CONTROL DESIGN FOR DISTRIBUTED-PARAMETER SYSTEMS VIA PARAMETRIZATION

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Abstract: A methodology, called by the name parametrization, is presented here for calculating open-loop controls for linear distributed-parameter systems. It is based on the theory of certain type of pseudo-differential operators (ΨDO). These operators have features which are useful from practical and computational viewpoints. The operators form an algebra, and their inverses belong to the calculus, too. Boundary conditions always present in real-life problems are naturally included in the operator structures. Consequently, their defining elements, so-called symbols, have transfer function like properties. However, more complicated than only rational symbols are included in the calculus.

Keywords: Partial differential systems, Parametrization, Pseudo-differential operators, Heat system, Open-loop control design.

1. INTRODUCTION

Industrial controller design needs more and more sophisticated methodologies. This is due to increasing possibilities for complicated calculations in sufficiently short time intervals for control purposes. Control design for processes and systems described by nonlinear ordinary differential equation (ODE) model systems is nowadays very frequently solved based on the concept of *flatness*. It has its origin in differential-algebraic formulation of control systems introduced, terminologically coined, and developed by Michel Fliess and co-workers at the beginning of 90th (Fliess *et al.*, 1992 & 1995; Martin, 1992). Further, and most recent applications have been described in (Fliess *et al.*, 1999; Fleck *et al.*, 2004; Ratering and Eberhard, 2004), just to mention a few.

According to the latest knowledge of the authors, no constructive methods have been presented for evaluating whether a given ODE model system is flat or not. However, several researchers have applied this concept to linear, as well as nonlinear, partial differential equation (PDE) model systems controlled on the boundary (Lynch and Rudolph, 2000; Petit and Rouchon, 2001; Dunbar *et al.*, 2003; Meurer and Zeitz, 2004). The direction of applications is clearly towards real industrial processes and systems away from so-called academic toy model applications.

Here a methodology, called by the name *parametrization* is presented for calculating open-loop controls for linear distributed-parameter system models. It has its background in the flatness concept. The methodology is based on the theory of certain type of pseudo-differential operators

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(ΨDO). These operators have features which are useful from practical and computational viewpoints, *i.e.*

- dynamics (*e.g.* PDE) is included
- boundary conditions are included
- the operators form an algebra, *i.e.* it is closed under multiplication
- their inverses belong to the same algebra.

Actually, ΨDOs are described by their symbols, which in the ODE framework correspond to transfer functions but have several independent variables (*c.f.* Laplace variable in ordinary transfer functions) corresponding to spatial variables, say $x_1, x_2, \ldots, x_n$ in addition to the time t. Furthermore, a symbol needs not be a multivariable polynomial (as they are in PDE systems, because different partial differentiations correspond to different variables in the polynomial) but can be their solution operators, *i.e.* rational or even transcendental functions. Detailed descriptions of the necessary components in the operators are described in (Nihtilä *et al.*, 2005) based on the theory developed and refined *a.o.* by (Boutet de Monvel, 1971; Grubb, 1986 & 1995; Schrohe, 2001).

In the parametrization the goal is to find a function ($\sim$parameter), say z, or a set of differentially independent functions, depending on the time t or on several independent variables $t, x_{k_1}, \ldots, x_{k_j}$. Then the system variables, distributed like $u = u(x,t)$ and localized like $y(t) = u(1,t)$, can be expressed as functions of these parameters in the form, which may include dependence on partial derivatives and pseudo-differentially operated forms of the parameter function z. This rough description is analogous to the definition of flatness in ODE systems (Fliess *et al.*, 1995), where the parameter z stands for the *flat output*.

In the paper it is first given an introductory example on a heat model with a theoretical solution via ΨDOs. Then a general form of a linear control system of distributed-parameter models is described. Motion planning for the heat system considered is studied next. Finite-time transient results are demonstrated via a simulated piecewise differentiable response. This is based on parametrization of the heat system by the output to be controlled. In the discussion chapter possibilities to develop closed-loop control systems, applicability of parametrization to nonlinear PDE (and ΨDO) systems, and to quantum control are dicussed and possible tools are emphasized.

2. SYSTEM MODELS

An example on the heat model is presented in the framework of ΨDOs. Then a more general class of control systems is introduced.

2.1 *Example*

As an example consider a spatially 1-dimensional model of a metal rod heated at one end. The goal is to control the temperature at the other end of the rod. The PDE model of the system is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = g_0(x), \tag{1}$$

$$u(0,t) = 1, \quad \frac{\partial u}{\partial x}(0,t) = -w(t), \tag{2}$$

$$y(t) = u(1,t), \tag{3}$$

where u stands for the temperature, w the control, and y the output to be controlled. The function g_0 ($g_0(0) = 1$) stands for the initial temperature distribution along the rod, and $(x,t) \in G \times I = [0,1] \times [0,T]$.

By defining $v_1 = u - g_0$, the system (1)-(3) transforms to

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0, \tag{4}$$

$$v_1(0,t) = 0, \quad v_1(1,t) = y(t) - y(0), \tag{5}$$

$$v_1(x,0) = 0. \tag{6}$$

Define further a partial differential operator $r^+\mathcal{A}$, a trace operator $\mathcal{T}$, and their vector $\mathbb{L}$ by

$$\mathbb{L}v_1 = \begin{bmatrix} r^+\mathcal{A}v_1 \\ \mathcal{T}v_1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial v_1}{\partial t} - \dfrac{\partial^2 v_1}{\partial x^2} \\ (v_1(0,t), v_1(1,t)) \end{bmatrix}. \tag{7}$$

Then the system (4)-(6) with the initial profile becomes

$$\mathbb{L}v_1 = \begin{bmatrix} 0 \\ (0, y(t) - y(0)) \end{bmatrix}; \quad v_1(x,0) = 0. \tag{8}$$

The system operator $\mathbb{L}$ is unifomly parabolic. The system operator is in a sense invertible. The inversion, *i.e.* the solution of the system (8), is obtained according to (Grubb, 1995) in the form

$$\begin{aligned} v_1 &= [\, \mathcal{Q} + \mathcal{G} \quad \mathcal{K} \,] \begin{bmatrix} 0 \\ (0, y(t) - y(0)) \end{bmatrix} \\ &= \mathcal{B}\, y, \end{aligned} \tag{9}$$

where the operators are as follows:

$\mathcal{Q}$ is a pseudo-differential operator $X \to X$
$\mathcal{G}$ is a singular Green (SG) operator $X \to X$
$\mathcal{K}$ is a potential operator $Y \to X$

where $X = C^\infty(\overline{G} \times I)$ and $Y = C^\infty(\partial G \times I)$. Theory of pseudo-differential operators has been explained in detail *e.g.* in (Grubb, 1986 & 1995).

2.2 *Control system*

Let G be an open bounded set in $\mathbb{R}^n$ with a smooth boundary ∂G, and let I be and interval in $\mathbb{R}$. The control system considered in componentwise form ($i = 1, \ldots, N_1$ in (10), and $p = 1, \ldots, m_1$ in (11)) is

$$\sum_{j=1}^{N} \left(r^+ \mathcal{A}_{ij} + \mathcal{B}_{ij}\right) v_j + \sum_{\ell=1}^{m} \mathcal{K}_{i\ell} w_\ell = 0, \quad (10)$$

$$\sum_{j=1}^{N} \mathcal{T}_{pj} v_j + \sum_{\ell=1}^{m} \mathcal{Q}_{p\ell} w_\ell = 0, \quad (11)$$

where (10) is the domain equation and (11) is the boundary equation. The operators are of the following type:

- $r^+ \mathcal{A}_{ij} : C^\infty(\overline{G} \times I) \to C^\infty(\overline{G} \times I)$, ψDO
- $\mathcal{T}_{pj} : C^\infty(\overline{G} \times I) \to C^\infty(\partial G \times I)$, trace op.
- $\mathcal{K}_{j\ell} : C^\infty(\partial G \times I) \to C^\infty(\overline{G} \times I)$, potential op.
- $\mathcal{B}_{ij} : C^\infty(\overline{G} \times I) \to C^\infty(\overline{G} \times I)$, SG op.
- $\mathcal{Q}_{p\ell} : C^\infty(\partial G \times I) \to C^\infty(\partial G \times I)$, ψDO.

2.3 *Examples of the operators*

Let $\overline{G} = [0, 1]$, $I = [0, T]$. Potential operators describe the boundary effects on the domain G. As an example

$$(\mathcal{K}\varphi)(x, t) = (1 - x)\, \varphi_1(t) + x\, \varphi_2(t)$$

where $\varphi = (\varphi_1, \varphi_2)$ describes variations on the boundary $\partial G = \{0, 1\}$. A trace operator can be defined by

$$(\mathcal{T}u)(t) = u\Big|_{\partial G} = (u(0, t), u(1, t)).$$

Partial differential operators, like

$$r^+ \mathcal{A} = \frac{\partial^2}{\partial t^2} - c(t)\frac{\partial^2}{\partial x^2} + a(x)\frac{\partial}{\partial x}$$

form a subclass of ψDOs. In addition to PD operators their inverses are in a sense also ψDOs. Then integral representations are used for definitions. A typical representation of this class is given by

$$(r^+ \mathcal{A}v)(x, \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,\xi x} a(\xi, \lambda)\, \widehat{(e^+ v)}(\xi, \lambda)\, \mathrm{d}\xi$$

The operator e^+ stands for the extension by zero of the function (in the space argument x) from G to $\mathbb{R}$, and r^+ restriction back to G. The wide-hat denotes Fourier transformation (with respect to x). The symbol λ represents a parameter in the function $v(x, \lambda)$. It may correspond to the Laplace variable for the original time argument t, or the time argument itself. The function $a(\xi, \lambda)$ is the symbol of the operator. In the case of linear parabolic constant coefficient PD operators represented in the integral form, the symbol is of the form $\lambda - L(\xi)$, where $L(\xi)$ a polynomial in ξ. The symbol of the inverse of this PD operator is simply

$$b(\xi, \lambda) = \left(\lambda - L(\xi)\right)^{-1},$$

when boundary conditions are not included in the problem. However, boundary conditions are always present in real-life problems. The we have to consider a larger class of operators. These operators originate in the algebra of (Boutet de Monvel, 1971), enlarged by (Grubb, 1986 & 1995). The variables in (10)-(11): $v_1, \ldots, v_N$ and $w_1, \ldots, w_m$ are indeterminates, *i.e.*

$$v_j \in C^\infty(\overline{G} \times I); \quad w_\ell \in C^\infty(\partial G \times I). \quad (12)$$

Denote now $\mathfrak{R}^{N,m} = C^\infty(\overline{G} \times I)^N \times C^\infty(\partial G \times I)^m$. A compact form of the model (10)-(11) is given by $\mathbb{L}u = 0$. Then $\mathbb{L}$ can be interpreted as a linear operator

$$\mathbb{L} : \mathfrak{R}^{N,m} \to \mathfrak{R}^{N_1, m_1}, \quad (13)$$

$$\text{where } \mathbb{L} = \begin{bmatrix} \left(r^+ \mathcal{A}_{ij} + \mathcal{B}_{ij}\right) & \left(\mathcal{K}_{i\ell}\right) \\ \left(\mathcal{T}_{pj}\right) & \left(\mathcal{Q}_{p\ell}\right) \end{bmatrix}; \quad (14)$$

$$u = \begin{bmatrix} v \\ w \end{bmatrix}; \; v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}; \; w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \quad (15)$$

The matrix operator $\mathbb{L}$ is of the type $(N_1 + m_1) \times (N + m)$ having submatrices made by the operators of the type above. The individual operator quadruples (2×2-matrices) of the form $\begin{bmatrix} r^+ \mathcal{A} + \mathcal{B} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{bmatrix}$ form an operator algebra, say $\mathfrak{D}$, under approppriate assumptions. Then we can form in many cases formal adjoints, which are in the same algebra. Adjoints are needed in an approach to construct parametrization operators, see *e.g.* (Nihtilä *et al.*, 2000 & 2003). Here, however, an inversion technique is applied.

3. MOTION PLANNING FOR THE HEAT SYSTEM

The goal is to drive the output y of the heat system (1-3) from one steady-state value y_0 to another y_1 in a finite time T, and keep it in this new value. This corresponds to the driving of system's state u from one steady-state profile g_0 to another, say g_1 in a finite time. The steady-state profiles, because $\frac{\partial u}{\partial t} = 0$ and due to the boundary conditions, are

$$g_i(x) = 1 + (y_i - 1)x, \; i = 0, 1. \quad (16)$$

It is seen from (9) that the solution

$$u(x, t) = v_1(x, t) + g_0(x) = (\mathcal{B}\, y)(x, t)$$

is a function of y. Consequently, the solution and its values on the other boundary are parametrized by y. In the form of a theorem a result is obtained.

Theorem 1. A parametrization of the linear heat system (1)-(3) with the parameter (function) $y(t) = u(1,t)$ is given by

$$u(x,t) = (\mathcal{B}\, y)(x,t) + g_0(x) \tag{17}$$

$$w(t) = -\frac{\partial}{\partial x}\Big((\mathcal{B}\, y)(x,t) + g_0(x)\Big)\Big|_{x=0}, \tag{18}$$

where the operator $\mathcal{B}$ is described in Appendix.

3.1 *Simulation*

In order to construct the operator $\mathcal{B}$ applied to y we have in the general solution (37) of Appendix $f = 0$ and $g = (0, V(\lambda))$. Furthermore, then

$$\begin{aligned}
(\mathcal{K}'g)(x,\lambda) &= x\,V(\lambda),\\
(\mathcal{A}-\lambda\mathcal{I})(\mathcal{K}'g)(x,\lambda) &= -x\,\lambda\,V(\lambda),\\
(\mathcal{K}_\lambda g)(x,\lambda) &= (\mathcal{K}'g)(x,\lambda)\\
&\quad + (\mathcal{Q}_\lambda + \mathcal{G}_\lambda)(x\,\lambda\,V(\lambda))\\
&= x\,V(\lambda) + (\mathcal{Q}_\lambda + \mathcal{G}_\lambda)(x)\,\lambda\,V(\lambda). \quad (19)\\
\text{where } \; V(\lambda) &= \mathcal{L}\{y(t) - y(0)\} \quad (20)
\end{aligned}$$

Via some technicalities including residual calculus and Cauchy's principal value techniques we calculate the operator values of $(\mathcal{Q}_\lambda f)(x,\lambda)$ and $(\mathcal{G}_\lambda f)(x,\lambda)$ for the function $f(x,\lambda) = (\mathcal{K}'g)(x,\lambda)$. By using of these results we obtain the Laplace transformed temperature U and control W in the form ($\mu = \sqrt{\lambda}$)

$$U(x,\lambda) = \left(2x - \frac{\sin(\mu x)}{\sin\mu}\right)V(\lambda) + \frac{g_0(x)}{\lambda} \tag{21}$$

$$W(\lambda) = -\left(2 - \frac{\mu}{\sin\mu}\right)V(\lambda) - \frac{g_0'(0)}{\lambda}. \tag{22}$$

These then must be inverted numerically due to transcendental nature of $\mathcal{K}_\lambda(0, V(\lambda))$ in the variable λ. As a correctness test we have

$$\begin{aligned}
U(1,\lambda) &= \mathcal{L}\{y(t)\},\\
U(0,\lambda) &= \mathcal{L}\{u(0,t)\}.
\end{aligned}$$

To demonstrate possibility of nonsmooth trajectories a piecewise linear function is applied for the parameter y:

$$y(t) = \begin{cases} y_0, & 0 \le t \le T_1\\ y_0 + k\,(t - T_1), & T_1 \le t \le T_2\\ y_1, & T_2 \le t \le T \end{cases} \tag{23}$$

$$k = \frac{y_1 - y_0}{T_2 - T_1}.$$

The following values were applied in simulation:

$$T_1 = 10; \quad T_2 = 15; \quad T = 25; \quad y_0 = 2; \quad y_1 = 3.$$

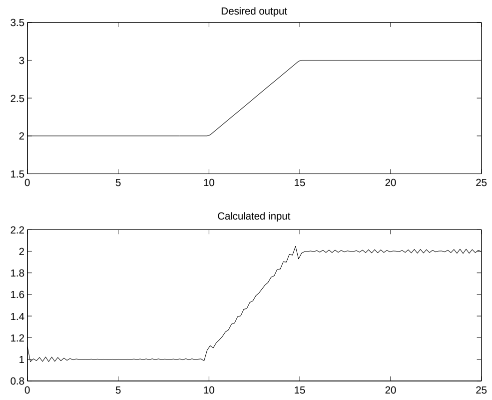

Fig. 1. Upper curve represents the desired ramp output y (the parameter). Lower figure depicts the corresponding (reference) input w.

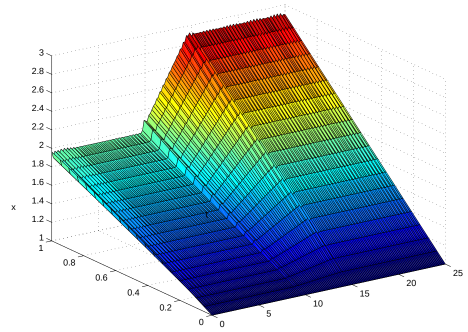

Fig. 2. Solution $u(x,t)$ of the heat system (1)-(3) calculated by the finite difference scheme using the input of Fig.1. when $\overline{G} = [0,1]$ and $t \in [0, 25]$.

Difference method used. Suppose the control $w(t)$ is given. $\overline{G} = [0,1]$ and the time interval $[0,T]$ are discretized as $0 = x_1 < x_2 < \cdots < x_n = 1$, and $0 = t_1 < t_2 < \cdots < t_m = T$. The spatial step length is $h = 1/n$. Initial conditions are discretized, for $j = 1,\ldots,m$, as

$$u(x_1,t_j) = 1, \quad u(x_2,t_j) = -hw(t_j) + u(x_1,t_j).$$

Derivatives $\frac{\partial u}{\partial t}(x_i,t_j) \hat{=} \frac{Du}{Dt}(x_i,t_j)$ are calculated as

$$\begin{aligned}
\frac{Du}{Dt}(x_i,t_1) &:= \frac{u(x_i,t_2) - u(x_i,t_1)}{t_2 - t_1},\\
\frac{Du}{Dt}(x_i,t_j) &:= \frac{u(x_i,t_{j+1}) - u(x_i,t_{j-1})}{t_{j+1} - t_{j-1}},\\
&\quad j = 2,\ldots,m-1,\\
\frac{Du}{Dt}(x_i,t_m) &:= \frac{u(x_i,t_m) - u(x_i,t_{m-1})}{t_m - t_{m-1}},
\end{aligned}$$

for $i = 1,\ldots,n$. The heat equation is solved as

$$u(x_{i+1},t_j) = h^2\frac{Du}{Dt}(x_i,t_j) + 2u(x_i,t_j) - u(x_{i-1},t_j).$$

Calculation proceeds as follows. First initial data is obtained from the first equation. Thus we know $u(x_i,t_j)$ for $i = 1,2$ and $j = 1,\ldots,m$. Suppose inductively we know $u(x_i,t_j)$ for $i \le k$ and

$j = 1, \ldots, m$. Then from the second equation we calculate discretized derivatives $\frac{Du}{Dt}(x_k, t_j)$, $j = 1, \ldots, m$ and then from the last equation we obtain $u(x_{k+1}, t_j)$, for $j = 1, \ldots, m$. As we get $k = n$ we terminate the algorithm.

4. DISCUSSION

Applicability to extend industrially attractive and relatively easily implementable flatness-based control of ODE models to PDE systems is demonstrated. An example of the heat equation with the necessary boundary conditions is studied by using the parametrization concept.

Pseudo-differential operator theory offers a well-founded means to study control issues for PDE systems. From the mathematical viewpoint the domains of ψDOs have been extended from C^∞-functions to temperate distributions of Laurent Schwartz (Horváth, 1966) on $\mathcal{S}'(\mathbb{R}^n)$ and to some other, say somewhat tricky Sobolev spaces. Consequently, we do not need to worry about ordinary differentiability of the functions at hand.

Closed-loop control is always desirable in industrial environment. The parametrization-based open-loop control methodology can be implemented in closed-loop form by using some invariances of the PDE systems at hand. These considerations are, however, at the beginnning.

Applicability of parametrization-based control in nonlinear systems is based on global linearization of PDE systems. As a test example control design (or motion planning) was carried out for nonlinear viscous Burgers' system via parametrization in (Nihtilä *et al.*, 2004).

Control of quantum mechanically describable systems offers a real challenge for open-loop control (LeBris, 2000; Brown and Rabitz, 2002). Then laser-controlled electric fields are used for exciting states of molecular systems. This issue, if feasibly solved, may be long-ranging application possibilities in future telecommunication and computer systems.

The invariances, global linearization, and quantum control need, however, deep knowledge of the properties of PDE systems and their symmetries (Olver, 1993; Bocharov *et al.*, 1999; Hydon, 2000) offering this way collaboration possibilities between trained mathematicians and industrial engineers.

5. APPENDIX

Define between suitable Sobolev spaces H_1 and H_2 the operator

$$\mathbb{A}_\lambda = \begin{bmatrix} \mathcal{A}_\lambda \\ \mathcal{T} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial^2}{\partial x^2} - \lambda \mathcal{I} \\ \mathcal{T} \end{bmatrix} \tag{24}$$

for $\lambda \in \mathbb{C}_{-\epsilon} = \{\, \lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda \geq -\epsilon \,\}$, where $\mathcal{I}$ is the identity operator, and the trace operator $\mathcal{T}$ is given in (33). Then $\mathbb{A}_\lambda^{-1}$ exists and it has the form

$$\mathbb{A}_\lambda^{-1} = [\, \mathcal{Q}_\lambda + \mathcal{G}_\lambda \quad \mathcal{K}_\lambda \,] \tag{25}$$

The operator $\mathcal{B}$ is given by

$$\mathcal{B} = \mathcal{L}^{-1} \mathbb{A}_\lambda^{-1} \mathcal{L}, \tag{26}$$

where $\mathcal{L}$ is the Laplace transform, and $\mathcal{L}^{-1}$ its inverse.

5.1 *Operators $\mathcal{Q}_\lambda$, $\mathcal{G}_\lambda$, and $\mathcal{K}_\lambda$*

Let $\mathcal{A} : L_2(G) \to L_2(G)$ be the operator

$$D(\mathcal{A}) = H_0^1(G), \tag{27}$$

$$\mathcal{A}\, v = -\frac{\partial^2 v}{\partial x^2} \quad \text{(in weak sense)}. \tag{28}$$

Then for $\lambda \in \mathbb{C}_{-\epsilon}$ ($\epsilon > 0$), the resolvent operator of $\mathcal{A}$ exists and is given by

$$(\mathcal{A} - \lambda \mathcal{I})^{-1} = \mathcal{Q}_\lambda + \mathcal{G}_\lambda, \tag{29}$$

where the operators are ($\mu = \sqrt{\lambda}$)

$$(\mathcal{Q}_\lambda\, f)(x, \lambda) = -\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\xi x}}{\xi^2 - \lambda} \widehat{e^+ f}(\xi, \lambda)\, \mathrm{d}\xi \tag{30}$$

$$(\mathcal{G}_\lambda\, f)(x, \lambda) = \frac{\sin(\mu x)}{2\pi \sin \mu} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\xi}}{\xi^2 - \lambda} \widehat{e^+ f}(\xi, \lambda)\, \mathrm{d}\xi$$
$$+ \frac{\sin(\mu(1-x))}{2\pi \sin \mu} \int_{\mathbb{R}} \frac{1}{\xi^2 - \lambda} \widehat{e^+ f}(\xi, \lambda)\, \mathrm{d}\xi. \tag{31}$$

The inverse $\mathbb{A}_\lambda^{-1}$ is constructed as follows. The solution v of the problem

$$\mathcal{A}_\lambda\, v = -\frac{\partial^2 v}{\partial x^2} - \lambda\, v = f \tag{32}$$

$$\mathcal{T}\, v = (v(0, \lambda),\, v(1, \lambda)) = g \tag{33}$$

for the given $f(x, \lambda)$ and $g(\lambda) = (g_1(\lambda),\, g_2(\lambda))$ is decomposed as $v = v_1 + v_2$, where v_1 is the solution for $g = 0$, and v_2 for $f = 0$. Substituting $V_2 = v_2 - \mathcal{K}'\, g$ to (32-33) for $f = 0$, where the potential operator is defined by

$$(\mathcal{K}' g)(x, \lambda) = (1 - x)\, g_1(\lambda) + x\, g_2(\lambda), \tag{34}$$

transforms the problem (with $f = 0$) into

$$\mathcal{A}_\lambda\, V_2 = -\, \mathcal{A}_\lambda\, \mathcal{K}'\, g \tag{35}$$

$$\mathcal{T}\, V_2 = 0. \tag{36}$$

Hence the solution $v = v_1 + v_2 = v_1 + V_2 + \mathcal{K}'\, g$ is

$$v = (\mathcal{Q}_\lambda + \mathcal{G}_\lambda)\, f + \mathcal{K}_\lambda\, g \tag{37}$$

The potential component is

$$\mathcal{K}_\lambda = [\mathcal{I} - (\mathcal{Q}_\lambda + \mathcal{G}_\lambda)(\mathcal{A} - \lambda \mathcal{I})]\mathcal{K}'. \tag{38}$$

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