THE UNIFORM GLOBAL OUTPUT REGULATION PROBLEM FOR DISCONTINUOUS SYSTEMS

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Abstract: In this paper we formulate and study the uniform global output regulation problem for systems with discontinuous right-hand sides. A characterization of all controllers solving this problem is provided. Based on this characterization, the regulator equations for discontinuous systems are derived. Solvability of these regulator equations is a necessary condition for the solvability of the uniform global output regulation problem. The developed theory is illustrated with an example.

Keywords: Output regulation, discontinuous systems, nonlinear systems, convergent systems, invariant manifold theorem.

1. INTRODUCTION

The nonlinear output regulation problem, which deals with asymptotic tracking of reference signals and/or asymptotic rejection of disturbances in the output of a nonlinear dynamical system, has received a lot of attention for the last fifteen years, see e.g. (Isidori and Byrnes, 1990), (Huang and Lin, 1994), (Byrnes *et al.*, 1997), (Isidori and Byrnes, 2003), (Chen and Huang, 2004), where different variants of the local, semiglobal and global variants of the output regulation problem are studied. To the best of our knowledge, all existing results on this problem deal with systems with *continuous* right-hand sides.

In this paper we study the output regulation problem for nonlinear systems with *discontinuous* right-hand sides. This study is motivated by a large variety of regulation problems for this class of systems, which includes mechanical systems with friction and electrical systems with switchings. Our treatment of the problem is based on the notion of convergent systems presented in (Demidovich, 1967), see also (Pavlov *et al.*, 2004*a*). First, we present preliminary notions and results on discontinuous systems and on convergent systems (Section 2). Then we formulate the uniform global output regulation problem for systems with discontinuous right-hand sides (Section 3). This problem setting is an extension of the uniform global output regulation problem for continuous systems formulated and studied in (Pavlov *et al.*, 2004*b*; Pavlov, 2004). Further, we derive certain invariant manifold theorems which serve as a main tool for the analysis of this problem (Section 4). Based on these theorems, we obtain necessary and sufficient conditions under which a controller solves the uniform global output regulation problem (Section 5). This result yields a variant of the regulator equations for discontinuous systems. Solvability of these equa-

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tions is, under minor assumptions, a necessary condition for the solvability of the uniform global output regulation problem. The developed theory is illustrated with an example (Section 6). Finally, the paper is finished with conclusions (Section 7).

2. PRELIMINARIES

Regular systems

In this paper, we will consider systems of the form

$$\dot{z} = \mathcal{F}(z, w), \tag{1}$$

with the state $z \in \mathbb{R}^d$ and piecewise-continuous bounded inputs $w(t) \in \mathbb{R}^m$. The right-hand side of system (1) is allowed to be discontinuous. In this case, a solution of system (1) is understood in the sense of Filippov (Filippov, 1988). In the scope of the output regulation problem, we will deal with so-called regular systems, as defined below.

Definition 1. System (1) is called *regular* if for any piecewise-continuous input $w(t)$ and any initial condition $z_0 \in \mathbb{R}^d$, $t_0 \in \mathbb{R}$ there exists a solution $z(t, t_0, z_0)$ of system (1) in the sense of Filippov defined on its maximal interval of existence $I \subset \mathbb{R}$. This solution is right-unique and for any $t \in I$, $t \geq t_0$ the solution $z(t, t_0, z_0)$ continuously depends on the initial condition z_0.

Remark. It is worth mentioning that right-uniqueness of Filippov solutions implies their continuous dependency on the initial conditions, see e.g. (Filippov, 1988).

Convergent systems

In many control problems and in particular in the output regulation problem, it is desirable to achieve closed-loop dynamics with the property that all solutions of the closed-loop system "forget" their initial conditions and converge to some steady-state solution which is determined only by the input. Such property of a system can be formalized in the notion of convergent systems presented below, see (Demidovich, 1967) or (Pavlov *et al.*, 2004*a*).

Definition 2. System (1) with a given piecewise-continuous input $w(t)$ defined for all $t \in \mathbb{R}$ is said to be *convergent* if

i. all solutions $z(t)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $z(t_0) \in \mathbb{R}^d$,
ii. there exists a unique solution $\bar{z}_w(t)$ defined and bounded for all $t \in \mathbb{R}$,
iii. the solution $\bar{z}_w(t)$ is globally asymptotically stable.

System (1) is said to be convergent (for all inputs) if it is convergent for every bounded piecewise-continuous input $w(t)$.

We will refer to $\bar{z}_w(t)$ as the steady-state solution. In the context of the output regulation problem, we will need a stronger convergence property, which is formulated in the following definition.

Definition 3. System (1) is said to be *uniformly convergent* if it is convergent, for every bounded piecewise-continuous input $w(t)$ the corresponding steady-state solution $\bar{z}_w(t)$ is *uniformly* globally asymptotically stable (UGAS) and for any $\rho > 0$ there exists $\mathcal{R} > 0$ such that

$$|w(t)| \leq \rho \ \forall\, t \in \mathbb{R} \ \Rightarrow \ |\bar{z}_w(t)| \leq \mathcal{R} \ \forall\, t \in \mathbb{R}. \tag{2}$$

The uniform convergence property is an extension of stability properties of asymptotically stable linear time-invariant systems. One can easily show that the system $\dot{z} = Az + Bw(t)$ with a Hurwitz matrix A is uniformly convergent.

Sufficient conditions for the uniform convergence property can be found in several papers. In (Demidovich, 1967) (see also (Pavlov *et al.*, 2004*a*)) a simple sufficient condition for this property was obtained for systems with smooth right-hand sides. In (Yakubovich, 1964), sufficient conditions for the uniform convergence property were obtained for systems of the form

$$\dot{x} = Ax - D\phi(y) + g(w) \tag{3}$$
$$y = Cx,$$

where $\phi(y)$ is a (possibly discontinuous) scalar nonlinearity and $g(w)$ is a continuous function. The next theorem is a particular case of the result from (Yakubovich, 1964).

Theorem 1. Suppose the matrix A is Hurwitz, the nonlinearity is non-decreasing and the following two conditions are satisfied

$$\mathrm{Re}(C(i\omega I - A)^{-1}D) > 0, \quad \forall\, \omega \geq 0, \tag{4}$$

$$\lim_{\omega \to +\infty} \omega^2 \mathrm{Re}(C(i\omega I - A)^{-1}D) > 0. \tag{5}$$

Then system (3) is regular and uniformly convergent.

3. THE UNIFORM GLOBAL OUTPUT REGULATION PROBLEM

Consider systems modelled by equations of the form

$$\dot{x} = f(x, u, w) \tag{6}$$

$$e = h_r(x, w), \quad y = h_m(x, w), \tag{7}$$

with the state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^p$, regulated output $e \in \mathbb{R}^l$, measured output $y \in \mathbb{R}^k$ and exogenous signal $w \in \mathbb{R}^m$. The functions $h_r(x, w)$ and $h_m(x, w)$ are assumed to be continuous. The function $f(x, u, w)$ is continuous in u and w, but may be discontinuous in x. The exogenous signal $w(t)$, which can be viewed as a disturbance in equation (6) or as a reference signal in (7), is generated by the exosystem

$$\dot{w} = s(w). \tag{8}$$

The function $s(w)$ is assumed to be locally Lipschitz. We will consider solutions of the exosystem starting in a compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$.

The uniform global output regulation problem is formulated in the following way: *find, if possible, a feedback of the form*

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y), \quad \xi \in \mathbb{R}^q \\ u &= \theta(\xi, y), \end{aligned} \tag{9}$$

with some q such that the closed-loop system

$$\dot{x} = f(x, \theta(\xi, h_m(x, w)), w), \tag{10}$$

$$\dot{\xi} = \eta(\xi, h_m(x, w)) \tag{11}$$

satisfies the following three conditions:

- **a)** *system (10), (11) is regular,*
- **b)** *system (10), (11) is uniformly convergent,*
- **c)** *for any solution of the exosystem starting in $w(0) \in \mathcal{W}_+$ and any solution of the closed-loop system starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ it holds that $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to +\infty$.*

Notice, that in this problem setting discontinuities are allowed not only in the system (6), but also in the controller (9). In the problem formulation presented above, condition **a)** means that all solutions of the closed-loop system are well-defined in the sense of Filippov, they are right-unique and continuously depend on their initial conditions. Condition **b)** means that for any piecewise-continuous bounded input $w(t)$ the closed-loop system has a unique UGAS steady-state solution. Finally, condition **c)** means that for any solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}_+$ and for any solution of the closed-loop system corresponding to this $w(t)$ the regulated output $e(t)$ tends to zero. This problem setting is an extension of the uniform global output regulation problem formulated and studied in (Pavlov *et al.*, 2004*b*) for continuous systems. It was shown in (Pavlov *et al.*, 2004*b*) that such problem formulation is a natural extension of the conventional problem settings for the linear and local nonlinear output regulation problems to the global output regulation problem for nonlinear systems.

It should be mentioned, that the requirement of uniform convergence implies that for any solution of the exosystem $w(t)$ all solutions of the closed-loop system corresponding to this $w(t)$ are bounded in forward time.

4. INVARIANT MANIFOLD THEOREMS

In this section, we present two results on the dynamics of uniformly convergent systems of the form

$$\dot{z} = \mathcal{F}(z, w), \quad z \in \mathbb{R}^d \tag{12}$$

with inputs $w(t)$ generated by the autonomous system

$$\dot{w} = s(w). \tag{13}$$

System (12) may be discontinuous, but it is assumed to be regular (see Definition 1). System (13) has a locally Lipschitz right-hand side. First, we will consider the case of system (13) satisfying the following boundedness assumption.

(BA) All solutions of system (13) are defined for all $t \in (-\infty, +\infty)$ and for every $r > 0$ there exists $\rho > 0$ such that

$$|w_0| < r \quad \Rightarrow \quad |w(t, w_0)| < \rho \ \ \forall\, t \in \mathbb{R}. \tag{14}$$

A simple example of a system satisfying this assumption is a linear harmonic oscillator. The next theorem provides sufficient conditions for the existence of a continuous globally asymptotically stable invariant manifold of the form $z = \alpha(w)$.

Theorem 2. Consider system (12) and system (13) satisfying the boundedness assumption **(BA)**. Suppose system (12) is regular and uniformly convergent. Then there exists a unique continuous function $\alpha : \mathbb{R}^m \to \mathbb{R}^d$ such that for any solution $w(t)$ of system (13) the function $z(t) = \alpha(w(t))$, $t \in \mathbb{R}$, is a UGAS solution (in the sense of Filippov) of system (12).

The proof of this theorem is given in Appendix. In the output regulation problem we may deal with exosystems that do not satisfy the boundedness assumption **(BA)**. For example, it can be an exosystem with a limit cycle or any other attractor with an unbounded domain of attraction. Therefore, we need to relax the conditions of Theorem 2 in order to include exosystems with complex dynamics. This is done in the next theorem.

Theorem 3. Consider systems (12) and (13). Suppose system (12) is regular and uniformly convergent. Let $\mathcal{W}_+$ be a bounded positively invariant set of system (13) and $\mathcal{W}_\pm \subset \mathcal{W}_+$ be an invariant subset of $\mathcal{W}_+$. Then there exists a continuous mapping $\alpha : \mathbb{R}^m \to \mathbb{R}^d$ such that for any solution $w(t)$ of system (13) starting in $w(0) \in \mathcal{W}_+$ the function $z(t) = \alpha(w(t))$ for $t \geq 0$ is a UGAS solution of system (12). In general, the mapping $\alpha(w)$ with such properties is not unique, but it is uniquely defined for all $w \in \mathcal{W}_\pm$.

The proof of this theorem is omitted here, because it repeats the proof of a similar result for continuous systems given in (Pavlov *et al.*, 2004*b*).

5. CONDITIONS FOR UNIFORM GLOBAL OUTPUT REGULATION

In this section, we apply the invariant manifold theorems to study the solvability of the uniform global output regulation problem. Denote by

$\Omega(w_0) \subset \mathbb{R}^m$ the set of all ω-limit points of the trajectory $w(t, w_0)$ satisfying the initial condition $w(0, w_0) = w_0$. Recall that $w^* \in \Omega(w_0)$ if there exists a sequence $\{t_k\}$ such that $t_k \to +\infty$ and $w(t_k, w_0) \to w^*$ as $k \to +\infty$. Denote

$$\Omega(\mathcal{W}_+) := \bigcup_{w_0 \in \mathcal{W}_+} \Omega(w_0).$$

Since the set $\mathcal{W}_+$ is a compact positively invariant set, then $\Omega(\mathcal{W}_+)$ is a nonempty invariant subset of $\mathcal{W}_+$. Moreover, $\Omega(\mathcal{W}_+)$ attracts all solutions of the exosystem (8) starting in $\mathcal{W}_+$, see (Bhatia and Szego, 1970). The next theorem, which is based on Theorem 3, establishes necessary and sufficient conditions for a controller (9) to solve the uniform global output regulation problem.

Theorem 4. Suppose controller (9) is such that the closed-loop system (10), (11) is regular and uniformly convergent. Then controller (9) solves the uniform global output regulation problem if and only if there exist continuous mappings $\pi(w)$ and $\sigma(w)$ defined in some neighborhood of $\Omega(\mathcal{W}_+)$ such that for any solution of the exosystem starting in $w(0) \in \Omega(\mathcal{W}_+)$ the function $(x(t), \xi(t)) = (\pi(w(t)), \sigma(w(t)))$ is a solution of the closed-loop system (10), (11) for all $t \in \mathbb{R}$ and $h_r(\pi(w), w) = 0$ for all $w \in \Omega(\mathcal{W}_+)$.

Proof: Since the closed-loop system is regular and uniformly convergent, by Theorem 3 there exist continuous mappings $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ such that for any solution of the exosystem $w(t)$ staring in $w(0) \in \mathcal{W}_+$ the function $(x(t), \xi(t)) = (\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is a UGAS solution of the closed-loop system.
Necessity: Set $\pi(w) := \tilde{\pi}(w)$ and $\sigma(w) := \tilde{\sigma}(w)$. Consider a point $w_* \in \Omega(\mathcal{W}_+)$. By the definition of $\Omega(\mathcal{W}_+)$, there exists $w_0 \in \mathcal{W}_+$ and $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to +\infty$ and $w(t_k, w_0) \to w_*$ as $k \to +\infty$. Therefore, for this $w(t) := w(t, w_0)$ the regulated output $e(t)$ along the solution $(\pi(w(t)), \sigma(w(t)))$ satisfies $e(t_k) \to h_r(\pi(w_*), w_*)$ as $k \to +\infty$. Since $e(t) \to 0$ as $t \to +\infty$, then $h_r(\pi(w_*), w_*) = 0$. Due to the arbitrary choice of $w_* \in \Omega(\mathcal{W}_+)$, we obtain that $h_r(\pi(w), w) = 0$ for all $w_* \in \Omega(\mathcal{W}_+)$.
Sufficiency: Since $(\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is UGAS, every solution of the closed-loop system converges to $(\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$. At the same time, $(\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ tends to the set $\tilde{\pi}(\Omega(\mathcal{W}_+)) \times \tilde{\sigma}(\Omega(\mathcal{W}_+))$, because $w(t)$ tends to the ω-limit set $\Omega(\mathcal{W}_+)$. Since $\Omega(\mathcal{W}_+)$ is an invariant subset of $\mathcal{W}_+$, by Theorem 3 the mappings $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ are uniquely defined for all $w \in \Omega(\mathcal{W}_+)$. This implies that $\tilde{\pi}(w) = \pi(w)$ and $\tilde{\sigma}(w) = \sigma(w)$ for all $w \in \Omega(\mathcal{W}_+)$, where $\pi(w)$ and $\sigma(w)$ are the mappings from the formulation of the theorem. Uniting these observations, we conclude that every solution of the closed-loop system tends to the set $\pi(\Omega(\mathcal{W}_+)) \times \sigma(\Omega(\mathcal{W}_+))$. Since $h_r(\pi(w), w) = 0$ for all $w \in \Omega(\mathcal{W}_+)$, this implies that the regulated output $e(t)$ tends to zero as $t \to +\infty$.□

Theorem 4 provides a characterization of all controllers solving the uniform global output regulation problem for discontinuous systems. For continuous systems similar results were obtained in (Pavlov *et al.*, 2004*b*) and (Isidori and Byrnes, 2003). If a controller is sought in the form (9) with a continuous mapping $\theta(\xi, y)$, then by denoting $c(w) := \theta(\sigma(w), h_m(\pi(w), w))$, we obtain the following necessary condition for the solvability of the problem.

Lemma 1. The uniform global output regulation problem is solvable with a controller (9) with a continuous mapping $\theta(\xi, y)$ only if there exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W}_+)$ such that for any solution of the exosystem $w(t)$ lying in $\Omega(\mathcal{W}_+)$ the function $\bar{x}_w(t) := \pi(w(t))$, $t \in \mathbb{R}$, is a solution (in the sense of Filippov) of the system

$$\dot{x} = f(x, c(w(t)), w(t)) \tag{15}$$

$$\text{and} \qquad h_r(\pi(w), w) = 0, \quad \forall\, w \in \Omega(\mathcal{W}_+) \tag{16}$$

For systems with continuous right-hand sides, it was shown in (Pavlov *et al.*, 2004*b*) that solvability of the *regulator equations* (see e.g. (Byrnes *et al.*, 1997)) is a necessary condition for the solvability of the uniform global output regulation problem. The result of Lemma 1 is an extension of this necessary condition to the case of discontinuous systems. In the discontinuous case, equations (15), (16) are a counterpart of the regulator equations.

6. EXAMPLE

We illustrate the application of Theorem 4 with the following example. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3 \\ \dot{x}_2 &= x_3 - x_2 - \mathrm{sign}(x_2) \\ \dot{x}_3 &= u \\ e &= x_1 - w_1, \quad y = (x, w), \end{aligned} \tag{17}$$

with the exogenous signal generated by the exosystem

$$\dot{w}_1 = w_2, \quad \dot{w}_2 = -w_1. \tag{18}$$

The set of initial conditions for the exosystem is given by $\mathcal{W}_+ := \{(w_1, w_2) : w_1^2 + w_2^2 \leq 1\}$. Any solution of the exosystem starting in $\mathcal{W}_+$ satisfies $|w_2(t)| \leq 1$ for all $t \in \mathbb{R}$. Moreover, $\Omega(\mathcal{W}_+) = \mathcal{W}_+$. We will try to find a continuous static state-feedback controller solving the uniform global output regulation problem. First, we need to check solvability of the regulator equations (see Lemma 1). One can easily check that for $c(w) = -w_1$, $\pi(w) = (w_1, 0, w_2)^T$ and for any solution of the exosystem (18) starting in $\mathcal{W}_+$ the function $\bar{x}_w(t) := \pi(w(t))$ is a solution (in the sense of Filippov) of system (15) and the equality (16) holds. We will look for a controller solving the global uniform output regulation problem in the form

$$u = c(w) + K(x - \pi(w)). \qquad (19)$$

System (17) in closed-loop with this controller is a system of the form (3), where $D = (0,1,0)^T$, $C = (0,1,0)$, $g(w) = (0,0,c(w) - K\pi(w))^T$ and the matrix A is computed from the parameters of system (17) and from the gain matrix K. The gain K is chosen such that conditions (4) and (5) are satisfied. For example, it can be chosen equal to $K = (-1,-1,-1)$. By Theorem 1, the closed-loop system is regular and uniformly convergent. Moreover, for any solution of the exosystem $w(t) = (w_1(t), w_2(t))$ lying in $\mathcal{W}_+$ the function $\bar{x}_w(t) := \pi(w(t))$ is a solution (in the sense of Filippov) of the closed-loop system and along this solution the regulated output equals zero. By Theorem 4 controller (19) solves the global uniform output regulation problem.

7. CONCLUSIONS

We have formulated and studied the uniform global output regulation problem for systems with discontinuous right-hand sides. Necessary and sufficient conditions for a controller to solve this problem are presented. It is shown that solvability of the (extended) regulator equations is a necessary condition for the solvability of the problem. These solvability results are based on certain invariant manifold theorems serving, in this case, as global discontinuous counterparts of the center manifold theorems, which form a foundation for the analysis of the local output regulation problem for smooth nonlinear systems.

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APPENDIX: PROOF OF THEOREM 2

Uniqueness: First, we show, that if there exists a continuous mapping $\alpha(w)$ such that $z(t) = \alpha(w(t))$ is a solution of system (12), then such mapping is unique. Suppose $\alpha(w)$ and $\tilde{\alpha}(w)$ are two such distinct mappings. Consider a solution $w(t)$ of system (13). Due to the boundedness assumption **(BA)**, this solution is bounded on $t \in \mathbb{R}$. Since $\alpha(w)$ and $\tilde{\alpha}(w)$ are continuous, they map bounded sets to bounded sets. Therefore, $\tilde{z}(t) := \tilde{\alpha}(w(t))$ and $z(t) := \alpha(w(t))$ are two distinct solutions of system (12) which correspond to the same input $w(t)$ and which are bounded for all $t \in \mathbb{R}$. This contradicts the convergence property of system (12). Thus, such mapping $\alpha(w)$, if it exists, is unique.

Existence: We prove the existence of $\alpha(w)$ by constructing this mapping. Due to the boundedness assumption **(BA)**, for every $w_0 \in \mathbb{R}$ the solution $w(t, w_0)$ of system (13) which satisfies the initial condition $w(0, w_0) = w_0$ is defined and bounded for all $t \in \mathbb{R}$. Since system (12) is convergent, for this $w(t) = w(t, w_0)$ there exists a unique UGAS steady-state solution $\bar{z}_w(t)$, which is defined and bounded for all $t \in \mathbb{R}$. For all w lying on the trajectory $w(t, w_0)$, $t \in \mathbb{R}$, define the mapping $\alpha(w)$ in the following way: $\alpha(w(t, w_0)) := \bar{z}_w(t)$. Repeating this process for all trajectories $w(t, w_0)$ of system (13), we define $\alpha(w)$ for all $w \in \mathbb{R}^m$. By the definition of $\alpha(w)$, for any solution $w(t) = w(t, w_0)$ of system (13) the function $z(t) = \alpha(w(t))$, $t \in \mathbb{R}$, is a UGAS solution (in the sense of Filippov) of system (12).

Continuity: It remains to show that the mapping $z = \alpha(w)$, constructed above, is continuous, i.e.

that for any $w_1 \in \mathbb{R}^m$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|w_1 - w_2| < \delta$ implies $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$. For simplicity, we will prove continuity in the ball $|w| < r$. Since r can be chosen arbitrarily, this will imply continuity in $\mathbb{R}^m$. In the sequel, we assume that $|w_1| < r$ and $\varepsilon > 0$ are fixed and the point w_2 varies in the ball $|w_2| < r$.

As a preliminary observation, notice that $|w_1| \le r$ and $|w_2| \le r$ imply, due to the boundedness assumption **(BA)**, that $|w(t, w_i)| \le \rho$ for $i = 1, 2$ and for all $t \in \mathbb{R}$. This, in turn, due to uniform convergence of system (12) (see (2)) and due to the construction of $\alpha(w)$, implies that $|\alpha(w(t, w_i))| \le \mathcal{R}$ for $i = 1, 2$ and for all $t \in \mathbb{R}$.

In order to prove continuity of $\alpha(w)$, we introduce the function

$$\varphi_T(w_1, w_2) := \hat{z}(0, -T, \alpha(w(-T, w_2)), w_1),$$

where the number $T > 0$ will be specified later and $\hat{z}(t, t_0, z_0, w_*)$ is the solution of the time-varying system

$$\dot{\hat{z}} = \mathcal{F}(\hat{z}, w(t, w_*)) \tag{20}$$

with the initial condition $\hat{z}(t_0, t_0, z_0, w_*) = z_0$. The function $\varphi_T(w_1, w_2)$ has the following meaning. First, consider the steady-state solution $\alpha(w(t, w_2))$, which is a solution of system (20) with the input $w(t, w_2)$ and initial condition $\alpha(w(0, w_2)) = \alpha(w_2)$. We shift along $\alpha(w(t, w_2))$ to time $t = -T$ and appear in $\alpha(w(-T, w_2))$. Then we switch the input to $w(t, w_1)$, shift forward to the time instant $t = 0$ along the solution $\hat{z}(t)$ corresponding to this $w(t, w_1)$ and starting in $\hat{z}(-T) = \alpha(w(-T, w_2))$ and appear in $\hat{z}(0) = \varphi_T(w_1, w_2)$. Notice, that $\varphi_T(w_0, w_0) = \alpha(w_0)$, because there is no switch of inputs and we just shift back and forth along the same solution $\alpha(w(t, w_0))$ – at this point we need the right-uniqueness of solutions of system (12), which is guaranteed by the regularity requirement on system (12). Thus,

$$\begin{aligned}\alpha(w_1) - \alpha(w_2) &= \varphi_T(w_1, w_1) - \varphi_T(w_2, w_2) \\ &= \varphi_T(w_1, w_1) - \varphi_T(w_1, w_2) \quad (21) \\ &\quad + \varphi_T(w_1, w_2) - \varphi_T(w_2, w_2).\end{aligned}$$

By the triangle inequality, this implies

$$\begin{aligned}|\alpha(w_1) - \alpha(w_2)| \le\, &|\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| \\ &+ |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)|.\end{aligned}$$

As follows from Lemma 2 (see below), there exist $T > 0$ such that

$$|\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| < \varepsilon/2 \quad \forall \ |w_2| < r. \tag{22}$$

It follows from Lemma 3 (see below), that given a number $T > 0$, there exists $\delta > 0$ such that

$$|\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2 \quad \forall \ w_2 : |w_1 - w_2| < \delta. \tag{23}$$

Unifying inequalities (22) and (23), we obtain $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$ for all w_2 satisfying $|w_1 - w_2| < \delta$. Due to the arbitrary choice of $\varepsilon > 0$ and $|w_1| < r$, this proves continuity of $\alpha(w)$ in the ball $|w| < r$. This completes the proof of continuity of $\alpha(w)$ and the proof of the theorem.□

Lemma 2. There exists $T > 0$ such that inequality (22) holds.

Proof: In order to prove inequality (22), notice that $\varphi_T(w_1, w_1) = \hat{z}_1(0)$ and $\varphi_T(w_1, w_2) = \hat{z}_2(0)$, where $\hat{z}_1(t)$ and $\hat{z}_2(t)$ are solutions of system (20) with the input $w(t, w_1)$ satisfying the initial conditions $\hat{z}_1(-T) = \alpha(w(-T, w_1))$ and $\hat{z}_2(-T) = \alpha(w(-T, w_2))$. By the definition of $\alpha(w)$, $\hat{z}_1(t) = \alpha(w(t, w_1))$ is a bounded UGAS solution of system (20). This impliess that it attracts all other solutions $\hat{z}(t)$ of system (20) uniformly over the initial conditions $t_0 \in \mathbb{R}$ and $\hat{z}(t_0)$ from any compact set. In particular, for the compact set $K(\mathcal{R}) := \{z : |z| \le \mathcal{R}\}$ and for the fixed $\varepsilon > 0$ there exists $\tilde{T}_\varepsilon(\mathcal{R})$ such that $\hat{z}(t_0) \in K(\mathcal{R})$ implies

$$|\hat{z}_1(t) - \hat{z}(t)| < \varepsilon/2, \quad \forall\ t \ge t_0 + \tilde{T}_\varepsilon(\mathcal{R}),\ t_0 \in \mathbb{R}. \tag{24}$$

Set $T := \tilde{T}_\varepsilon(\mathcal{R})$. By the definition of $\hat{z}_2(t)$, $\hat{z}_2(-T) = \alpha(w(-T, w_2))$. Since $\alpha(w(t, w_2)) \in K(\mathcal{R})$ for all $t \in \mathbb{R}$ and all $|w_2| < r$ (see above), then $\hat{z}_2(-T) \in K(\mathcal{R})$. Thus, for $t_0 = -T$ and $t = 0$ formula (24) implies

$$|\hat{z}_1(0) - \hat{z}_2(0)| < \varepsilon/2, \tag{25}$$

which is equivalent to (22).□

Lemma 3. Given a number $T > 0$ there exists a number $\delta > 0$ such that inequality (23) is satisfied.

Proof: In order to show (23), notice that for a fixed $T > 0$, the function $\hat{z}(0, -T, z_0, w_0)$ is continuous with respect to z_0 and w_0. This is due to the requirement of continuous dependence on initial conditions from the regularity assumption on system (12). Thus, $\hat{z}(0, -T, z_0, w_0)$ is uniformly continuous over the compact set $G := \{(z_0, w_0) : |z_0| \le \mathcal{R}, |w_0| \le r\}$. Hence, there exists $\delta > 0$ such that if $|z_0| \le \mathcal{R}$, $|w_1| \le r$, $|w_2| \le r$ and $|w_1 - w_2| < \delta$, then

$$|\hat{z}(0, -T, z_0, w_1)) - \hat{z}(0, -T, z_0, w_2)| \le \varepsilon/2. \tag{26}$$

Recall, that by the definition of $\varphi_T(w_1, w_2)$

$$\begin{aligned}&\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2) = \\ &\hat{z}(0, -T, z_0, w_1)) - \hat{z}(0, -T, z_0, w_2),\end{aligned} \tag{27}$$

where $z_0 := \alpha(w(-T, w_2))$. Notice, that $|w_1| \le r$, $|w_2| \le r$ and $|\alpha(w(-T, w_2))| \le \mathcal{R}$. Hence, as follows from (26) and (27),

$$|w_1 - w_2| < \delta \ \Rightarrow\ |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2.$$

Thus, we have shown (23).□