ROBUST ESTIMATION VERSUS OPTIMAL ESTIMATION AND THEIR APPLICATION FOR AIRBORNE GRAVIMETRY

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Abstract: The efficiency of the robust estimation approach based on the information about the variances of the signal itself and its several derivatives is analyzed. Comprehensive comparison of robust and optimal estimators is performed. It is shown that the robust estimator suggested provides the reliable and precise signal estimation in the presence of noise. The obtained theoretical results are illustrated with an example of an airborne gravimetry problem. Advantages and drawbacks of robust and optimal estimators are demonstrated for various operating conditions.

Keywords: Optimal and robust estimation, accuracy, sensitivity, airborne gravimetry.

1. INTRODUCTION

For many applications it is necessary to estimate the valid signal using noisy measurements $y(t)$ of the form

$$y(t) = x(t) + n(t)\,, \qquad (1)$$

where the valid signal $x(t)$ and the measurement noise $n(t)$ are assumed to be stationary, centered and uncorrelated relative to each other stochastic processes.

Specifying stochastic models for $x(t)$, $n(t)$ the estimation problem can be formulated in the framework of the optimal estimation theory (Van Trees, 1968). According to this theory, the optimal (minimum variance) estimate $\hat{x}$ provides the minimum for the following criterion:

$$D_e = E\left[(x - \hat{x})\right]^2\,,$$

where E - the mathematical expectation.

Minimizing this expression for the known spectral densities of the signal $S_x(\omega)$ and measurement noise $S_n(\omega)$, it is not difficult to derive the optimal estimator transfer function (TF) $H_{opt}(j\omega)$ and the error variance corresponding to it (Van Trees, 1968). Unfortunately, it is not infrequent that the reliable information about the spectral densities $S_x(\omega)$ and $S_n(\omega)$ is not available.

It is known that the disagreement between the actual model and the accepted mathematical one often causes not only difference between expected accuracies and their actual values, but also divergence of the filter, i.e., unlimited error in filtering. In order to cope with this problem the design of estimators which provides an upper bound on the error variance for all admissible modeling uncertainty has received much attention over the past years (Geromel, 1998; Shaked and Souza, 1995; Kassam and Lim, 1977). These estimators are referred to as cost-guaranteed (often robust) estimators and can be regarded as an extension of the standard optimal estimator to the case of uncertain models. In addition great interest was attracted to the estimators which minimize the H_∞ – norm of the spectral density of the estimation error (Nagpal and Khargonekar, 1991; Grimble and Elsayed, 1990). The main advantage of using an H_∞ estimator in comparison with a minimum variance estimator (Wiener or Kalman) is that no statistical models of process and measurement noises are needed. The H_∞ estimator also tends to be insensitive to the model uncertainty (Theodor, *et. al.,* 1994). Moreover, the model uncertainty can be included in the H_∞ estimator synthesis to guarantee an H_∞ performance (Li and Fu, 1997).

It is remarkable that all these estimators are designed for the "nominal" signal model and for the given uncertainty structure. The main distinction of the approach suggested in this paper is that instead of spectral densities to describe signals, the a priori information about signals is represented by the numerical characteristics such as variances of the signal itself and its several derivatives. Only these numerical characteristics are used in synthesizing an estimator. Such information about signals is reliable and can be obtained even from the heuristic considerations. Henceforward such estimator will be called robust.

The subsequent discussion is presented as follows. Chapter 2 is devoted to the statement and solution of the estimation problem in the framework of the robust approach suggested. In Chapter 3 the robust and optimal estimators are compared from the viewpoint of accuracy and sensitivity by an example of the most commonly encountered class of problems. In the last chapter the robust estimator design approach as outlined in the previous chapters is applied to the problem of airborne gravimetry (Stepanov, *et. al.,* 2002, Kulakova, *et al*., 2004).

2. ROBUST APPROACH

In the suggested robust approach the model for the signal is represented by the variances of the signal itself and its several derivatives (Nebylov, 2004), related to a particular spectral density by the following expressions:

$$D_i = \frac{1}{\pi}\int_0^\infty \omega^{2i} S\ (\omega)d\omega,\ \ i = 0, K. \qquad (2)$$

It is clear that numerous spectral densities will fit these characteristics.

The problem of the estimator development in the framework of the robust approach is reduced to the determination of the estimator TF when only these numerical characteristics are given.

It is obvious that in this case it is impossible to find the error variance D_e, however, it is possible to estimate its upper bound $\overline{D_e}$ for the prescribed TF (Nebylov, 2004) as follows:

$$H(s) = \frac{1 + b_1 s + \ldots + b_l s^l}{a_0 + a_1 s + \ldots + a_m s^m},\ \ l \le m-1 \qquad (3)$$

where $\{a_i\}_0^m$, $\{b_j\}_1^l \in [0,\infty)$ are the TF coefficients, $s = j\omega$ is a Laplas operator. The value of $\overline{D_e}$ depends on these coefficients and the specified numerical characteristics of the signal (2). The problem of robust estimator development is reduced to the determination of the TF coefficients which minimize the upper bound $\overline{D_e}$. This problem is usually solved with the use of numerical search methods for the function extremum (for example, by using the function "fminsearch" in MATLAB).

The orders l and m can be determined from the heuristic considerations for each specific problem to be solved. In the general case the order of the highest of the specified derivatives of the valid signal determines the spectral density inclination in the high-frequency domain, and hence, the order of the estimator that should be used for the estimation of the signal with such spectral density. Therefore, generally, the order of the estimator must not be lower than the order of highest specified derivatives of the valid signal.

Since it is the upper bound $\overline{D_e}$ that should be minimized in synthesizing the estimator, the estimator will provide the guaranteed estimation of the signals that are given by model (2). In so doing it should be remembered that the actual error variance D_e will not exceed the value $\overline{D_e}$.

3. COMPARISON OF THE ROBUST AND OPTIMAL ALGORITHMS

In this chapter the robust and optimal estimators are compared for the case when the variances of the valid signal itself and its first derivative are given. The measurement noise is assumed to be an N intensity white noise. The robust estimator TF for this problem will have the first order and will be represented as:

$$H_r(s) = \frac{b}{a+s}. \qquad (4)$$

The values of the parameters a and b depend on the specific values of the N, D_0 and D_1.

In order to design an optimal estimator it is necessary to specify the spectral density for the valid signal. In so doing the only restriction is that the spectral density must comply with condition that the signal be differentiable. For definiteness, the valid signal is assumed to have the spectral density as the differentiability second-order process with the correlation function

$$k(\tau) = \sigma^2 e^{-\alpha|\tau|}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|), \qquad (5)$$

where σ^2 is the signal variance; β is the prevailing frequency and α is the attenuation factor. Such signal can describe, in particular, the change of object altitude. The expressions below hold for it: $D_0 = \sigma^2 \;\; D_1 = \sigma^2\left(\alpha^2 + \beta^2\right)$.

The peculiar features of the robust approach will be investigated and compared to those of the optimal approach when the signal described by the model (5) and measurements contain a white noise. In this case the following parameter values are assumed to be given: $\alpha = 1$ s^{-1}; $\beta = 2\pi$ s^{-1}; $\sigma = 1$ m; $N = 0.01$ m^{2}s.

The estimator accuracy and its sensitivity to the disagreement between the actual model and the accepted mathematical one were investigated by simulation.

The comparison in accuracy has shown that the loss in the accuracy of the robust estimator against the optimal estimator does not exceed 20 - 30 % in the rms (root- mean-square) error. As this takes place the rms estimation error for the optimal estimator is $\sigma_e = 0.34$ m, the upper bound of the rms estimation error for the robust estimator is $\overline{\sigma}_e = 0.44$ m, and the rms error that ensures the robust estimator for the signal described by model (5) is $\sigma_{er} = 0.41$ m.

In the investigation of the estimator sensitivity under the conditions of varying signal parameters it was assumed that the values β (prevailing frequency) and σ^2 (variance) are exactly known but the value of the attenuation factor α (or the correlation interval $\tau = 1/\alpha$) is uncertain. The influence of the parameter α on the estimation accuracy is shown in Fig. 1 when the uncertainty range is given as $\alpha = 0.1 - 15$ s^{-1}.

From Fig.1 it is clear that the robust estimator is low-sensitive to the change of the attenuation factor α. In the whole uncertainty range its rms error only increased by the 24%, at the same time the rms error of the optimal estimator increased by two times.

In Fig.1 the influence of the parameter α on the estimation accuracy is also shown for the H_∞ estimator (Nagpal and Khargonekar, 1991) and minimax estimator (Kassam and Lim, 1977). In the framework of the minimax approach the optimal solution is developed for the worst input signal belonging to the preset class. As follows from Fig.1, the worst input signal (that provides the maximum

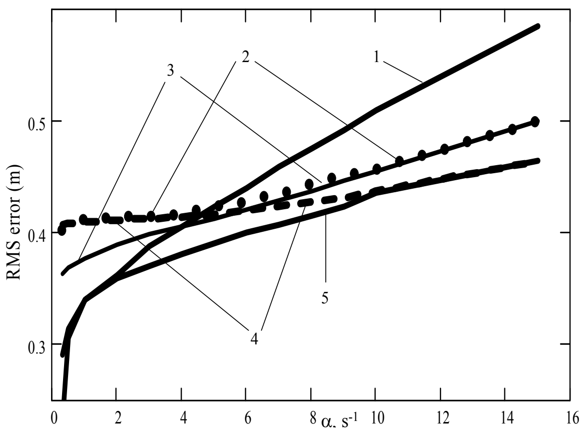

Fig.1. Influence of the parameter α on the estimation rms error. 1 – The optimal estimator for $\alpha = 1$; 2 – the robust estimator; 3 – the H_∞ estimator (level of noise attenuation $\gamma = 0.12$); 4 – the minimax estimator; 5 – the estimator optimal for the current value of α.

rms estimation error) for the case that the β value is known, is the signal with the minimum correlation interval. This means that the minimax estimator will correspond to the optimal estimator when $\alpha^* = 15$ s^{-1}. The H_∞ estimator is designed to minimize the peak of the estimation error spectrum and can be also interpreted as a minimax problem where the effect of the worst disturbances (noises) of finite energy on the estimation error is minimized. The H_∞ and Kalman estimators have a similar observer structure, but different estimator gain values. From Fig.1 it follows that for the example under consideration the robust and minimax estimators are practically the same in the left part of Fig. 1, while in the right part the robust estimator is close to the H_∞ estimator.

It is interesting to find the spectrum of the valid signal $S_r(\omega)$, observed together with a white noise, for which the synthesized robust estimator will be optimal. The solution of this problem leads to the "robust" spectral density $S_r(\omega)$ of the form:

$$S_r(\omega) = \frac{(2a-b)bN}{\omega^2 + (b-a)^2}. \qquad (6)$$

Thus the robust estimator is optimal for the first-order process.

If the a priori information about the valid signal is represented only by the parameter D_1, the coefficients a and b in TF (4) will be equal. Then the "robust" spectral density $S_r^{'}(\omega)$ will correspond to the random walk (integral of the white noise).

It should be noted that neither the model $S_r(\omega)$ nor the model $S_r^{'}(\omega)$ belongs to the preset class of input signals (2) because the first model has an unlimited

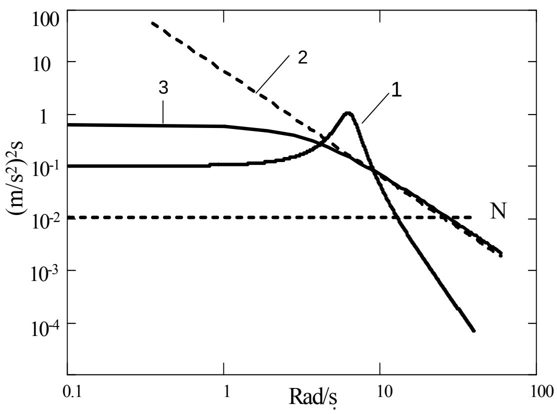

Fig.2. Signal spectral density $S(\omega)$ (1) and the "robust" spectral densities $S_r'(\omega)$ (2), $S_r(\omega)$ (3).

D_1 and the second, in addition to it, an unlimited D_0.

Fig. 2 shows the spectral density of the valid signal and the "robust" spectral densities $S_r(\omega)$ and $S_r'(\omega)$. In order to estimate the value of the information about the additional variances of the derivatives of the valid signal, special consideration must be given to how close the curves $S_r(\omega)$ and $S_r'(\omega)$ are in the vicinity of their cross point with the measurement errors. It is clear that the curve $S_r'(\omega)$ is an approximation of the spectral density $S_r(\omega)$ (that is, it represents the rectified curve $S_r(\omega)$ in the logarithmic scale) at its cross point with the line, corresponding to the level of the measurement noise. Thus, the robust algorithm is mainly determined by the highest of the specified variances of the derivatives of the signal.

It should be noted that in the case when the variance of the second or the third derivative of the signal is given (only one), the "robust" spectral density will correspond to the second or third integral of the white noise, respectively.

The solutions of the applied problems show that the application of the models in the form of integrals of the white noise for the sensor description is a very efficient because it allows developing reliable and simple algorithms (Chelpanov, *et.al., 1978*, Tupysev, 2004). But it is necessary to remember that in the robust approach suggested the properties of the signal under estimation are initially assumed to be different from those of the models in the form of integrals of the white noise.

3. THE PROBLEM OF AIRBORN GRAVIMETRY

The gravity anomaly (GA) estimation problem aboard an aircraft is solved by using the data from a gravimeter and phase measurements of altitude from the differential satellite navigation system (DSNS) (Stepanov, *et al.*, 2002, Kulakova, *et al.*, 2004). The data from the DSNS is used to eliminate the unknown altitude values h. For this purpose the difference between the second integral of the gravimeter readings and the altitude from the DSNS is formed. The differential measurements can be presented as follows (Stepanov, *et al.*, 2002, Kulakova, *et al.*, 2004):

$$z_h = \frac{\tilde{g}^{\,\mathrm{gr}}}{s^2} - h^{\mathrm{SNS}} = \frac{\tilde{g} + \delta g}{s^2} - \delta h\,, \qquad (7)$$

where $\tilde{g}$ is the gravity anomaly; δg is the gravimeter error; h is the unknown altitude values and δh is the error in DSNS measurements.

The estimation problem consists in obtaining GA, using the differential measurements (7).

The survey data for the investigations was obtained with the use of the gravimeter developed in the CSRI Elektropribor (Stepanov, *et al.*, 2002) and dual-frequency geodetic *Novatel* receivers. It was shown that the errors in phase measurements are mainly of white-noise character with the intensity $(0.005\mathrm{m})^2$s and the gravimeter errors can be described as a white noise with the intensity $R_g = (5\mathrm{mGal})^2$s.

To obtain the estimation algorithm it is necessary to specify a model for GA. A lot of various GA spectral densities have been used to describe it (Jin, *et al.*, 1997). But usually only two parameters are preset for them: the variance of GA $D_{\tilde{g}0} = \sigma_{\tilde{g}}^2$ and the value reverse to the correlation interval α (which is equivalent that the variance of the first derivative for GA $D_{\tilde{g}1}$ is given). At the same time the adequacy of the model is of great relevance for airborne gravimetry, as GA changes with the change of altitude. Below are the three models for GA: the Jordan model (Jordan, 1972) (widely used in the problems that require stochastic description of GA), random walk (the simplest of the models used to describe GA) and the model represented by two parameters $D_{\tilde{g}0}$, $D_{\tilde{g}1}$.

GA spectral density corresponding to the Jordan model is shown in Fig.3 and is given by the following expression

$$S_{\tilde{g}}(\omega) = 2\alpha^3 \cdot \sigma_{\tilde{g}}^2 \cdot \frac{5\cdot\omega^2 + \alpha^2}{(\omega^2 + \alpha^2)^3}\,. \qquad (8)$$

The random walk is an unsteady process, but formally it can be presented as a process with spectral density of the form: (see Fig.3) (Chelpanov, *et al.*, 1978):

$$S_{\tilde{g}w}(\omega) \approx q_{\tilde{g}w} / \omega^2\,. \qquad (9)$$

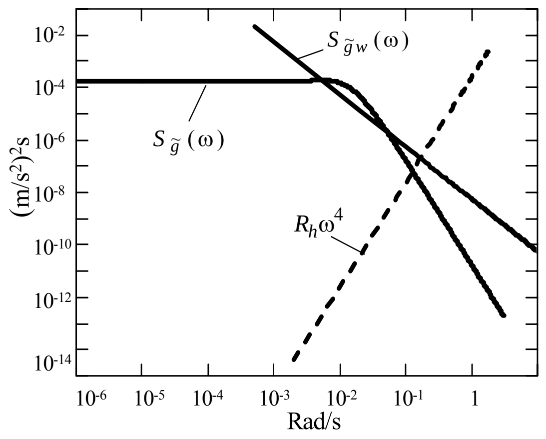

Fig.3. GA spectral densities and measurement errors for $\nabla\tilde{g}$ =10 mGal/km, $\sigma_{\tilde{g}}$ =30 mGal, V =50 m/s.

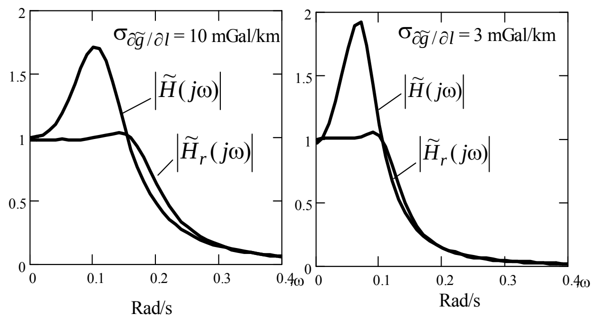

Fig.4. AFC for vertical acceleration for the estimator optimal for the Jordan model $\left|\tilde{H}(j\omega)\right|$ and for the robust estimator $\left|\tilde{H}_r(j\omega)\right|$.

The process noise intensity $q_{\tilde{g}w}$ in model (9) can be found from the equation $q_{\tilde{g}w} = (\nabla\tilde{g})^2 Vl$, where V is the speed of the aircraft, $\nabla\tilde{g} = \sigma_{\partial\tilde{g}/\partial l}$ is the rms value of the gravity increment at a distance of l, which is assumed to be equal to 1 km. For the Jordan model the value $\nabla\tilde{g}$ is related to the values $\sigma_{\tilde{g}}$ and α by the expression $\nabla\tilde{g}^2 = 2\alpha^2\sigma_{\tilde{g}}^2$.

The optimal estimator for the GA described by the Jordan model will have the fifth order, whereas in the case of the GA represented as the integral of the white noise, it will have the third order.

The solution of the problem in the framework of the robust approach suggested has led to the third-order estimator. At the same time the TF for the vertical acceleration $\tilde{H}_r(s) = \frac{1}{s^2} H_r(s)$ turned out to be similar to the classical low-pass filter. It had flat vertex and an inclination of 60 dB/dec in the high-frequency domain (Kulakova, *et al.*, 2004). The amplitude-frequency characteristics (AFC) for the robust estimator and the estimator optimal for the Jordan model are shown in Fig. 4. The existence of peaks for the optimal estimator AFC makes it too sensitive to an increased noise power at the frequencies where the AFC is high. It is also sensitive to the change of the accepted model that causes decrease of the valid signal energy in this frequency range. Therefore, from the intuitive point of view, the robust estimator is low-sensitive. Although no special measures are taken with this aim when it is being synthesized.

Table 1 presents the rms estimation errors for different rms values of the derivative for GA – 3, 5 and 10 mGal/km. In calculations it was assumed that $\sigma_{\tilde{g}}$ =30mGal. The aircraft speed was assumed to be equal to V =50m/s, which is typical for an aerogravimetric survey.

Table 1. Rms estimation errors for GA, mGal

$\nabla\tilde{g}$, mGal/km	3	5	10
Optimal algorithm for the Jordan model, σ_e	2	3.1	5.5
Optimal algorithm for Wiener process, σ_e	2.9	4.4	7.8
Robust algorithm, $\bar{\sigma}_e$	2.8	4.1	6.7
Robust algorithm, σ_{er}	2.8	3.9	6.4

The robust estimator is characterized in Table 1 by two parameters: the upper bound of the rms estimation error $\bar{\sigma}_e$ and the rms error σ_{er} that ensures the robust estimator for GA described as the Jordan model. The comparison in accuracy shows that the estimators have close rms error values. For the three selected values of the GA derivatives $\nabla\tilde{g}$ = 10, 5, 3 mGal/km, the loss in accuracy of the robust estimator against the estimator optimal for the Jordan model is 20, 25 and 30% in the rms error, respectively. This difference in accuracy between the estimators at variation of the derivative value $\nabla\tilde{g}$ is caused by the fact that the decrease of the value $\nabla\tilde{g}$ results in narrowing of the spectral density $S_{\tilde{g}}(\omega)$. This allows narrowing the pass band of the estimator optimal for the Jordan model, but the robust estimator must take into account all the signals belonging to the preset class.

It should be noted that for the entire class of input signals with the preset values $D_{\tilde{g}0}$, $D_{\tilde{g}1}$ accepted at the synthesis of the robust estimator , the estimation error will not exceed the value $\bar{\sigma}_e$. It should also be taken into consideration that the estimator adjusted for the random walk provides the rms error even higher than the value $\bar{\sigma}_e$.

It is interesting to note that the information about GA variance $\sigma_{\tilde{g}}^2$ had little influence on the synthesis of the robust algorithm when the variance of the first derivative for GA ($D_{\tilde{g}_1}$) was known. This means that if only this parameter containing the information about both the variance and the correlation interval is preset, the upper bound $\overline{D_e}$ will vary slightly in comparison with the case when two parameters ($D_{\tilde{g}_0}$, $D_{\tilde{g}_1}$) are preset.

Besides, the investigations have shown that the robust estimator can be approximately considered as the estimator corresponding to the solution of the optimal estimation problem when the GA is described as a random walk. The "robust" spectral density only differs from model (9) in the value of the process noise intensity $q_{\tilde{g}w}$.

Thus, the robust approach suggested allows developing an estimator that will have the features similar to those of the estimator adjusted for the model represented as the integrals of the white noise, but at the same time it will take better account of the real signal properties.

7. CONCLUSIONS

Currently it is more and more frequent that the designers focus their attention on the efficiency of signal estimation in the case when the reliable information about the characteristics of these signals is not available. To achieve guaranteed accuracy, estimators can be adjusted to unsteady models represented as integrals of a white noise. Then as shown in the work by Tupysev (2004) the estimators remain steady in operation even with piecewise constant behavior of real processes and in the same time the estimation error for stationary processes will not exceed a preset level.

However, there arises a problem of choosing the process noise intensity in a simplified model and the order of integration of the white noise. One of the possible ways for specifying a model of the signal as integrals of a white noise is rectification of the spectral density of the valid signal in the vicinity of its cross point with measurement errors (Chelpanov, *et.al., 1978*). In this case the simplified model will have the same order as the ancestor model. It is clear that the inadequacy of the ancestor model will affect the simplified model. The robust approach suggested for the estimator designing makes use of such numerical characteristics as the signal variance, variance of speed variation or variance of acceleration variation. It is much easier to obtain these values from the available experimental data than the spectral density of the signal. The robust estimator will lose to the optimal estimator in accuracy to a small extent, but it will ensure reliable and efficient estimation of signals, because it is adjusted to the unsteady models presented as integrals of a white noise. That has been demonstrated by an example of an airborne gravimetry problem (Kulakova, *et al*., 2004).

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