.*, 1995). However, neither of them do not take into account the specific simple structure of the extended system (3). If we take this structure into account, and assume integrability of $\mathcal{H}_k$, $k = 1, \ldots, s+1$, the following recursive explicit algorithm can be obtained to compute the basis of $\mathcal{H}_{k+2} = \operatorname{span}_{\mathcal{K}}\{\omega_{1,1}^{[k+1]}, \ldots, \omega_{1,n_1}^{[k+1]}, \ldots, \omega_{p,1}^{[k+1]}, \ldots, \omega_{p,n_p}^{[k+1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-k-1)}\}$ from definition (5):

$$\begin{aligned} \omega_{i,l_i}^{[0]} &= \omega_{i,l_i}^{[1]} := \mathrm{d}y_{l_i}^{(l_i-1)}, \\ \omega_{i,l_i}^{[k+1]} &= \omega_{i,l_i}^{[k]} \\ &\quad -(-1)^k \sum_{j=1}^{m} \left\langle \omega_{i,l_i}^{[k]}, L_f^k \partial/\partial u_j^{(s)} \right\rangle \mathrm{d}u_j^{(s-k)}, \\ & i = 1, \ldots, p, l_i = 1, \ldots, n_i \quad k = 1, \ldots, s. \end{aligned} \tag{9}$$

At the kth step of the algorithm the one-form $\omega_{i,l_i}^{[k]}$, obtained at the previous step, is orthogonalized with respect to the vector fields $L_f^k(\partial/\partial u_j^{(s)})$, $j = 1, \ldots, m$. From direct computation we get that $\omega_{i,l_i}^{[k+1]}$ annihilate, or equivalently, the subspace of one-forms $\mathcal{H}_{k+2}$ annihilates all the vector fields $L_f^l(\partial/\partial u_j^{(s)})$, $l = 0, \ldots, k$, $j = 1, \ldots, m$.

Alternatively, instead of (9), another formula can be derived to compute $\omega_{i,l_i}^{[k+1]}$, $k = 1, \ldots, s$ in terms of Lie derivatives of one-forms, and not in terms of Lie derivatives of vector fields as in (9):

$$\omega_{i,l_i}^{[k+1]} = \omega_{i,l_i}^{[k]} - \sum_{j=1}^{m} \left\langle L_f^k \omega_{i,l_i}^{[k]}, \frac{\partial}{\partial u_j^{(s)}} \right\rangle \mathrm{d}u_j^{(s-k)}. \tag{10}$$

The advantage of using algorithms (9) or (10) lies in the fact that they can be directly and easily implemented in computer algebra program *Mathematica*. However, integration of the subspace $\mathcal{H}_{s+2}$ to obtain the state coordinates can be difficult.

The formulas (9) and (10) are equivalent. The proof is a straightforward extension of the proof in the SISO case ((Kotta and Mullari, 2003)) and therefore omitted.

3.4 Algorithmic realizability conditions

We extend to the MIMO case the constructive algorithm (up to the solution of the set of partial differential equations) for finding, if possible, the state coordinates from the input-output differential equations, given in (Crouch and Lamnabhi-Lagarrigue, 1988; Glad, 1989). Define $\bar{y} = (y_1, \ldots, y_1^{(n_1-1)}, \ldots, y_p, \ldots, y_p^{(n_p-1)})$ and $\bar{u} = (u_1, \ldots, u_1^{(s-1)}, \ldots, u_m, \ldots, u_m^{(s-1)})$. The starting point for the algorithm are not equations (1), but the equations where the highest derivatives of controls, $u_j^{(s)}$ appear already linearly

$$\begin{aligned} y_i^{(n_i)} &= \sum_{j=1}^{m} \alpha_{ij}(\bar{y}, \bar{u}) u_j^{(s)} + \beta_i(\bar{y}, \bar{u}), \\ i &= 1, \ldots, p \end{aligned} \tag{11}$$

The goal of the first step of the algorithm is to find the new generalized state variables $\tilde{z}_1, \ldots, \tilde{z}_n$ such that $\dot{\tilde{z}}$ does not depend on $u^{(s)}$. Note that only the n_1th, the $(n_1 + n_2)$th, …and the nth equations of (4) depend on $u^{(s)}$. So, one can define $\tilde{z}_i = z_i$, for $i = 1, \ldots, n_1 - 1, n_1 + 1, \ldots, n_1 + n_2 - 1, \ldots, n_1 + \ldots + n_{p-1} + 1, \ldots, n - 1$ and find for $k = n_1, n_1 + n_2, \ldots, n$

$$\tilde{z}_k = r_k(\bar{y}, \bar{u}) \tag{12}$$

such that $\dot{\tilde{z}}_k = \sum_{i=1}^{p}((\partial r_k/\partial y_i)\dot{y}_i + \ldots + (\partial r_k/\partial y_i^{(n_i-2)})y_i^{(n_i-1)} + (\partial r_k/\partial y_i^{(n_i-1)})[\alpha_{ij}(\cdot)u_j^{(s)} + \beta_i(\cdot)]) + \sum_{j=1}^{m}((\partial r_k/\partial u_j)\dot{u}_j + \ldots + (\partial r_k/\partial u_j^{(s-1)})u_j^{(s)})$ does not depend on $u^{(s)}$, which means that $r_k(\cdot)$ has to be a solution of the set of m partial differential equations in variables $\bar{y}$ and $\bar{u}$

$$\begin{aligned} &\left\langle \mathrm{d}r, -L_f \partial/\partial u_j^{(s)} \right\rangle = \\ &\sum_{i=1}^{p} \alpha_{ij}(\cdot) \frac{\partial r}{\partial y_i^{(n_i-1)}} + \frac{\partial r}{\partial u_j^{(s-1)}} = 0. \end{aligned} \tag{13}$$

The equation (13) is solvable if (8) is satisfied for $0 \leq q, r \leq 1$. Then there exist, at least locally, $n + (s-1)m$ independent solutions $r_1 = y_1, \ldots, r_{n_1-1} = y_1^{(n_1-2)}, \ldots,$ $r_{n_1+\ldots+n_{p-1}+1} = y_p, \ldots, r_{n-1} = y_p^{(n_p-1)}$, $r_{n+j} = u_j, \ldots, r_{n+m(s-1)+j} = u_j^{(s-2)}$ and p solutions $r_{n_1}, r_{n_1+n_2}, \ldots, r_n$ of the form (12), whose Jacobian with respect to $y_i, \ldots, y_i^{(n_i-1)}, u_j, \ldots, u_j^{(s-1)}$ is nonsingular and that satisfy (13). The generalized state equations in the new coordinates become

$$\begin{aligned} \dot{\tilde{z}}_1 &= \tilde{z}_2 \\ &\vdots \\ \dot{\tilde{z}}_{n_1-2} &= \dot{\tilde{z}}_{n_1-1} \\ \dot{\tilde{z}}_{n_1-1} &= \tilde{\varphi}_{11}(\tilde{z}_1, \ldots, \tilde{z}_n, u, \dot{u} \ldots, u^{(s-1)}) \\ \dot{\tilde{z}}_{n_1} &= \tilde{\varphi}_{12}(\tilde{z}_1, \ldots, \tilde{z}_n, u, \dot{u}, \ldots, u^{(s-1)}) \\ &\vdots \\ \dot{\tilde{z}}_{n_1+\ldots+n_{p-1}+1} &= \tilde{z}_{n_1+\ldots+n_{p-1}+2} \\ &\vdots \\ \dot{\tilde{z}}_{n-2} &= \tilde{z}_{n-1} \\ \dot{\tilde{z}}_{n-1} &= \tilde{\varphi}_{p1}(\tilde{z}_1, \ldots, \tilde{z}_n, u, \dot{u}, \ldots, u^{(s-1)}) \\ \dot{\tilde{z}}_n &= \tilde{\varphi}_{p2}(\tilde{z}_1, \ldots, \tilde{z}_n, u, \dot{u}, \ldots, u^{(s-1)}) \end{aligned} \tag{14}$$

At the next step, if $\tilde{\varphi}_{ij}(z, u, \ldots, u^{(s-1)})$, $i = 1, \ldots, p$, $j = 1, 2$ are linear in the highest time derivative of controls $u_j^{(s-1)}$, then the same procedure can be repeated to them to produce a new generalized state space representation with $u^{(s-2)}$ as the highest time derivative of the input.

Next, we will demonstrate that the algorithm described above, constructs exact basis vectors for the subspaces of one-forms $\mathcal{H}_3$, whenever possible. Note that equation (13) is solvable iff $\mathcal{H}_3$ is integrable, and the solutions $r_k(\cdot)$, $k = n_1, n_1 + n_2, \ldots, n$ of the form (12) define the new state coordinates $\tilde{z}_k = r_k(\bar{y}, \bar{u})$. We will demonstrate that $\mathrm{d}r_k = \mathrm{d}\tilde{z}_k \in \mathcal{H}_3$. According to (9), $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-2)}, \omega_{i,n_i}^{[2]}, i = 1, \ldots, p, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-2)}\}$, where for (11)

$$\omega_{i,n_i}^{[2]} = \mathrm{d}y_i^{(n_i-1)} - \alpha_{ij}(\cdot)\mathrm{d}u_j^{(s-1)}. \tag{15}$$

Note that the one-form $\omega_{i,n_i}^{[2]}$ annihilates $L_f \dfrac{\partial}{\partial u_j^{(s)}}$. So, if $\omega_{i,n_i}^{[2]}$ is exact, the solution of (13) can be obtained by integrating $\omega_{i,n_i}^{[2]}$. Though the one-form (15) is not necessarily exact, from integrability of $\mathcal{H}_3$, it is possible to find the integrating factors that make the solution exact and equal to $\mathrm{d}r_i$, r_i being the solution of (13).

In the similar manner it can be demonstrated that the next steps of the algorithm construct the exact basis vectors for $\mathcal{H}_4, \ldots, \mathcal{H}_{s+2}$, whenever possible.

4. EXAMPLE

We will demonstrate on the example below the equivalence of the considered methods. Consider the system

$$\ddot{y}_1 = y_2 u_1 + \dot{u}_2, \quad \ddot{y}_2 = y_1 \dot{u}_1 \tag{16}$$

Note that for (16)

$$f = \dot{y}_1 \frac{\partial}{\partial y_1} + (y_2 u_1 + \dot{u}_2)\frac{\partial}{\partial \dot{y}_1} + \dot{y}_2 \frac{\partial}{\partial y_2} + y_1 \dot{u}_1 \frac{\partial}{\partial \dot{y}_2}$$

$$+\dot{u}_1\frac{\partial}{\partial u_1}+\dot{u}_2\frac{\partial}{\partial u_2}+\ddot{u}_1\frac{\partial}{\partial \dot{u}_1}+\ddot{u}_2\frac{\partial}{\partial \dot{u}_2}$$

In order to calculate the sequence of subspaces $\{\mathcal{H}_k\}$ by (9) we first find

$$L_f\frac{\partial}{\partial \dot{u}_1}=-\frac{\partial}{\partial u_1}-y_1\frac{\partial}{\partial \dot{y}_2}$$
$$L_f\frac{\partial}{\partial \dot{u}_2}=-\frac{\partial}{\partial u_2}-\frac{\partial}{\partial \dot{y}_1}$$

which also yields that Lie brackets based conditions (8) are satisfied. So, $\omega_{11}^{[2]}=\mathrm{d}y_1$, $\omega_{21}^{[2]}=\mathrm{d}y_2$, $\omega_{12}^{[2]}=\mathrm{d}\dot{y}_1-\mathrm{d}u_2$, $\omega_{22}^{[2]}=\mathrm{d}\dot{y}_2-y_1\mathrm{d}u_1$ and

$$\mathcal{H}_3=\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1,\mathrm{d}\dot{y}_1-\mathrm{d}u_2,\mathrm{d}y_2,\mathrm{d}\dot{y}_2-y_1\mathrm{d}u_1\}$$

which is obviously completely integrable.

Next calculate according to (6),

$$S_3=\mathrm{span}_{\mathcal{K}}\left\{\frac{\partial}{\partial u_1}+y_1\frac{\partial}{\partial \dot{y}_2},\frac{\partial}{\partial \dot{u}_1},\frac{\partial}{\partial u_2}+\frac{\partial}{\partial \dot{y}_1},\frac{\partial}{\partial \dot{u}_2}\right\}$$

which is involutive, and the maximal annihilator of $\mathcal{H}_3$. We can find the coordinates

$$\begin{array}{l} x_1=y_1 \quad x_2=\dot{y}_1-u_2,\\ x_3=y_2,\ x_4=\dot{y}_2-u_1y_1 \end{array} \qquad (17)$$

of the integrable classical state space realization by integrating the integrable basis vectors of $\mathcal{H}_3$, that is after changing $\omega_{22}^{[2]}$ by $\omega_{22}^{[2]}-u_1\omega_{11}^{[2]}$. So, the state equations are

$$\begin{array}{ll} \dot{x}_1=x_2+u_2 & \dot{x}_2=x_3u\\ \dot{x}_3=x_4+u_1x_1 & \dot{x}_4=-u_1(x_2+u_2) \end{array}$$

From (13), the state coordinates r_2 and r_4 can be found as the solutions of the set of two partial differential equations

$$\frac{\partial r}{\partial u_1}+y_1\frac{\partial r}{\partial \dot{y}_2}=0,\quad \frac{\partial r}{\partial u_2}+y_1\frac{\partial r}{\partial \dot{y}_1}=0$$

whereas $r_1=y_1$ and $r_3=y_2$. It is easy to see, that x_2 and x_4 in (17) provide the solution.

REFERENCES

Aranda-Bricaire, E., C.H. Moog and J.B. Pomet (1995). A linear algebraic framework for dynamic feedback linearization. *IEEE Trans Autom Control* **40**, 127–132.

Crouch, P.E. and F. Lamnabhi-Lagarrigue (1988). State space realizations of nonlinear systems defined by input-output differential equations. In: *Analysis and Optimization Systems. Lect. Notes in Contr. and Inf. Sci.* (A. Bensousan and J.L. Lions, Eds.). Vol. 111. pp. 138–149.

Crouch, P.E. and F. Lamnabhi-Lagarrigue (1992). Realizations of input-output differential equations. In: *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing II. Proc. MTNS-91.* pp. 259–264. Mita Press.

Delaleau, E. and W. Respondek (1995). Lowering the orders of derivatives of controls in generalized state space systems. *J Math Systems, Estim and Control* **5**, 1–27.

der Schaft, A.J. (1988). Transformations of nonlinear systems under external equivalence. In: *Lect. Notes in Contr. and Inf. Sci.*. Vol. 122. pp. 33–43.

der Schaft, A.J. Van (1987). On realization of nonlinear systems described by higher-order differential equations. *Math Syst Theory* **19(20)**, 239–275 (305–306).

der Schaft, A.J. Van (1989). Transformations and representations of nonlinear systems. In: *Perspectives in Control Theory* (B. Jakubczyk *et al.*, Ed.). Birkhäuser. Boston.

Glad, S.T. (1989). Nonlinear state space and input-output descriptions using differential polynomials. In: *New Trends in Nonlinear Control Theory. Lect. Notes in Contr. and Inf. Sci.* (J. Descusse, M. Fliess, A. Isidori and P. Leborne, Eds.). Vol. 122. pp. 182–189. Springer-Verlag. New York.

Kotta, Ü. (1998). Removing input derivatives in generalized state-space systems: a linear algebraic approach. In: *Proceedings of the 4th International Conference on Actual Problems of Electronics Instrument Engineering.* pp. 141–147. Novosibirsk.

Kotta, Ü. and T. Mullari (2003). Realisation of nonlinear systems described by input/output differential equations: equivalence of different methods. In: *Proc. of the European Control Conference*. UK, Cambridge.

Liu, P. and C.H. Moog (1994). Minimal realization for single-input single-output nonlinear systems. In: *Proc. ACC.* pp. 551–552.

Moog, C. H., Y-F Zheng and P. Liu (2002). Input-output equivalence of nonlinear systems and their realizations. In: *Proc. of the 15th IFAC World Congress*. Barcelona.

Popov, V.M. (1969). Some properties of the control systems with irreducible matrix transfer functions. In: *Lecture Notes in Mathematics.* Vol. 144. pp. 169–180.

Sontag, E.D. (1988). Bilinear realizability is equivalent to existence of a singular affine differential i/o equation. *Systems and Control Letters* **11**, 181–187.

Wang, Y. and E.D. Sontag (1992). Algebraic differential equations and rational control systems. *SIAM J. Contr. and Opt.* **30**, 1126–1149.