STABILIZATION OF LINEAR SYSTEMS BY DYNAMIC HIGH-GAIN ROTATION

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Abstract: We discuss stabilization of linear systems by dynamic high-gain rotation. The existence of stabilizing rotations is established for systems with negative trace, and an adaptive method to choose the controller gain is presented. The stabilization is robust with respect to arbitrary (possibly time-varying) skew-symmetric perturbations, which is also illustrated by a numerical example.

Keywords: stabilisation, Lyapunov function, robust stability, adaptive control

1. INTRODUCTION

Consider a system

$$\dot{x} = A\,x + u \quad \text{for } A \in \mathbb{R}^{n\times n} \text{ with } \operatorname{tr} \mathrm{A} < 0. \qquad (1)$$

We study the problem of stabilizing (1) by rotation in the sense that a state feedback $u(t) = S(t)x(t)$ with a skew-symmetric matrix S yields stability of the closed-loop system

$$\dot{x} = [A + S(t)]x\,. \qquad (2)$$

Recall that the trace of A, which is the sum of the diagonal elements of A, is the exponential growth rate of n-volumes under the linear flow generated by the equation. A negative trace thus means that n-volumes decay exponentially to zero. Adding of S to the nominal system matrix A amounts to imposing an additional rotation. If the nominal system arises from a mechanical system, this can, up to a certain extent, be interpreted as an exchange of energy between different modes of the system.

The idea of stabilization by rotation is not new. It has been investigated for random and for stochastic linear differential equations by Arnold et al. (1983) (see also Arnold et al. (1996)). In their approach, an essential assumption is sufficient 'richness' of the noise in the sense that enough rotations have to be excited by the noise. Another approach, using periodic excitations by skew-symmetric matrices, goes back to Meerkov (1980), whilst for example Morgan and Narendra (1977) and Čelikovský (1993) analyze stability in special cases, when $S(t)$ is given.

In the present note, we establish, for systems (1), the existence of merely a single deterministic time-independent skew-symmetric matrix S and a scalar (possibly time-varying) gain parameter so that $\dot{x} = [A+k(t)S]x$ is stable in a sense which will be made precise below. The essential mechanism in this approach of "energyless stabilization" is a mixing of stable and unstable modes, exploiting the fact that under the condition $\operatorname{tr} \mathrm{A} < 0$ the stable modes dominate the dynamic behaviour as soon as the mixing is strong enough.

More precisely, we present the following results for (1).

(i) There exists a skew-symmetric $\Sigma_A \in \mathbb{R}^{n\times n}$ and some $k_0 \geq 0$, such that $u(t) = k\,\Sigma_A x(t)$ applied to (1) yields, for any $k \geq k_0$, an asymptotically stable closed-loop system $\dot{x} = [A + k\,\Sigma_A]x$.
(ii) There exists a skew-symmetric $\Sigma_A \in \mathbb{R}^{n\times n}$ and some $k_0 \geq 0$ such that, for any monotonically increasing and continuous function $k : [0,\infty) \to [k_0,\infty)$, the application of $u(t) = k(t)\,\Sigma_A x(t)$ to (1) yields an asymptotically stable system $\dot{x} = [A + k(t)\,\Sigma_A]x$.
(iii) If monotonicity of k in (ii) is dropped, then the system is not necessarily stable.
(iv) The dynamical state feedback

$$u(t) = k(t)\,\Sigma_A\,x(t)$$
$$\dot{k}(t) = \min\{\varepsilon, \|x(t)\|\}, \quad k(0) = k^0$$

applied to (1) yields, for any $\varepsilon > 0$ and initial data $k^0 \in \mathbb{R}$, $x(0) \in \mathbb{R}^n$, a closed-loop initial value problem which has a unique solution (x,k) on the whole of $\mathbb{R}_{\geq 0}$ and this solution satisfies: $k(t)$ converges to a finite limit as $t \to \infty$ and $\lim_{t\to\infty} x(t) = 0$.

Note that in (i) the matrix Σ_A and the scalar k_0 depend on A, or – as we will see – only on the symmetric part of A. Having established (i), it is not straightforward to show that k can be replaced by a time-varying $k(\cdot)$ which satisfies $k(t) \geq k_0$ for all $t \geq 0$. In fact, the latter is not sufficient and monotonicity of $k(\cdot)$ is necessary for the result in (ii). In (iv) we show how to determine the scalar parameter k_0 needed in (i) adaptively. The idea is to increase $k(t)$ as long as $\|x\|$ is not integrable so that $k(t)$ finally becomes sufficiently large and ensures that $\|x\|$ becomes integrable.

2. PRELIMINARIES

For $n \in \mathbb{N}$ consider the skew-symmetric matrix

$$\Sigma_n = \begin{bmatrix} 0 & & -1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} = (\sigma_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n},$$

with $\sigma_{ij} = \begin{cases} 0\,, & i = j\,, \\ -1\,, & i < j\,, \\ 1\,, & i > j\,. \end{cases}$

The eigenvalues $i\omega_j$ of Σ_n are imaginary and distinct. We write $\Omega_n = \operatorname{diag}(\omega_1,\ldots,\omega_n)$. Throughout this paper, we will use the following notation. I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. Let $A \in \mathbb{R}^{n\times n}$ be given, and let U be an orthogonal matrix containing the eigenvectors of $A + A^T$; then $U^T(A + A^T)U = D$ is diagonal and we set

$$\Sigma_A = U^T \Sigma_n U\ , \tag{3}$$

which is skew-symmetric. We will make use of the eigenstructure of the matrix pencil

$$A_k = A + k\Sigma_A\ , \quad k \in \mathbb{R} \tag{4}$$

for large k.

Lemma 1. Let $A \in \mathbb{R}^{n\times n}$. Then there exist $k_0 > 0$ and some analytic matrix-valued function $S.\, : [k_0,\infty[\to \mathrm{GL}_n(\mathbb{C})$ with the properties:

(a) $S_k^{-1}A_kS_k = ik\Omega_n + \frac{\operatorname{tr} A}{n}I_n + \Delta_k$, where $\Delta_k = \operatorname{diag}(\delta_1(k),\ldots,\delta_n(k)) = O(1/k)$ as $k \to \infty$.
(b) $\lim_{k\to\infty} S_k = S_\infty$, where $S_\infty^* S_\infty = I_n$ and $S_\infty^*\Sigma_A S_\infty = i\Omega_n$.
(c) $S_k^*S_k - I_n = O(1/k)$, as $k \to \infty$.

The proof of Lemma 1 is based on the following general result in perturbation theory (e.g. Baumgärtel (1972); Demmel (1997)).

Theorem 2. Let $A, B \in \mathbb{R}^{n\times n}$ and assume that B has distinct eigenvalues $\lambda_1(B),\ldots,\lambda_n(B)$ with corresponding eigenvectors $v_1(B),\ldots,v_n(B)$.
Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, the matrix $\varepsilon A + B$ has also n distinct eigenvalues and, as $\varepsilon \to 0$,

$$\lambda_j(\varepsilon A + B) = \lambda_j(B) + \varepsilon\,\frac{v_j(B)^* A\, v_j(B)}{v_j(B)^* v_j(B)} + \mathcal{O}(\varepsilon^2).$$

For appropriate numbering, and $j = 1,\ldots,n$, we also have that

$$\varepsilon \mapsto \lambda_j(\varepsilon A + B) \quad \text{and} \quad \varepsilon \mapsto v_j(\varepsilon A + B)$$

are analytic on $(0,\varepsilon_0)$, and $\lim_{\varepsilon\to 0} v_j(\varepsilon A + B) = v_j(A)$.

Corollary 3. Let $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A < 0$. Then there exists $k_0 > 0$ such that, for all $k \geq k_0$, $\sigma(A + k\Sigma_A) \subset \mathbb{C}_-$.

Proof: Lemma 1 implies $\operatorname{Re}\lambda_j(A_k) = \frac{\operatorname{tr} A}{n} + O(1/k)$ for $j = 1,\ldots,n$. If k is large enough, then all the real parts are negative. □

3. STABILIZATION BY ROTATION

The result of Corollary 3 does not imply that for continuous $k(\cdot)$ with $k(t) \geq k_0$ for all $t \geq 0$ and sufficiently large $k_0 > 0$, the time-varying system

$$\dot{x} = (A + k(t)\Sigma_A)x \tag{5}$$

is asymptotically stable. This is illustrated in Example 9. However, we can show that if $k_0 > 0$ is sufficiently large and $k(\cdot)$ is *monotonically increasing* and $k(t) \geq k_0$ for all $t \geq 0$, then (5) becomes asymptotically stable.

Theorem 4. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\,\mathrm{A} < 0$. Then there exists some $k_0 \geq 0$ such that, for any monotonically non-decreasing and continuous $k : [0,\infty) \to [k_0,\infty)$, the zero solution of system (5) is uniformly asymptotically stable.

Note that in Theorem 4 the scalar k_0 depends on A. This drawback can be resolved by determining $k(\cdot)$ adaptively. Loosely speaking, k is adaptively tuned such that it increases as long as $\|x\|$ is not integrable, and settles to a finite limit when it is stabilizing.
As a prerequisite we also need a variation of Theorem 4, where the monotonicity assumption is replaced by boundedness condition on the derivative of k.

Theorem 5. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\,\mathrm{A} < 0$. Then for any differentiable $k : [0,\infty) \to \mathbb{R}$ with $k(t) \to \infty$ as $t \to \infty$ and $\limsup_{t\to\infty} |\dot{k}(t)| < \infty$, the zero solution of

$$\dot{x} = [A + k(t)\,\Sigma_A]x \tag{6}$$

is uniformly exponentially stable.

The Theorems 4 and 5 constitute the backbone of the following adaptive stabilization result.

Theorem 6. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\,\mathrm{A} < 0$, and $\varepsilon > 0$. Then the state feedback

$$u(t) \;=\; k(t)\,\Sigma_n\,x(t)$$

together with gain adaptation

$$\dot{k}(t) \;=\; \min\{\varepsilon, \|x(t)\|\}, \qquad k(0) = k^0, \tag{7}$$

applied to (1) yields, for any initial data $x(0) = x^0 \in \mathbb{R}^n$, $k^0 \in \mathbb{R}$, the closed-loop initial value problem

$$\left.\begin{aligned} \dot{x}(t) &= [A + k(t)\,\Sigma_A]x(t),\ x(0) = x^0, \\ \dot{k}(t) &= \min\{\varepsilon, \|x(t)\|\}, \qquad k(0) = k^0, \end{aligned}\right\} \tag{8}$$

which has a unique solution (x,k) on the whole of $\mathbb{R}_{\geq 0}$, and this solution satisfies:

(i) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$,
(ii) $\lim_{t\to\infty} x(t) = 0$.

Remark 7.

(i) Note that the result in Theorem 6 does not say that the system $\dot{x} = [A + k(t)\,\Sigma_A]x$ becomes asymptotically stable; nor is the so called "limit system" $\dot{x} = [A + k_\infty\,\Sigma_A]x$ necessarily stable. The dynamic gain adaptation (8) ensures only that the trajectory (x,k) converges.
(ii) Note further that the increase of k is at most linear. This may be advantageous when compared to $\dot{k}(t) = \|x(t)\|^p$ for $p \geq 1$. The latter is a valid alternative to (7) but omitted here for brevity. The gain adaptation $\dot{k}(t) = \|x(t)\|^2$ has been introduced for high-gain stabilizable linear input-output systems, see for example the seminal work by Morse (1983), Willems and Byrnes (1984). The gain adaptation (7) has been introduced in Ilchmann and Ryan (2004).
(iii) We also omit for brevity to show that this dynamic stabilization is robust with respect to arbitrary bounded skew-symmetric perturbations of A. It can be shown that A can be replaced by $A + \Sigma(t)$ for any measurable $\Sigma : t \mapsto \mathbb{R}^{n\times n}$ with $\Sigma(t) = -\Sigma(t)^T$ and $\sup_{t>0} \|\Sigma(t)\| < \infty$.

For the proof of Theorem 4 we need the following observation.

Lemma 8. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}\,\mathrm{A} < 0$. Then by Corollary 3 there exists $k_0 \geq 0$ such that

$$\sigma(A_k) \subset \mathbb{C}_- \,, \quad \text{for all } k \geq k_0,$$

and therefore (see, for example, Sontag (1998))

$$P_k \;:=\; \int_0^\infty \mathrm{e}^{A_k^T s}\mathrm{e}^{A_k s}\,ds \tag{9}$$

is the unique positive definite solution of

$$A_k^T P_k + P_k A_k \;=\; -I_n\,, \quad \text{for all } k \geq k_0\,. \tag{10}$$

Furthermore, there exist numbers $a > 0$ and $M > 1$, such that for all $k, m \geq k_0$:

$$\kappa_2(P_k) = \|P_k\|\|P_k^{-1}\| \leq 1 + \frac{a}{k}\,, \tag{11}$$

and

$$A_m^T P_k + P_k A_m \leq \left(-1 + \tfrac{|k-m|}{k}\,M\right) I_n\,. \tag{12}$$

Proof of Theorem 4: Let $x : \mathbb{R}_+ \to \mathbb{R}$ denote a solution of (5) for a given function k as specified. For arbitrary $m \geq k_0$, let P_m be given by (9). Assume that, for some fixed m and all t in a given interval $[t_0, t_1]$, we have (with M defined in Lemma 8)

$$|k(t) - m| \leq \frac{m}{2M}\,. \tag{13}$$

A standard Lyapunov argument yields

$$\|x(t_1)\|^2 \leq \kappa_2(P_m)\, e^{\frac{-1}{2\|P_m\|}\,(t_1 - t_0)}\|x(t_0)\|^2\,. \tag{14}$$

Now we distinguish between the two cases, when k either is bounded or unbounded. In the first case $k(t)$ converges monotonically to some number k_*. We set $m = k_*$, and by convergence there exists some $t_0 > 0$ such that (13) holds for all $t \geq t_0$. Hence

$$\|x(t)\|^2 \leq \kappa_2(P_{k_*})\,\mathrm{e}^{\frac{-1}{2\|P_{k_*}\|}(t-t_0)}\|x(t_0)\|^2 \to 0$$

as $t \to \infty$.
The second case, when $k(t) \to \infty$ as $t \to \infty$, is more subtle. Now we do not have a joint uniform static quadratic Lyapunov function for all $k(t) \in \mathbb{R}$. To overcome this problem, we write the interval $[k_0, \infty[$ as a disjoint union of subintervals $[k_j, k_{j+1}[$, such that (13) holds for $m = (k_j + k_{j+1})/2$ and all $k(t) \in [k_j, k_{j+1}[$.
Such a partition is obtained as follows: Condition (13) is equivalent to

$$k(t) \in [m - \tfrac{m}{2M}, m + \tfrac{m}{2M}] = m[\tfrac{2M-1}{2M}, \tfrac{2M+1}{2M}]\,.$$

Setting

$$k_j = \gamma^j k_0, \quad \gamma := \frac{2M+1}{2M-1}\,, \quad \text{for } j = 0, 1, 2, \ldots,$$

we find the intervals $[k_j, k_{j+1}[$ to be suitable. With these k_j we define $t_j = \sup\{t \geq 0 \mid k(t) \leq k_j\}$, such that obviously $t_{j+1} \geq t_j$ for all j, $t_j \to \infty$ as $j \to \infty$ and $k(t) \in [k_j, k_{j+1}]$ for all $t \in [t_j, t_{j+1}]$. Finally, we define $m_j := \frac{k_j + k_{j+1}}{2} = \gamma^j m_0$, $m_0 := \frac{1+\gamma}{2} k_0$, and consider (14) for an interval $[t_j, t_{j+1}]$ instead of $[t_0, t_1]$ with $m = m_j$.
From Lemma 8 we recall that

$$\kappa_2(P_{m_j}) \leq 1 + \frac{a}{m_j} = 1 + \frac{a}{m_0}\,\frac{1}{\gamma^j}\,. \qquad (15)$$

Moreover, since $\|P_{m_j}\| \to \frac{n}{-2\,\mathrm{tr}\,\mathrm{A}}$ as $j \to \infty$, there exists a number $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have $\|P_{m_j}\| \leq \frac{n}{-\mathrm{tr}\,\mathrm{A}}$, which means

$$-\frac{1}{2\|P_{m_j}\|} \leq \frac{\mathrm{tr}\,\mathrm{A}}{2n} < 0\,. \qquad (16)$$

Inserting (15) and (16) in (14), we obtain, for $j \geq j_0$,

$$\|x(t_{j+1})\|^2 \;\leq\; \left(1 + \frac{\alpha}{\gamma^j}\right) e^{-\beta\delta_j}\|x(t_j\|^2\,, \qquad (17)$$

where we have set $\alpha = \frac{a}{m_0}$, $\beta = -\frac{\mathrm{tr}\,\mathrm{A}}{2n} > 0$, and $\delta_j = t_{j+1} - t_j$. Note that $\sum_{j=0}^{\infty}\delta_j = \infty$. For simplicity of notation and without loss of generality, we assume that $j_0 = 0$. Then for all j we have

$$\|x(t_{j+1})\|^2 \leq \prod_{\ell=0}^{j}\left(1 + \tfrac{\alpha}{\gamma^\ell}\right) e^{-\beta\delta_\ell}\|x(t_0)\|^2\,.$$

We finally conclude that $\prod_{\ell=0}^{j}\left(1 + \frac{\alpha}{\gamma^\ell}\right) e^{-\beta\delta_\ell}$ converges to 0 as $j \to \infty$, because

$$\ln\left(\prod_{\ell=0}^{j}\left(1 + \tfrac{\alpha}{\gamma^\ell}\right) e^{-\beta\delta_\ell}\right) \leq \sum_{\ell=0}^{j}\tfrac{\alpha}{\gamma^\ell} - \sum_{\ell=0}^{j}\beta\delta_\ell$$

diverges to $-\infty$ as $j \to \infty$.

Thus $x(t_j) \to 0$ as $j \to \infty$. Moreover, $\|x(t)\|^2 \leq (1+a)\|x(t_j)\|^2$ for $t \in [t_j, t_{j+1}]$. Hence, in fact, $x(t) \to 0$ as $t \to \infty$, which completes the proof. □

It is noteworthy that we cannot dispense with the monotonicity assumption in Theorem 4. We present an example of a system of the form (5) which is destabilized by periodic switching between two values k and m of $k(t)$. These values can be chosen arbitrarily large, i.e. for each $k_0 > 0$ we can find appropriate $k, m > k_0$. In fact, we can even destabilize the system by switching between gain values k_j and m_j, where both k_j and m_j tend monotonically to infinity.

Example 9. Let $A = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$, $\Sigma_A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
The eigenvalues of $A_k = A + k\Sigma_A$ are $-1 \pm i\alpha_k$ with

$$\alpha_k = \sqrt{k^2 - 9} \in \mathbb{R}\,, \quad \text{if } k \geq 3\,. \qquad (18)$$

Consider the product $e^{A_k t_k} e^{A_m t_m}$, where

$$t_k = \frac{\pi}{2\alpha_k}\,,\; t_m = \frac{\pi}{2\alpha_m}\,.$$

A straighforward computation yields

$$e^{A_k t_k} e^{A_m t_m} = \frac{e^{-(t_k+t_m)}}{-\alpha_k\alpha_m}\begin{bmatrix} km - 9 & 3(k-m) \\ 3(k-m) & km - 9 \end{bmatrix}\,.$$

We denote the spectral radius of this matrix by $\rho(k, m)$. By elementary estimates, it follows that for $k = 9m$ we have $\rho(k, m) \geq 1 + \frac{\nu}{m}$ with some $\nu > 0$ independent of m. For a given sequence $m = m_1, m_2, m_3, \ldots$ we set $k_j = 9m_j$, and for the corresponding time intervals t_{k_j} and t_{m_j} we switch the gain value $k(\cdot)$ between k_j and m_j. This leads to the transfer operator

$$\underbrace{e^{A_{k_1}t_{k_1}} e^{A_{m_1}t_{m_1}}}_{=:\Phi_1}\underbrace{e^{A_{k_2}t_{k_2}} e^{A_{m_2}t_{m_2}}}_{=:\Phi_2}\cdots$$

with $\rho(\Phi_j) \geq 1 + \frac{\nu}{m_j}$ and corresponding normalized eigenvector $v = \frac{1}{\sqrt{2}}[1, 1]^T$ of Φ_j for all j. Hence $\|(\Phi_1\Phi_2\ldots\Phi_\ell)\,v\| \geq \prod_{j=1}^{\ell}\left(1 + \frac{\nu}{m_j}\right) \to \infty$ for $\ell \to \infty$. The latter is satisfied if $\sum_{j=1}^{\infty}\frac{1}{m_j} = \infty$, e.g. for $m_j \leq rj$ for some $r > 0$. We conclude that e.g. for $m_j = 9j$ the zero solution of system (5) is unstable. □

Clearly, we could approximate the step function k in Example 9 by some smooth function which destabilizes the system, too. But an important feature of such a destabilizing smooth function lies in the fact that its derivative takes arbitrarily large values as $k \to \infty$. It is therefore not surprising that an alternative to the monotonicity condition in Theorem 4 is provided by a boundedness condition on the derivative of k, as given in Theorem 5.

Proof of Theorem 5:
We follow the proof of Theorem 4 for the case of unbounded k up to inequality (17). Choosing j_0 large enough, we may assume $\frac{\alpha}{\gamma^j} \leq 1$ for all $j \geq j_0$. Moreover, since $k_{j+1} - k_j \to \infty$, and $\limsup_{t\to\infty} |\dot{k}(t)| < \infty$, we may also assume $\delta_j \geq \frac{2\ln 2}{\beta}$ for all $j \geq j_0$, such that

$$\begin{aligned}\|x(t_{j+1})\|^2 &\leq 2e^{-\frac{\beta}{2}(t_{j+1}-t_j)}e^{-\frac{\beta}{2}(t_{j+1}-t_j)}\|x(t_j)\|^2 \\ &\leq e^{-\frac{\beta}{2}(t_{j+1}-t_j)}\|x(t_j)\|^2 .\end{aligned}$$

This proves uniform exponential stability. □

It is quite instructive to see, how Theorems 4 and 5 play together in Step 2 of the following proof.

Proof of Theorem 6:
Step 1: Since the right hand side of (8) is locally Lipschitz, the initial value problem has a unique solution $(x, k) : [0, \omega) \to \mathbb{R}^n \times \mathbb{R}$ for a maximal $\omega \in (0, \infty]$. Furthermore, k has at most linear growth and therefore a possible finite escape time can only occur in the x-dynamics, which, however, is a linear system. Therefore, $\omega = \infty$.
Step 2: We show that $k \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$, whence Assertion (i).
Suppose that $k \notin L^\infty([0,\infty), \mathbb{R})$. By (7) it follows that $k(t)$ tends monotonically to ∞ for $t \to \infty$. Now Theorem 4 ensures that $x(t)$ tends to 0 as $t \to \infty$. By the gain-adaptation law (7), there exists $t_0 \geq 0$ such that $0 \leq \dot{k}(t) \leq \varepsilon$ for all $t \geq t_0$. Therefore, Theorem 5 yields that $x(t)$ tends to 0 *exponentially*, and, invoking again (7), k – as the integral of an exponentially decaying function – is bounded. This contradicts the assumption $k \notin L^\infty([0,\infty), \mathbb{R})$.
Step 3: We show that $x \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$.
Seeking a contradiction, suppose that x is unbounded. Observe, however, that by boundedness of k and (5), there exists $c_1 > 0$ so that

$$\forall t > 0 : \quad \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\| \;\leq\; c_1 \|x(t)\| .$$

Choose some $t_0 \geq 0$ such that $\|x(t_0)\| \geq \varepsilon$.
For arbitrary $r > 0$, set

$$\begin{aligned}\tau_r &:= \inf\{t > t_0 \mid \|x(t)\| = e^r\|x(t_0)\|\} , \\ \sigma_r &:= \sup\{t \in [t_0, \tau_r) \mid \|x(t)\| = \|x(t_0)\|\} .\end{aligned}$$

Then

$$e^r\|x(t_0)\| = \|x(\tau_r)\| \leq e^{c_1(\tau_r - \sigma_r)}\|x(t_0)\| ,$$

and thus $\tau_r - \sigma_r \geq r/c_1$. Since k is monotonically increasing, and

$$\dot{k}(t) = \varepsilon , \quad \text{for all} \;\; t \in [\sigma_r, \tau_r] ,$$

we have

$$k(\tau_r) = k(\sigma_r) + \varepsilon\,(\tau_r - \sigma_r) \geq k^0 + \frac{\varepsilon}{c_1} r .$$

Since r is arbitrary, the latter contradicts boundedness of k. Therefore, x is bounded.

Step 4: We show Assertion (ii).
Since x and k are bounded, it follows that $\dot{x}$ is bounded, and so x is uniformly continuous. Consequently, also $t \mapsto \min\{\varepsilon, \|x(t)\|\}$ is uniformly continuous. Thus we may apply Barbălat's lemma (Barbălat (1959)) to conclude that $k_\infty - k^0 = \int_0^\infty \min\{\varepsilon, \|x(t)\|\}\,\mathrm{d}t \in \mathbb{R}$ yields $\min\{\varepsilon, \|x(t)\|\} \to 0$ as $t \to \infty$, which is Assertion (ii). This completes the proof of the Theorem. □

4. NUMERICAL EXAMPLE

Just to get an impression we apply the gain adaptation of Theorem 6 to a system of the form

$$\dot{x} = (A + \delta\Sigma(t) + k(t)\Sigma_A)x .$$

As an example we choose $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ and $\Sigma(t) = \sin t\Sigma_0 - \cos(\sqrt{2}t)\Sigma_A$. For the values $\delta = 0$, $\delta = 10$, $\delta = 20$ we plot (in Fig. 1) the norm of the solution $\|x(t)\|$ and the size of the adaptation parameter $k(t)$. Analogous results are obtained for different matrices and higher dimensions; one may note the fast oscillations in the solution as k increases.

5. CONCLUSIONS

We have shown numerous stabilization results of linear systems by rotation. Our main achievements are the following results:

(a) For any A with $\mathrm{tr}\,\mathrm{A} < 0$, there exists a skew-symmetric matrix Σ_A, such that $A + k\Sigma_A$ is stable for all large k. The matrix Σ_A depends only on the symmetric part $A + A^T$ of A.
(b) The system $\dot{x} = A + k(t)\Sigma_A$ is stable, if $k(t)$ is sufficiently large and k grows monotonically. If k is not monotone, then the system may be unstable, even if k is arbitrarily large.
(c) A stabilizing controller gain function k can be chosen adaptively.
(d) The dynamic state feedback controller is robust with respect to skew-symmetric perturbations.

It is clearly a drawback of our approach that we require the full state vector to be available for control. Therefore our results can only be seen as a first step towards the design of an adaptive controller using rotations. Questions for further research are for example:

(e) Characterize, for a given matrix A, all skew-symmetric matrices Σ which are stabilizing in the sense of (a).
(f) Can a suitable Σ be found adaptively?
(g) Give conditions for a system to be stabilizable by rotations, if one does not have full access to the state vector.

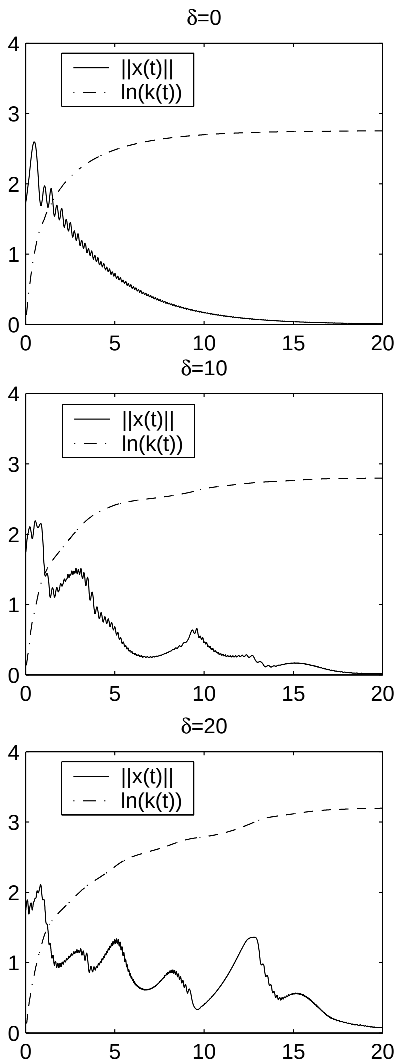

Fig. 1. Dynamic gain adaptation for the time varying system $\dot{x} = (A + \delta\Sigma(t) + k(t)\Sigma_A)x$ with different values δ.

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