PASSIVITY-BASED APPROACH TO PROBLEMS OF ROBUST SPATIAL MOTION CONTROL

Iliya V. MIROSHNIK

Laboratory of Cybernetics and Control Systems
State University of Inf. Technologies, Mechanics and Optics
Sablinskaya 14, Saint-Petersburg, 197101, Russia
miroshnik@yandex.ru

Abstract. The paper addresses problems of analysis and control of spatial behavior of uncertain dynamical systems associated with properties of invariance and attractivity of smooth geometric objects (goal sets). The use of geometric control theory and inequalities of passivity allows one to reduce the problems to input-to-state stability with respect to part of the variables, construct Lyapunov-like storage functions and propose simplified local solutions. On this basis, design procedures, static and dynamic control laws ensuring the desired properties of spatial dynamics are proposed.

Keywords: Nonlinear control, attractors, passivity, partial input-to-state stabilization

I. INTRODUCTION

The required performance of a dynamical system is often associated with achieving a desired mode of *spatial motion* $x(t) \in R^n$ prescribed by equations of curves, surfaces or other nontrivial geometric objects (goal sets) $\mathcal{Z}^*$ of the state space R^n given as

$$\varphi(x) \ = \ 0. \tag{1}$$

The relevant problems of *spatial motion control* (see Fradkov *et al.*, 1999; Korolev *et al.*,2000; Miroshnik, 2002a) consist in finding a control that provides invariance and attractivity of $\mathcal{Z}^*$. They are directly related to disturbance attenuation, qualitative and optimal control, output stabilization, coordination and curve/surface-following (Isidori, 1995; Elkin, 1998; Fradkov *et al.*,1999, *etc.*). On the other hand, the problems are closely connected with *partial stability*, or stability with respect to function $\xi = \varphi(x)$ (Vorotnikov, 1998 and 2004; Fradkov *et al.*,1999; Miroshnik, 2002b and 2004).

Usually, the know solutions (Isidori, 1995; Vorotnikov, 1998; Fradkov *et al.*, 1999), providing attractivity of submanifolds and partial stabilization of the system, are not robust, and even small variation of the models can induce a considerable change of the system's behaviour. Moreover, even in "good" cases there are a variety of *internal* disturbing factors such as functional and parametric uncertainties caused by absence of an exact description and slow variation of the models of the plant and the geometric object, as well as *external* disturbances and signal uncertainties. All of them can prevent a desired performance of the system that causes the necessity of studying the robust properties of the spatial motion control systems and finding constructive conditions, under which attractivity of the relevant sets and partial stability of uncertain systems are achieved.

The main purpose of this paper [1] is to represent an approach to the analysis of attractivity properties of multidimensional sets of smooth uncertain dynamical systems and methodologies of the design of control laws which provide an "approximate" partial stability, or more exactly, their (partial) *input-to-state stabilization* (ISS). The use of known techniques of geometric theory, allows one to reduce the problems under consideration to stability with respect to part of the variables, construct Lyapunov and storage functions and propose simplified local solutions. The results to be discussed are based on the general principles of input-to-state stability (Sontag *et al.*, 1995; Khalil, 1996) and *inequalities of passivity* (Fradkov *et al.*,1999; Polushin *et al.*, 2000), which (for bounded disturbances) leads to standard conditions of robustness (so called *asymptotic gain*). By using *quadratic storage functions*, estimates of the disturbed processes, as well as static and dy-

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namic control laws, providing the required robust and ISS properties, are obtained.

2. ATTRACTIVITY AND PARTIAL STABILITY

In the beginning, we analyze the behaviour of the smooth autonomous nonlinear system

$$\dot{x} \;=\; f_c(x), \tag{2}$$

where f_c is the smooth vector field supposed to be complete in an open set $\mathcal{X} \subset R^n$, with respect to a connected geometric object (1), or ν-dimensional smooth hypersurface $\mathcal{Z}^* \;=\; \{x \in \mathcal{X} :\;\; \varphi(x) = 0\}$. Here a smooth vector function $\varphi = \{\varphi_i\}$ ($i = 1, 2, \ldots, n-\nu$, $\nu < n$), is supposed to satisfy the local *regularity condition*, and therefore the set $\mathcal{Z}^*$ is *an embedded submanifold* of $\mathcal{X}$ (Isidori, 1995; Fradkov *et al.*, 1999). For the sake of simplicity, we also suppose that $\mathcal{Z}^*$ is one-sheeted and define the vector *of local coordinates* $z = \{z_i\} \in \mathcal{Z} \subset R^\nu$ as

$$z \;=\; \psi(x), \tag{3}$$

where ψ is a smooth mapping from $\mathcal{Z}^*$ to the open simply connected set $\mathcal{Z}$. In order to analyze the motion in the vicinity of $\mathcal{Z}^*$, we define a neighborhood of $\mathcal{Z}^*$ as an open simply connected set $\mathcal{E}(\mathcal{Z}^*) = \{x \in \mathcal{X} :\; \psi(x) \;\in \mathcal{Z}\} \supset\; \mathcal{Z}^*$, and introduce the vector of external dynamics (of deviations) $\xi \in \Xi \subset R^{n-\nu}$ as

$$\xi \;=\; \varphi(x). \tag{4}$$

The invariant set $\mathcal{Z}^*$ is called *an attracting submanifold* of the system (2) when

$$\lim_{t\to\infty}\; \mathrm{dist}(x(t,x_0), \mathcal{Z}^*) \;=\; 0 \tag{5}$$

uniformly with respect to $x_0 \in \mathcal{E}(\mathcal{Z}^*)$.

The system (2),(4) at the partial equilibrium point $\xi = 0$ is called *partially (uniformly) asymptotically stable* when

$$\lim_{t\to\infty}\; \xi(t,x_0) \;=\; 0 \tag{6}$$

uniformly with respect to $x_0 \in \mathcal{E}(\mathcal{Z}^*)$.

Find the Jacobian matrix of the mapping (3),(4) $J(x) \;=\; \begin{vmatrix} \Phi(x) \\ \Psi(x) \end{vmatrix} \;=\; \begin{vmatrix} \partial\varphi/\partial x \\ \partial\psi/\partial x \end{vmatrix}$ and the metric matrix $Q(x) = (J(x)J^T(x))^{-1}$. The invariant submanifold $\mathcal{Z}^*$ of a partially stable system is an attracting set under the following condition of *metric regularity* (Fradkov *et al.*, 1999; Miroshnik, 2004).

Assumption 1. For all $x \in \mathcal{E}(\mathcal{Z}^*)$, it holds that

$$q_1^2 I \;\le\; Q(x) \;\le\; q_2^2 I, \quad q_2 \ge q_1 > 0.$$

In order to obtain simplified conditions of attractivity of the submanifold $\mathcal{Z}^*$ and to analyze partial stability of the system (2), (4), we transform the system and reduce the problem to that of stability with respect to part of the variables (Miroshnik, 2001 and 2002a). By using the regular coordinate change (3)-(4) (diffeomorphism) with the smooth inverse $x = r(z,\xi)$, we obtain the *task-oriented* model

$$\dot{\xi} \;=\; f_\xi(\xi, z), \tag{7}$$
$$\dot{z} \;=\; f_z(\xi, z), \tag{8}$$

where $f_\xi = (\Phi\; f) \circ r(\xi, z)$, $f_z = (\Psi\; f) \circ r(\xi, z)$, and $f_\xi\;(0, z) \;=\; 0$. Perform an expansion of f_ξ and rewrite (7) in the form

$$\dot{\xi} \;=\; A_c(z)\xi + o(\xi, z), \tag{9}$$

where $A_c = (\partial f_\xi/\partial\xi)|_{\xi=0}$, assuming that $o(\xi,z)/|\xi| \to 0$ as $\xi \to 0$ uniformly in $z \in \mathcal{Z}$. Now the transformed system is represented by the linear-like nonstationary part (9) which parameters are generated by the nonlinear model (8).

Suppose that for all $z \in \mathcal{Z}$ the matrix $A_c(z)$ is bounded and define a function $\lambda(z)$ such that

$$\lambda(z) \;>\; \max_i\; \mathrm{Re}\; \lambda_i\; \{A_c(z)\}. \tag{10}$$

Then we can choose a Lyapunov-like function

$$V(x) = \xi^T P(z)\xi, \tag{11}$$

where the matrix $P = P^T$ is found as a solution of Lyapunov-like equation

$$A_c(z)^T P(z) + P(z)A_c(z) = -\epsilon I + 2\lambda(z)P(z), \tag{12}$$

$\epsilon > 0$. Consider the matrix $\dot{P}(z)$ and write

$$\dot{P} \;=\; \Pi(z) + O(\xi, z) \tag{13}$$

where $\Pi(z) \;=\; \{(\partial P_i/\partial z)f_{z0}(z)\}$, supposing that $O(\xi, z) \to 0$ as $\xi \to 0$ uniformly in $z \in \mathcal{Z}$.

By using Lyapunov Lemma, one can prove the following result (see Miroshnik, 2001 and 2002b).

Lemma 1. Suppose that, for all $z \in \mathcal{Z}$, the function $\lambda(z)$ obeys inequality (10), then there exist numbers $\pi_2 \ge \pi_1 > 0$ such that

$$\pi_1^2 I \le P(z) \le \pi_2^2 I. \tag{14}$$

If, additionally, there exists $\lambda_0 > 0$ obeying, for all all $z \in \mathcal{Z}$, the inequality

$$\Pi(z) \;+\; 2(\lambda(z) + \lambda_0)P(z) \;\le\; 0, \tag{15}$$

than there exists a number $\widehat{\lambda}_0 \in (0, \lambda_0)$ such that

$$\dot{V}(x) + 2\;\widehat{\lambda}_0 V(x) \le\; 0. \tag{16}$$

Inequality (16) leads to

$$|\dot{\xi}|_P + \widehat{\lambda}_0|\xi|_P \;\le\; 0 \tag{17}$$

and therefore

$$|\xi(t)|_P \;\leq\; e^{-\widehat{\lambda}_0 t}\,|\xi_0|_P. \qquad (18)$$

In the view of Assumption 1, we can formulate the following result (see Miroshnik, 2001 and 2002b).

Theorem 1. Suppose that Assumption 1 holds and in the neighborhood $\mathcal{E}(\mathcal{Z}^*)$ the system (2) is complete and satisfies the conditions of Lemma 1. Then the system (2), (4) at the point $\xi = 0$ is asymptotically stable with respect to ξ, and the set $\mathcal{Z}^*$ is an attractor of the system (2).

3. PARTIAL PASSIVITY AND ROBUSTNESS

Consider the disturbed system

$$\dot{x} \;=\; f_c(x) + \Delta(x,t), \qquad (19)$$

where Δ is an n-dimensional vector of disturbances supposed to be continues (in x and time) and, at least locally (in time), bounded in some domain $\mathcal{E}(\mathcal{Z}^*)$. We restrict the consideration to a case when, in the vicinity of $\mathcal{Z}^*$, the functions ψ, φ and Δ satisfies the following.

Assumption 2. There exists a smooth vector field $g_\xi(x) \in \mathbb{R}^n$ such that, for all $x \in \mathcal{E}(\mathcal{Z}^*)$, it holds that

$$\frac{\partial\varphi}{\partial x}\,\Delta(x,t) \;\in\; span\{\frac{\partial\varphi}{\partial x}\,g_\xi(x)\}, \quad \frac{\partial\psi}{\partial x}\,\Delta \;=\; 0.$$

Under the assumption the task-oriented model takes the form

$$\dot{\xi} \;=\; f_\xi(\xi,z) + g_\xi(\xi,z)w_1, \qquad (20)$$

and (8), where $g_\xi = (\Phi\ g)\circ r(z,\xi)$, $w_1 = w_1(x,t)$ is a scalar disturbance.

Let the point $\xi = 0$ be a partial equilibrium of system (19) when $\Delta = 0$, and, at this point, the latter be partially (uniformly) asymptotically stable. Behavior of the disturbed system in the vicinity of the submanifold $\mathcal{Z}^*$ is connected with the following property of input-to-state stability.

The system (19), (4) is called *partially input-to-state stable (partially ISS)* when for all continues $w_1(x,t) \in \mathcal{L}_p$ it holds that $|\xi(t| \in \mathcal{L}_p$, $p \in [1,\infty]$.

In the case under consideration, the definition implies, in particular, that $|\xi(t)|$ is square integrable (and tends to zero as $t \to \infty$) when $w_1(x,t)$ is square integrable, and if $w_1(x,t)$ is globally bounded, so is $|\xi(t)|$. The latter corresponds to the following notion.

The system (19), (4) is called *partially robust* when for all (globally) bounded disturbances $w_1(x,t) \in \mathcal{L}_\infty$ it holds that

$$\overline{\lim_{t\to\infty}}\ \xi(t,x_0) \;\leq\; \gamma(\|w_1\|_\infty), \qquad (21)$$

where γ is $\mathcal{K}$ function (*nonlinear asymptotic gain*), uniformly with respect to $x_0 \in \mathcal{E}(\mathcal{Z}^*)$.

Introduce expansions of f_ξ and g_ξ and write:

$$f_\xi \;=\; A_c\xi + o_1(\xi,z),\ g_\xi \;=\; b(z) + B(z)\xi + o_2(\xi,z),$$

where $A_c = (\partial f_\xi/\partial\xi)|_{\xi=0}$, $B = (\partial g_\xi/\partial\xi)|_{\xi=0}$, $b(z) = g_\xi(0,z)$, and $o_1(\xi,z)/|\xi| \to 0$, $o_2(\xi,z)/|\xi| \to 0$ as $\xi \to 0$ uniformly in $z \in \mathcal{Z}$. The transformed system is represented by the linear-like nonstationary part

$$\begin{aligned}\dot{\xi} \;&=\; A_c(z)\xi + \Big(b(z) + B(z)\xi\Big)w_1 + \\ &+\; \Big(o_1(\xi,z) + o_2(\xi,z)w_1\Big), \qquad (22)\end{aligned}$$

parameters of which are generated by the nonlinear model (8).

In order to investigate the partial ISS property of the system, we choose a quadratic *storage function* $V(x)$ of the form (11)-(12) and introduce *the virtual output*

$$v \;=\; q^T(z)\xi, \qquad (23)$$

where $q^T(z)$ is a row matrix defined as

$$q^T(z) \;=\; b^T(z)P(z). \qquad (24)$$

Suppose that for $z \in \mathcal{Z}$ the matrix $A(z)$ is bounded and there exists a function $\lambda(z)$ obeying inequality (10). Then, by using Lyapunov Lemma, one can prove the following result.

Lemma 2. Suppose that the function $\lambda(z)$ obeys inequality (10), then there exist numbers $\pi_2 \geq \pi_1 > 0$ such that inequality (14) holds.

If, additionally, for all $w_1(x,t)$ and $z \in \mathcal{Z}$, there exists $\lambda_0 > 0$ obeying the inequality

$$\begin{aligned}\Pi(z) \;&+\; (B^T(z)P(z) + P(z)B(z))w_1 + \\ &+\; (\lambda(z) + \lambda_0)P(z) \;\leq\; 0, \qquad (25)\end{aligned}$$

then there exists numbers $\widehat{\lambda}_0 \in (0,\lambda_0)$, such that

$$\dot{V}(x) + 2\widehat{\lambda}_0 V(x) \;\leq\; 2vw_1. \qquad (26)$$

Inequality (26) leads to

$$|\dot{\xi}|_P + \widehat{\lambda}_0|\xi|_P \;\leq\; |b(z)|_P|w_1|. \qquad (27)$$

Suppose that for all $z \in \mathcal{Z}$ the matrix $b(z) = g(0,z)$ is bounded, and $|b(z)|_P \leq \beta$, where $\beta > 0$. Then inequality (27) yields the estimate

$$|\xi(t)|_P \;\leq\; e^{-\widehat{\lambda}_0 t}\,|\xi_0|_P + \beta\overline{w}(t), \qquad (28)$$

where $\overline{w} = \int_0^t e^{-\widehat{\lambda}_0(t-\tau)}|w_1(\tau)|d\tau$. The latter leads to the following conclusion.

Theorem 2. Suppose that Assumption 1–2 hold and, in the neighborhood $\mathcal{E}(\mathcal{Z}^*)$, the system (19) is complete and satisfies the conditions of Lemma 2. Then the system (19), (4) is partially input-to-state stable and, for the globally bounded disturbances w_1, is partially robust.

Note that for the case considered the *asymptotic gain* in relation (21) is linear and found as

$$\gamma \;=\; \frac{\beta}{\lambda_0}\|w_1\|_\infty.$$

4. STABILIZATION AND STATIC ROBUST CONTROL

First, consider the smooth undisturbed system

$$\dot{x} \;=\; f(x) + g(x)u, \qquad (29)$$

where $u \in R^1$ is the input (control), f and g are the smooth vector fields defined in $\mathcal{X}$. The general problems of control is stated as follows.

Control problem 1. Find a control law which provides partial stabilization of the system (29) with respect to the equilibrium point $\xi = 0$ and attractivity of the submanifold $\mathcal{Z}^*$.

The solution of the problem is given by the control (Fradkov *et al.*, 1999; Miroshnik, 2002a)

$$u \;=\; -\,w_2(z) - v, \qquad (30)$$

where w_2 is an *invariant control* and v is a *stabilizing control*, satisfying the condition $v \;\to 0$ as $\xi \to 0$. Let us introduce additional hypotheses.

Assumption 3. For all $x \in \mathcal{Z}^*$ it holds that

$$\frac{\partial\varphi}{\partial x}f(x) \in \text{span}\left\{\frac{\partial\varphi}{\partial x}\;g(x)\right\},\;\; \text{rank}\Big(\frac{\partial\varphi}{\partial x}\;g(x)\Big) = 1.$$

The assumption sets, in particular, that the invariant control w_2 can be found as a solution of the equation

$$f_\xi(0,z) - g_\xi(0,z)w_2(z) \;=\; 0. \qquad (31)$$

Carry out transformation into the coordinates (ξ, z) and obtain the task-oriented model of the controlled system

$$\dot{\xi} \;=\; f_\xi(\xi,z) + g_\xi(\xi,z)u, \qquad (32)$$
$$\dot{z} \;=\; f_z(\xi,z), \qquad (33)$$

where $f_\xi = (\Phi\; f)\circ r(z,\xi)$, $g_\xi = (\Phi\; g)\circ r(z,\xi)$. Introduce an expansion of f_ξ and g_ξ:

$$f_\xi \;=\; f_\xi(0,z) + \frac{\partial f_\xi}{\partial\xi}\Big|_{\xi=0}\xi + o_1(\xi,z), \qquad (34)$$
$$g_\xi \;=\; g_\xi(0,z) + \frac{\partial g_\xi}{\partial\xi}\Big|_{\xi=0}\xi. + o_2(\xi,z). \qquad (35)$$

The model (32) under the control (30), where w_2 satisfies (31), takes the form of the linear-like nonstationary system

$$\dot{\xi} \;=\; A(z)\xi - b(z)v - \Big(B(z)\xi + o_2(\xi,z)\Big)v$$
$$+\;\; \Big(o_1(\xi,z) + o_2(\xi,z)w_2\Big), \qquad (36)$$

where $A(z) = (\partial f_\xi/\partial\xi) - B(z)w_2$, $B(z) = (\partial g_\xi/\partial\xi)$, $b(z) = g_\xi(0,z)$. The main control problem is reduced to finding a stabilizing control v.

Suppose that for $z \in \mathcal{Z}$ the matrices $A(z)$ and $b(z)$ are bounded and choose v in the form (Fradkov *et al.*,1999; Miroshnik, 2002a)

$$v \;=\; k^T(z)\xi, \qquad (37)$$

where $k^T(z)$ is the row matrix of varying coefficients defined as

$$k^T(z) \;=\; b^T(z)P(z), \qquad (38)$$

$P = P^T$ is a solution of the algebraic Riccati equation

$$A^T(z)P(z) + P(z)A(z) \;-$$
$$-2P(z)b(z)b^T(z)P(z) \;=\; 2\lambda(z)P(z) \qquad (39)$$

and the function $\lambda(z)$ satisfies the inequality

$$\lambda(z) \;<\; \min_i\; \text{Re}\;\lambda_i\;\{A(z)\}. \qquad (40)$$

The model (36) takes the form (9), where $A_c = A - bk^T = A - bb^TP$. In order to find conditions for partial stability of the closed loop system (9),(33), we can choose a Lyapunov-like function $V(x)$ of the form (11). Then, by using properties of algebraic Riccati equations and Lemma 1, one can prove the following result (Miroshnik, 2002a).

Lemma 3. Suppose that for all $z \in \mathcal{Z}$, the pair $A(z), b(z)$ is completely controllable, and the function $\lambda(z)$ obeys inequality (40), then there exist numbers $\pi_2 \geq \pi_1 > 0$ such that equation (14) holds,

If, additionally, there exists $\lambda_0 > 0$ obeying, for all all $z \in \mathcal{Z}$, inequality (15) then Lyapunov function $V(x)$ satisfies inequality (16).

Inequality (16) leads to the exponential estimate (18). The latter yields the following conclusion.

Theorem 3. Suppose that Assumptions 1, 3 hold and, in the neighborhood $\mathcal{E}(\mathcal{Z}^*)$, the system (29) is complete and satisfies the conditions of Lemma 3. Then under the nonlinear control (30), (37) the set $\mathcal{Z}^*$ is an invariant set and an attractor of the system (29).

Now consider a more general case and the disturbed controlled system

$$\dot{x} \;=\; f(x) + g(x)u + \Delta(x,t), \qquad (41)$$

assuming that parameters of the system and the gaol set, as well as the disturbance $\Delta = \Delta(x,t)$, are not known exactly. Then the spatial motion control problem is reduces to the following.

Control problem 2. Find a control law which provides
(i) partial input-to-state stabilization of the system (41), (4);
(ii) partial robustification of the system (41), (4) with respect to the disturbance $\Delta(x,t)$ and uncertainties of the system.

Suppose that the functions Δ and g satisfy Assumption 2. It sets requirements for the disturbance Δ which lead to simplification of the task-oriented model and ensure existence of an exact control, solving the problem. Under Assumption 1-3, the task-oriented model of the disturbed system (41) takes the form

$$\dot{\xi} = f_\xi(\xi, z) + g_\xi(\xi, z)(u + w_1) \qquad (42)$$

and (33), where $w_1 = w_1(x,t)$, and there exists a function $w_2(z)$ derived from equation (31). Then an exact solution of the control problem is given by the control law

$$u = -w - v, \qquad (43)$$

where $w = w_1 + w_2$, and v is a stabilizing control, chosen in the form (37)-(38).

When w_1 and w_2 are unknown one can make use of an approximate static control law of the form

$$u = -\widehat{w} - v, \qquad (44)$$

where $\widehat{w}$ is an estimate of w. Perform partial linearization of the system and rewrite (42) in the form

$$\dot{\xi} = A(z)\xi - b(z)v + b(z)\widetilde{w}, \qquad (45)$$

where $\widetilde{w}$ is a residual defined as

$$\widetilde{w} = w - \widehat{w}.$$

When the stabilizing control v is chosen in the form (37)-(38), the error model (45) takes the form

$$\dot{\xi} = A_c(z)\xi + b(z)\widetilde{w}. \qquad (46)$$

Under appropriate conditions of Lemmas 2-3 we obtain inequalities (14) and

$$\dot{V}(x) + 2\widehat{\lambda}_0 V(x) \leq 2v\widetilde{w}. \qquad (47)$$

The latter expression leads to

$$|\dot{\xi}|_P + \widehat{\lambda}_0|\xi|_P \leq |b(z)|_P|\widetilde{w}| \qquad (48)$$

and the estimate

$$|\xi(t)|_P \leq e^{-\widehat{\lambda}_0 t}|\xi_0|_P + \beta\overline{w}(t), \qquad (49)$$

where $\overline{w} = \int_0^t e^{-\widehat{\lambda}_0(t-\tau)}|\widetilde{w}(\tau)|d\tau$. For bounded residuals $\widetilde{w}$, it holds that

$$\overline{\lim_{t\to\infty}}\, \xi(t, x_0) \leq \frac{\beta}{\widehat{\lambda}_0}\|\widetilde{w}\|_\infty. \qquad (50)$$

Therefore the following is proved.

Theorem 5. Suppose that Assumption 1-3 hold and, in the neighborhood $\mathcal{E}(\mathcal{Z}^*)$, the system (41) is complete and satisfies the conditions (25),(40). Then the system (41), (4) under the control (44) is partially input-to-state stable and, for the globally bounded disturbances $\widetilde{w} = w - \widehat{w}$, is partially robust.

5. DESIGN OF DYNAMIC ROBUST CONTROL

Boundedness of the error ξ is achieved for bounded residuals $\widetilde{w} = w - \widehat{w}$. This usually implies that we know a sufficiently exact estimate $\widehat{w}$ of the disturbance w. In a possible case of an unboundeded residial, one can make use of dynamic control laws.

Let the disturbance w be represented as an output of the linear model:

$$\dot{\zeta} = F\zeta + d\delta_1, \qquad (51)$$
$$y = c^T\zeta, \quad w = \delta_0 + y, \qquad (52)$$

where $\zeta \in \mathbb{R}^m$, (F, d, c) is a nondegenerate triple of known matrices, and $\zeta(0) = \zeta_0$, $\delta_0(t), \delta_1(t)$ are considered as unknown components. We use a control law of the form

$$u = -\widehat{w} - v = -\widehat{\delta}_0 - c^T\widehat{\zeta} - v, \qquad (53)$$

Here the stabilizing signal v is found in the form (37), where the matrix of feedback gains $k(z)$ is computed according (38), and the estimates of ζ, y, w are generated by using the model

$$\dot{\widehat{\zeta}} = F\widehat{\zeta} - \kappa v + d\widehat{\delta}_1, \qquad (54)$$
$$\widehat{y} = c^T\widehat{\zeta}, \quad \widehat{w} = \widehat{\delta}_0 + \widehat{y} \qquad (55)$$

where κ is a column matrix of the gains to be found later, $\widehat{\delta}_0(t), \widehat{\delta}_1(t)$ are *a priori* known estimates of the appropriate functions.

Defining the residuals

$$\widetilde{\zeta} = \zeta - \widehat{\zeta}, \; \widetilde{y} = y - \widehat{y}, \quad \widetilde{w} = w - \widehat{w},$$

one can obtain an $n+m$-dimensional error model represented by equations (46), (37) and

$$\dot{\widetilde{\zeta}} = F\widetilde{\zeta} + \kappa v + d\widetilde{\delta}, \qquad (56)$$
$$\widetilde{y} = c^T\widehat{\zeta}, \quad \widetilde{w} = \widetilde{y} + \widetilde{\delta}_0, \qquad (57)$$

where $\widetilde{\delta}_0 = \delta_0 - \widehat{\delta}_0$, $\widetilde{\delta} = \delta - \widehat{\delta}$. Estimates of the errors depend on the choice of the matrix κ which can be carried out as follows.

Introduce a partial coordinate change for the error model (46),(37),(56),(57), defining the transformed vector

$$\overline{\xi} = \xi + ac^T\widetilde{\zeta} \qquad (58)$$

and the transformed output signal

$$\overline{v} = k^T\overline{\xi} = v + \alpha\widetilde{y}, \qquad (59)$$

where $\alpha = k^Ta$. Here the column vector a must be found as a solution of the quadratic equation

$$-\overline{A}_c ac^T + ac^TF - bc^T = 0, \qquad (60)$$

where

$$\overline{A}_c = A_c + c^T\kappa ad^T. \qquad (61)$$

The model of the transformed error model takes the form

$$\dot{\overline{\xi}} = \overline{A_c \xi} + b\widetilde{\delta}_0, \quad (62)$$

$$\dot{\widetilde{\zeta}} = \overline{F}\widetilde{\zeta} + \kappa\ k^T \overline{\xi} + d\widetilde{\delta}_1, \quad (63)$$

where

$$\overline{F} = F - \gamma \kappa c^T. \quad (64)$$

Thus, the transformed error model is represented as a serial connection of the block (62) with input $\widetilde{\delta}_0$ and output $\overline{v}$ and the block (63) with input $\widetilde{\delta}_1$ and output $\widetilde{y}$.

First, consider the block (63) and choose the storage function

$$\widetilde{V}(\widetilde{\zeta}) = \widetilde{\zeta}^T N \widetilde{\zeta}, \quad (65)$$

where $N = N^T$ is a solution of the algebraic equation

$$-(F + \widetilde{\lambda} I)^T N - N(F + \widetilde{\lambda} I) = -2cc^T, \quad (66)$$

and $\widetilde{\lambda} > 0$ satisfies the inequality

$$\widetilde{\lambda} \ > \ -\min_i \ \mathrm{Re}\ \lambda_i \ \{F)\}. \quad (67)$$

This requirement and observability of the pair (F, c) ensure the existence of the solution $N > 0$ of the equation (66) and therefore $\widetilde{V}(\widetilde{\zeta}) > 0$.

Choose the matrix κ as

$$\kappa \ = \ (\gamma)^{-1} N^{-1} c. \quad (68)$$

Substituting the latter into (61) and (64), we find

$$\overline{F} = F - N^{-1} cc^T, \quad (69)$$

$$\overline{A}_c = A_c + \frac{c^T N^{-1} c}{\alpha}\ ad^T. \quad (70)$$

The latter is used for finding the vector a (see (60)).

Now, we analyze the properties of the transformed error model. Differentiating the storage function (65), after appropriate substitutions one obtains

$$\dot{\widetilde{V}} + 2\widetilde{\lambda}\widetilde{V} \ = \ 2(\gamma)^{-1} \overline{v}\widetilde{y} + 2(d^T N \zeta)\widetilde{\delta}_1. \quad (71)$$

For the block (62),(59), define the storage function

$$\overline{V}(\overline{\xi}) \ = \ \overline{\xi}^T P \overline{\xi}, \quad (72)$$

where $P(z) > 0$ is a solution of equation (39). After the differentiation with respect to time and appropriate substitutions, we find

$$\dot{\overline{V}}(\xi) + 2\overline{\lambda V} \ \le \ 2\ \overline{v}\widetilde{\delta}_0, \quad (73)$$

where

$$\overline{\lambda} = \widehat{\lambda}_0 - \frac{c^T N^{-1} c}{\alpha} |a|_P |d|_P.$$

Relations (71) and (73) lead to the following differential inequalities

$$|\dot{\overline{\xi}}|_P + \overline{\lambda}|\overline{\xi}|_P \ \le \ |b|_P |\widetilde{\delta}_0|, \quad (74)$$

$$|\dot{\widetilde{\zeta}}|_N + \widetilde{\lambda}|\widetilde{\zeta}|_N \ \le \ |b|_P |\kappa|_N |\overline{\xi}|_P + |d|_N |\widetilde{\delta}_1|, \quad (75)$$

connected with the state ξ of the initial error model by means of the expression

$$|\xi|_P \ \le \ |\overline{\xi}|_P + \overline{\alpha}|\widetilde{\zeta}|_N, \quad (76)$$

where $\overline{\alpha} > 0$.

When $\widetilde{\delta}_1 = \widetilde{\delta}_0 = 0$, one can obtain that partial asymptotic stability of the system is achieved. In a more general case, when the functions $\widetilde{\delta}$ and $\widetilde{\delta}_0$ are globally bounded, from (74)-(76) it follows that

$$\overline{\lim_{t\to\infty}}\ |\xi(t, x_0)|_P \ = \ C_0 \|\widetilde{\delta}_0\|_\infty + C_1 \|\widetilde{\delta}_1\|_\infty, \quad (77)$$

where $C_0 > 0$, $C_1 > 0$.

Therefore the dynamic control (53)-(55) provides boundedness of the external dynamics $\xi(t)$ for the globally bounded inputs δ_0, δ_1 and the required partial robustification of the system (41).

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