VARIABLE STRUCTURE CONTROL WITH UNKNOWN INPUT-DELAY AND NONLINEAR DISSIPATIVE DISTURBANCE

Laura Levaggi * **Elisabetta Punta** **,[1]

* *DIMA - University of Genova*
Via Dodecaneso, 35 - 16146 Genova, Italy
phone: +39 010 3536929, fax: +39 010 3536752
e-mail: levaggi@dima.unige.it
** *ISSIA CNR - National Research Council of Italy*
Via De Marini, 6 - 16149 Genova, Italy
phone: +39 010 6475642, fax: +39 010 6475600
e-mail: punta@ge.issia.cnr.it

Abstract: In this paper a double integrator system with delayed input controlled by a sub-optimal second order sliding mode control law is considered. The case when the system is affected by a bounded, uncertain dissipative drift term is investigated. It is proven that, due to the input delay, the limit system trajectories are periodic. Moreover it is shown that the amplitude of the limit cycle is reduced by the presence of the disturbance. These results are exploited to synthesize a new control law which guarantees the simultaneous reduction of the amplitude of the oscillations and the reaching time of any neighbourhood of the limit cycle.

Keywords: Second-order systems, Variable-Structure Control, Time Delay, Bounded Disturbances, Robust Control

1. INTRODUCTION

In recent years both the engineering and mathematical communities have paid a great attention to the research area of systems with time delays (TDS). This increasing interest is motivated by several factors. From one side the technological progress allows the enhancement of the systems performances, nevertheless the analysis and control of systems requires the availability of more and more precise mathematical models. In this sense the previously neglected time delays must now be taken into account when modeling the systems dynamics. Furthermore communication networks and information technologies are rapidly spreading and, considering this kind of systems, delays play an important role.

From the control point of view, the effects of the time delays result in two main forms, either the evolution of the control systems is governed by equations of retarded type (delay in the state variable), or the time delays are introduced through the control channel (delay in the input variable). This second kind of delay can be caused by either the actuators, or the measurement devices, or both (Choi and Hedrick, 1996; Kolmanowskii and Myshkis, 1999). In this paper a class of uncertain second order control systems with delayed input is considered.

[1] Work partially supported by CNR, "Progetto Giovani", codice no. CNRG004E90, linea di ricerca: Fluidodinamica Industriale.

Classical control methods performances can be substantially deteriorated by delays, thus specific controllers have to be designed (for a survey see e.g. (Richard *et al.*, 2000)).

When considering control systems, important aspects, such as robustness with respect to external disturbances, have to be taken into account. The design of sliding mode control (SMC) for TDS has been studied in view of exploiting its properties. Many of these results consider delays in the state, where the SMC "philosophy" does not change, but the sliding surface must be suitably chosen.

The nature of the problem is instead completely altered if the effect of an input delay in a relay-type controller is considered. In this case the delay induces oscillations around the sliding surface. First order relay control systems with input delay are considered in (Fridman *et al.*, 2000; Fridman *et al.*, 2002). It is shown that the resulting motions are oscillatory and have finite limit frequency. Moreover zero limit frequency solutions coincide up to time shifts and the property of zero limit frequency is stable, while faster oscillations result in instability. In (Shustin *et al.*, 2003) the authors consider second order relay control with time delay. In this case it is still possible to prove that periodic slowly oscillating solutions exist and the property of zero limit frequency is stable. However, the wider variety of these solutions results in a more complicated structure. For example orbital asymptotic stability can be guaranteed only when the uniqueness of slowly oscillating solutions holds and this requires additional assumptions.

In the first part of this paper a double integrator with delayed input controlled according to a particular second order sliding mode control method (Bartolini *et al.*, 1997) is introduced. The effect of an input time delay on the closed loop system has been studied in (Levaggi and Punta, 2004) and (Levaggi and Punta, 2003*a*). It has been shown that, whenever the control modulus is constant throughout the evolution, for any choice of the control parameters and for any fixed constant input delay, the limit state evolution is periodic, thus giving rise in the phase plane to a limit cycle. Moreover this limit periodic trajectory is unique and it is globally attractive. The relevant results about this case are briefly presented in Section 2.

In Section 3 a bounded dissipative nonlinear term of the type $-F\operatorname{sign} y_2(t)$ acting on the control system is introduced. It is assumed that the modulus F is uncertain but a bound F_M for F is known. It is proved that in this case, once the control action is chosen to dominate the drift term, the system reaches a limit cycle. The amplitude and frequency of the oscillations of both position and velocity are uniquely determined by the control parameters, the delay τ and the drift modulus F. Moreover it is shown that, for any fixed feasible choice of the control parameters, the resulting limit cycle is smaller than the one obtained in the unperturbed case. Note that in (Levaggi and Punta, 2004) and (Levaggi and Punta, 2003*a*) it has been shown that if the control law is of a non-constant intensity, the amplitude and periodicity of the oscillation of position and velocity can depend on the starting point. Although in this case too the acceleration modulus is non-constant, the limit cycle is in contrast unique and stable.

In Section 4 a modification of the control algorithm in Section 2 is proposed. As in (Levaggi and Punta, 2003*b*) a time-dependent, piecewise constant control modulus is defined, which guarantees a step by step reduction of the size of the limit cycle. Due to the presence of the disturbance, the control modulus still has to satisfy some constraints. This implies that the lower limit for the size of the attainable limit cycle is non-zero. Therefore the asymptotic behaviour of the closed loop system is still a persisting oscillating one. Nonetheless the proposed control algorithm assures a faster damping of the oscillations of both position and velocity. These features are illustrated by simulation results.

2. A DOUBLE INTEGRATOR WITH DELAYED INPUT

The control system considered in this section is the following

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u(t-\tau), \quad u(\theta) = u_0(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{1}$$

which is a simple double integrator with scalar control u, with input subject to a fixed delay $\tau > 0$. This delay can be interpreted as either a delay in the actuators or in the sensors in the case of feedback control.

The following control law for system (1) is defined.

Algorithm 1:

Let $U > 0$, $\gamma \in (0,1)$. *When* $t = 0$, *set* $y_{1_M} = y_1(0)$.

For $t \in [0, \infty)$ *repeat the following steps:*
if $y_2(t) = 0$ *set* $y_{1_M} = y_1(t)$;
next apply the control

$$u(\theta) = -U \operatorname{sign}(y_1(\theta) - \gamma y_{1_M}). \tag{2}$$

The controlled system (1)-(2) has been studied in (Levaggi and Punta, 2003*a*), (Levaggi and Punta, 2003*b*): it is shown that the asymptotic behaviour of both position and velocity is a persisting oscillating one. The resulting limit cycle is globally attractive and stable. In this section the relevant results for this case are briefly presented.

The control law (2) is defined through the evolution of the trajectory's intercepts with the y_1 axis, which can be interpreted as a discrete dynamical system. Assume that at the initial time the position is $z_0 = y_1(0)$ and the velocity is zero; set

$$\delta = U\tau^2, \quad \alpha = 2\gamma - 1, \quad \beta = 2\sqrt{\delta(1-\alpha)}.$$

For $k = 0, 1, 2, \ldots$ the following discrete dynamical system is generated

$$\begin{aligned} \hat{z}_k &= \alpha z_k - \text{sign}\,(z_k)\,(\delta + \beta\sqrt{|z_k|}) \\ z_{k+1} &= \hat{z}_k + \delta\,\text{sign}\,(z_k)\,\max\{0, \text{sign}\,(z_k \hat{z}_k)\}. \end{aligned} \tag{3}$$

The elements z_k represent the useful intercepts with the y_1 axis : given z_k the next null-velocity point is $\hat{z}_k$. We skip it if sign $(z_k\hat{z}_k) = -1$, since it is not relevant for the convergence of the sequence. It can be proven that for all choices of the control parameters and any delay τ there exists $k_0 \geq 0$ such that $z_k\hat{z}_k < 0$ for all $k \geq k_0$. Therefore for $k \geq k_0$ we have

$$\begin{aligned} z_{k+1} &= -\text{sign}\,(z_k) f(|z_k|), \\ f(z) &= -\alpha\, z + \beta\sqrt{z} + \delta, \end{aligned}$$

which describes the sequence of the intercepts, after the finite transient to get to k_0. It is shown that the sequence $\{|z_k|\}$ is either eventually monotone or contractive and thus convergent.

Theorem 2.1. For all choices of the control parameters, any delay τ and any initial value z_0, the sequence

$$|z_{k+1}| = f(|z_k|) = f^{k+1}(|z_0|), \tag{4}$$

with $f(z) = -\alpha\, z + \beta\sqrt{z} + \delta$, converges to the unique fixed point of f

$$\bar{z} = \delta\, \frac{3 - \alpha + 2\sqrt{2(1-\alpha)}}{(\alpha+1)^2}. \tag{5}$$

Therefore for k tending to infinity the sequence bounces from the point $\bar{z}$ to its symmetric $-\bar{z}$. The solution of the closed loop (1)-(2) shows a limit cycle in the phase plane which is centered at the origin and symmetric with respect to the axes. The lengths of the intersections with the axes y_1 and y_2 are respectively

$$d_1 = U\tau^2\, g(\alpha), \qquad d_2 = U\tau\; 2\sqrt{2g(\alpha)}$$

where

$$g(\alpha) = \frac{2}{(\alpha+1)^2}(3 - \alpha + 2\sqrt{2(1-\alpha)}).$$

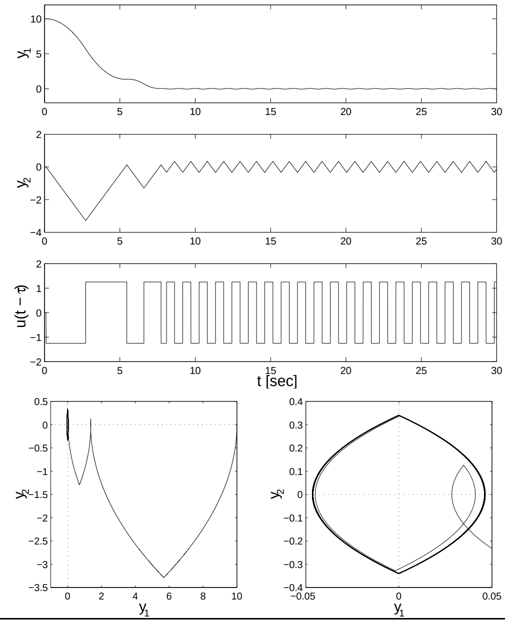

Fig. 1. The starting point is $(10, 0)$, $u_0 = 0$. The delay is $\tau = .1$ and the control parameters are set to $U = 1.25$ and $\gamma = 0.6$.

Proof. See Theorem 3.1 and Corollary 3.1 in (Levaggi and Punta, 2003*b*).

In Figure 1 both the phase plane plots and the evolution in time of position, velocity and control are shown for the simulation of a specific example. As proved, in the limit the evolution is periodic and the limit cycle symmetric.

3. A DOUBLE INTEGRATOR WITH DELAYED INPUT AND FRICTION

In the previous section it has been considered the effect of an input delay on a double integrator when controlled by means of a second order variable structure control law. Whatever the choice of the control parameters the system's trajectories are limited. Let us consider the case in which a dissipative nonlinear term of the type $-F\text{sign}\, y_2(t)$ acts on the control system. What can be expected, is that the system reaches a smaller limit cycle. In this sense, once the control action is chosen to dominate the drift term, it turns out that the action of the disturbance can be exploited to decrease the amplitude of the oscillations.

The considered control system is the following

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = -F\text{sign}\, y_2(t) + u(t-\tau), \end{cases} \tag{6}$$

with $u(\theta) = u_0(\theta)$, $\theta \in [-\tau, 0]$, and $F \geq 0$. The friction-like term $-F\text{sign}\, y_2(t)$, which acts on the system, is uncertain and is bounded, in modulus, by a known constant $F_M > F$.

The control law for system (6) is still designed according to Algorithm 1, under the control constraint

$$U \geq F_M. \tag{7}$$

As in the unperturbed case the evolution of the closed loop system depends on the distribution of the following set of extremal points

$$\{y_1(t) \,:\, y_2(t) = 0\}.$$

Assume, without loss of generality, that at the initial time the velocity is zero and $y_1(0) = x_0$. Set $u_0(\theta) = -U \operatorname{sign} x_0$, $\theta \in [-\tau, 0]$. For the closed loop system (6)-(2) the next extremal point is given by

$$\begin{aligned}\hat{x}_0 \;=\; & x_0 \frac{U(2\gamma - 1) + F}{U + F} - \operatorname{sign} x_0 \frac{U\tau^2(U - F)}{U + F} \\ & - \operatorname{sign} x_0 \frac{2U\tau\sqrt{2(1-\gamma)|x_0|(U - F)}}{U + F}.\end{aligned}$$

Now, if $\operatorname{sign}(x_0 \hat{x}_0) > 0$, once the position $\hat{x}_0$ is reached, the control structure changes, and so does the control sign. However, due to the input delay, this change will really affect the system after it has acquired a little velocity. The change of sign in the acceleration will then annihilate it and produce another extremal point, closer to x_0. As the behaviour of the closed loop is based on the convergence of the sequence of intercepts, in this case only the second one x_1 is significant

$$x_1 = \hat{x}_0 + \operatorname{sign} x_0 \max\{0, \operatorname{sign}(x_0 \hat{x}_0)\} U\tau^2 \frac{U - F}{U + F}.$$

Recursing the process, the following sequence is generated: given x_0 let

$$\begin{aligned} &\delta = U\tau^2, \quad r = \frac{F}{U}, \\ &\delta_r = \delta \frac{1 - r}{1 + r}, \quad \alpha_r = \frac{2\gamma - 1 + r}{1 + r}, \\ &\beta_r = 2\sqrt{\delta_r (1 - \alpha_r)}, \end{aligned}$$

and for $k = 0, 1, 2, \ldots$ define

$$\begin{aligned} \hat{x}_k &= \alpha_r x_k - \operatorname{sign} x_k \,(\delta_r + \beta_r \sqrt{|x_k|}) \\ x_{k+1} &= \hat{x}_k + \delta_r \operatorname{sign} x_k \,\max\{0, \operatorname{sign}(x_k \hat{x}_k)\}. \end{aligned} \tag{8}$$

Since the discrete dynamical system (8) has the same structure as (3), the analysis of the sequence of the extremal points in this case can be carried out along the same lines. The final results follow.

Theorem 3.1. For all choices of the control parameters, any delay τ and any initial value x_0, there exists $k_1 \geq 0$ such that $x_k \hat{x}_k < 0$ for all $k \geq k_1$. For $k \geq k_1$

$$\begin{aligned} x_{k+1} &= -\operatorname{sign}(x_k) f_r(|x_k|), \\ f_r(x) &= -\alpha_r\, x + \beta_r \sqrt{x} + \delta_r, \end{aligned}$$

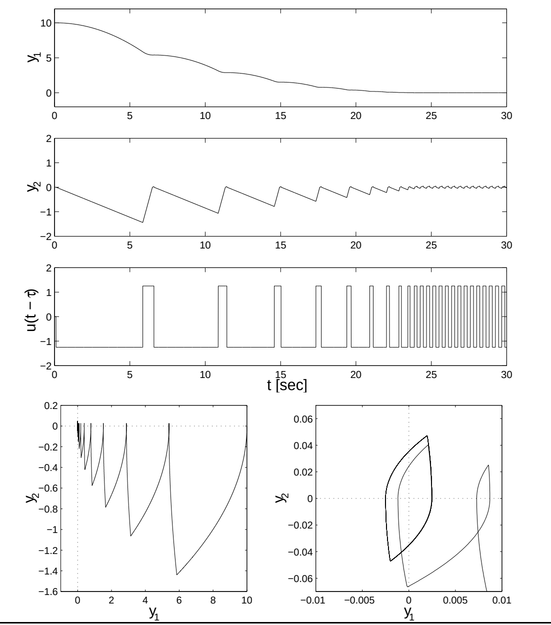

Fig. 2. The starting point is $(10, 0)$ and $u_0 = 0$. The delay is $\tau = .1$ and the control parameters are set to $U = 1.25$ and $\gamma = 0.6$. The drift modulus is $F = 1$.

and the sequence

$$|x_{k+1}| = f_r(|x_k|) = f_r^{k+1}(|x_0|), \tag{9}$$

converges to the unique fixed point of f_r

$$\bar{x}_r = \delta\,(1 - r) \frac{2 + r - \gamma + 2\sqrt{(1-\gamma)(1+r)}}{2(\gamma + r)^2}. \tag{10}$$

Therefore for k tending to infinity the sequence bounces from the point $\bar{x}_r$ to its symmetric $-\bar{x}_r$. The solution of the closed loop (6)-(2), under the constraint (7), shows a limit cycle in the phase plane which is symmetric with respect to the origin. The lengths of the intersections with the axes y_1 and y_2 are respectively

$$d_1 = U\tau^2\, g(r, \gamma), \qquad d_2 = U\tau\; 2\sqrt{2 g(r, \gamma)},$$

where

$$g(r, \gamma) = (1 - r) \frac{2 + r - \gamma + 2\sqrt{(1-\gamma)(1+r)}}{(\gamma + r)^2}.$$

It is easy to prove that for fixed $\gamma \in (0, 1)$ the function $r \mapsto g(r, \gamma)$ is decreasing, since it is the product of two non-negative decreasing functions. Therefore $g(r, \gamma) \leq g(0, \gamma)$, which means that the amplitude of the oscillations in the unperturbed case ($r = 0$) is greater than for $r = F/U \neq 0$.

Simulation results for an example are shown in Figure 2. Note that, apart from the drift term, the control parameters are chosen in accordance with the example in Figure 1. Comparisons between the two simulation results are also presented. In

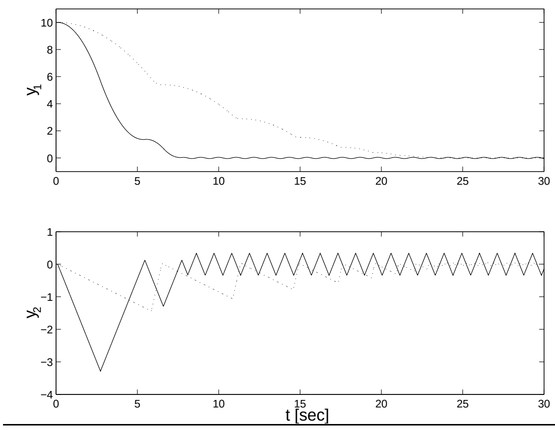

Fig. 3. Comparison between simulation results in Figure 1 (solid) and Figure 2 (dotted). The approaching of the limit cycle in the unperturbed case is faster.

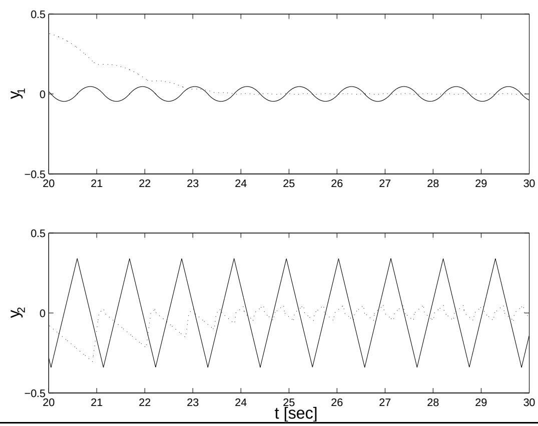

Fig. 4. Comparison between simulation results in Figure 1 (solid) and Figure 2 (dotted). The time axis has been zoomed in to show the different amplitude of the oscillations for the two cases.

Figure 3, it is possible to notice that the approaching of the limit cycle in the unperturbed case is faster, while in Figure 4 is possible to appreciate the different amplitude of the oscillations for the two cases.

4. LIMIT CYCLE REDUCTION

The analysis briefly reported in Section 2 shows that the delayed double integrator (1) cannot be stabilized by applying the control law (2), since the asymptotic behaviour of the closed loop system is a persisting oscillating one.

A control law for the regulation of system (1) that ensures asymptotic stability for any unknown input delay τ was proposed in (Levaggi and Punta, 2003*b*). In the unperturbed case the relevant parameters of the periodic motion are uniquely determined by the choice of the control modulus U and the size of the delay τ. The amplitude of the oscillations can thus be gradually decreased by a step by step reduction of the intensity of the control.

In this paper it is considered the case of a double integrator perturbed by a dissipative drift term which is bounded in modulus by a known constant. The presence of the uncertain friction-like action introduces some new elements which must be taken into account when analysing the controlled system (6)-(2). This time the control law must satisfy the constraint (7) and this fact must be considered in view of the limit cycle reduction. Moreover the periodic motion is determined not only by the choice of the control parameters U and γ and the size of the delay τ but even by the modulus F of the drift term which is uncertain. The knowledge of the bound F_M allows to decrease the amplitude of the oscillations; nevertheless the step by step reduction of the intensity of the control cannot be performed till the asymptotic stabilization. In fact because of the presence of the uncertainty, there exists a neighbourhood of the origin which is outside the set of reachable points of the closed loop system. The control algorithm used for the reduction of the limit cycle which will be here proposed, has to be compared with the one in Section 3 with respect to the time needed to reach a fixed point in the reachable set.

The control modulus $U(\theta)$ will now be a piecewise constant function of time. Fix a starting control modulus U_0 and choose $\rho \in (0,1)$. For $l > 0$, set $U_{l+1} = \rho(U_l - F_M) + F_M$. The definition could be given by performing the following steps: let $l = 0$. Then set the control modulus to U_l and keep it constant until the system is near the limit cycle. In practice, a tolerance $\varepsilon > 0$ is fixed and the sequence of singular values is checked to find a couple of consecutive points being "almost" symmetric with respect to the origin. The precision of course depends on ε. This can be done because the limit cycle is unique and thanks to the properties of the sequence of singular points showed in Theorem 3.1. In principle, once near the limit cycle, the control modulus can be reduced by setting $l = l + 1$ and recursing the procedure.

Algorithm 2:

When $t = 0$, *set* $x_0 = y_1(0)$, $k = 0$, $l = 0$, $U_l = U_0 > F_M$.

For all $t \in [0, \infty)$, *proceed as follows:*
  If $y_2(t) = 0$ *then*
    set $k = k + 1$ *and* $x_k = y_1(t)$;
    set $d = -\text{sign}\,(x_k\, x_{k+1})\,\big|\,|x_k| - |x_{k+1}|\,\big|$.
    If $d \in (0, \varepsilon)$ *then*
      set $l = l + 1$, $U_l = \rho(U_{l-1} - F_M) + F_M$.
  It is then applied the control

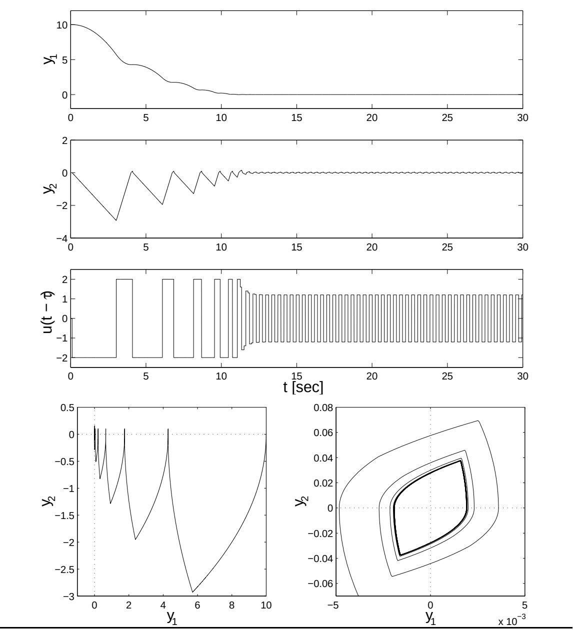

Fig. 5. The starting point is $(10, 0)$, $u_0 = 0$. The delay is $\tau = .1$ and the control parameters are set to $U_0 = 2$, $\gamma = 0.6$, $F_M = 1.2$ and $\rho = 0.5$. The drift modulus is $F = 1$.

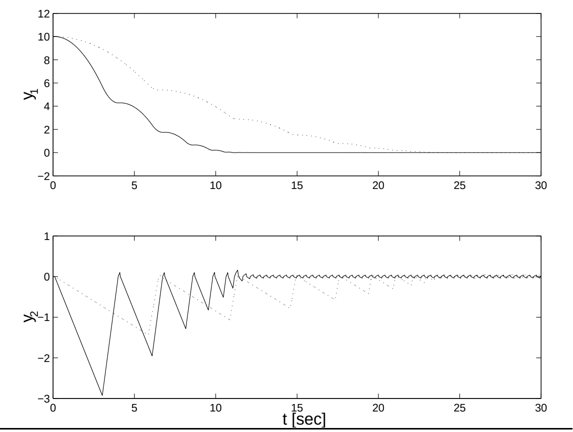

Fig. 6. Comparison between simulation results in Figure 2 (dotted) and Figure 5 (solid).

$$u(t) = -U_l \operatorname{sign}(y_1(t) - \gamma\, x_k). \tag{11}$$

In Figure 5 the simulation results obtained considering the system (6) and the control action (11) are shown. The system parameters refer to the example in Figure 2. Comparison between simulation results in Figure 2 and Figure 5 are presented in Figure 6. For the controlled system (6)-(11) the approaching time of the limit cycle is effectively reduced.

5. CONCLUSIONS

In this paper the coupled effect of an input delay τ and an uncertain dissipative disturbance on a sub-optimal second order sliding mode control is analysed. The system trajectories converge to a limit cycle, which is orbitally asymptotically stable and globally attractive. The amplitude of the oscillations of the periodic motions is shown to be smaller than in the unperturbed case. It is also proposed a modification of the sub-optimal control strategy which guarantees a step by step reduction of the size of the limit cycle, through the choice of a time-dependent, piecewise constant control modulus. Although the presence of the uncertainty prevents the asymptotic stabilization of the system, the proposed control algorithm assures a faster damping of the oscillations of both position and velocity.

REFERENCES

Bartolini, G., A. Ferrara and E. Usai (1997). Output tracking control of uncertain non-linear second order systems. *Automatica* **33**(12), 2203–2212.

Choi, S. B. and J. K. Hedrick (1996). Robust throttle control of automotive engines. *ASME J. Dynam. Systems Measurements Control* **118**, 92–98.

Fridman, E., L. Fridman and E. Shustin (2000). Steady modes in relay control systems with time delay and periodic disturbances. *ASME J. Dynam. Systems Measurements Control* **122**(4), 732–737.

Fridman, L., E. Fridman and E. Shustin (2002). Steady modes and sliding modes in relay control systems with delay. In: *Sliding Mode Control in Engineering* (W. Perruquetti (Eds.) J. P. Barbot, Ed.). Marcel Dekker. New York.

Kolmanowskii, V. B. and A. D. Myshkis (1999). *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer. Dordrecht, The Netherlands.

Levaggi, L. and E. Punta (2003*a*). On a second order discontinuous control system with delayed input. In: *Proc. 42th IEEE Conference on Decision and Control (CDC'03)*. Hawaii, USA.

Levaggi, L. and E. Punta (2003*b*). Stabilization of a second order variable structure system with unknown input delay. In: *Proc. IFAC Workshop on Time-Delay Systems (IFACTDS'03)*. INRIA, Rocquencourt, France.

Levaggi, L. and E. Punta (2004). Analysis of a second order sliding mode algorithm in presence of input delays. *Submitted for publication.*

Richard, J.-P., M. Dambrine, F. Gouaisbaut and W. Perruquetti (2000). Systems with delays: an overview of some recent advances. *SACTA* **3**(1), 3–23.

Shustin, E., E. Fridman and L. Fridman (2003). Oscillations in a second-order discontinuous system with delay. *Discrete Contin. Dyn. Syst.* **9**(2), 339–358.