MINIMIZING INTERACTION OF SUBSYSTEMS IN LARGE-SCALE INTERCONNECTED SYSTEMS USING GENERALIZED SAMPLING

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Abstract: This paper studies the application of generalized sampled-data hold functions in minimizing the strength of the interconnections between the subsystems of large-scale interconnected systems. To this end, it proposes a quadratic programming approach to the minimization of the magnitude of specific elements of the transfer function matrix of a discrete-time equivalent system. A quantitative measure for the degree of hierarchicalness of the discrete-time equivalent model is also given.

Keywords: Decentralized control, Large-scale system, Discretization, Operations research, Linear systems, Generalized Sampled-data Hold Functions.

1. INTRODUCTION

Application of generalized sampled-data hold functions (GSHF) in control was first introduced by Chammas and Leondes (Chammas and Leondes, 1979). The advantages of using GSHF's in control systems instead of conventional zero-order holds (ZOH) were investigated by Kabamba (Kabamba, 1987). The application of sampling in decentralized control systems was studied in (Ü. Özgüner and Davison, 1985). and the results obtained in (Aghdam and Davison, 1999*a*) showed that GSHF controllers can significantly improve the overall performance of certain classes of decentralized systems.

The results discussed above were all concentrated on performance improvement in control systems. A new application of GSHF's was investigated in (Aghdam and Davison, 2002), where it was shown that GSHF can be used to change the structure of the digraph of interconnected systems to simplify the control design problem, and conditions under which GSHF's can result in a hierarchical discrete-time equivalent system, were obtained. It turns out that for a system with a hierarchical structure, the decentralized control design problem simplifies, since it reduces to a centralized control design for each individual subsystem.

The results presented here are a continuation of the earlier work started in (Aghdam and Davison, 2002), which studied the conditions under which GSHF's can reduce a large decentralized control problem to several smaller centralized ones by eliminating the interaction between certain control agents. In order to cover a set of plants larger than the one defined in (Aghdam and Davison, 2002), the present development seeks to minimize the effects of the interconnections instead of completely removing them.

The paper is organized as follows. Section 2 presents the conditions under which GSHF's can minimize the

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strength of sets of interconnections to obtain an "approximate" hierarchical structure–defined in (Aghdam and Davison, 2002). The existence of global minimizers is studied in Subsection 2.1 and guidelines for an optimal rearrangement of the control agents are introduced in Subsection 2.2. Measures for interconnection strength and the degree of "hierarchicalness" appear also in Section 2. Section 3 presents numerical examples and Section 4 briefly discusses the importance of the results obtained.

2. MAIN RESULT

Consider the following strictly proper continuous-time decentralized LTI system with m control agents:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{m} \mathbf{b}_j \mathbf{u}_j(t) \quad (1a)$$

$$\mathbf{y}_j(t) = \mathbf{c}_j \mathbf{x}(t), \quad j \in \bar{\mathrm{m}} := \{1, \ldots, m\} \quad (1b)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector; $\mathbf{u}_j(t) \in \mathbb{R}^{q_j}$ and $\mathbf{y}_j(t) \in \mathbb{R}^{r_j}$ are the control input vector and output vector of agent $\#j$ respectively; and $\mathbf{A}$, $\mathbf{b}_j$, and $\mathbf{c}_j$ are matrices of appropriate dimensions. Assume that (1) is controllable and observable.

If the plant (1) is a large-scale interconnected system, the design of its decentralized controllers can be a demanding task. However, when certain conditions are satisfied, GSHF's can modify its structure so that the interactions between subsystems are eliminated or, at least, reduced. Centralized control techniques can then be applied to each subsystem to attain local control objectives. Thus, the goal now is to determine if and how much the interactions between certain control agents can be reduced by using generalized sampling.

2.1 Minimizing the strength of a set of interconnections

In (Aghdam and Davison, 2002) conditions under which the interconnections between subsystems can be completely removed using GSHF's were discussed. When these conditions are not satisfied, the question arises: is it possible to reduce the magnitude of certain interconnections by applying sampled-data hold functions? The motivation to this question is that systems with sufficiently weak interconnections often have the property that decentralized control can still effectively be applied, using centralized control design methods applied to the subsytems as illustrated in Example 1 of this paper.

For stable systems, the main results in (Aghdam and Davison, 2002) can be rewritten in terms of observability and controllability grammians. To this end, one may note that the continuous-time controllability grammians:

$$\mathbf{W}^2_{\mathrm{cc}_{i_j}} = \int_0^{\infty} \left(\mathrm{e}^{At}\,\mathbf{b}_{i_j}\right)\left(\mathrm{e}^{At}\,\mathbf{b}_{i_j}\right)' \, dt$$

give bases for the controllability subspaces of each input u_j, where $j \in \bar{\mathrm{m}}$ (defined in (1b)). Then, by applying a GSHF to the j-th control input of the system, one can design the associated j-th column of the discrete-time input matrix as a linear combination of the vectors defining the basis of the continuous-time j-th controllability subspace.

Proposition 1. If GSHF's are applied to (1), then the resulting discrete-time equivalent system, which is controllable and observable for almost all sampling periods, can be written as:

$$\mathbf{x}[k+1] = \mathbf{A}_{\mathrm{d}}\mathbf{x}[k] + \sum_{j=1}^{m} \mathbf{b}_{\mathrm{d}_j}\mathbf{u}_j[k]$$
$$\mathbf{y}_j[k] = \mathbf{c}_j\mathbf{x}[k], \quad j \in \bar{\mathrm{m}},$$

where each column of the discrete-time input matrix can be expressed as:

$$\mathbf{b}_{\mathrm{d}_j} = \mathbf{W}_{\mathrm{cc}_j}\chi_j \quad (2)$$

and $\chi_j \in \mathbb{R}^n$ is defined by the particular GSHF used.

PROOF. From Lemma 1 in (Aghdam and Davison, 1999*b*), the matrices $\mathbf{b}_{\mathrm{d}_j}$, $j \in \bar{\mathrm{m}}$, can be designed within the controllable subspaces of $(\mathbf{A}, \mathbf{b}_j)$, $j \in \bar{\mathrm{m}}$, or equivalently as a linear combination of the basis of such subspaces. Since the columns of the controllability grammians $\mathbf{W}_{\mathrm{cc}_j}$ provide bases for such subspaces, the matrices $\mathbf{b}_{\mathrm{d}_j}$ can be written as given in (2). ■

Define m distinct integers $i_1, \ldots, i_j \in \bar{\mathrm{m}}$. Let $\mathbf{C}_{\bar{\ell}_{i_j}}$ represent the matrix consisting of the $\bar{\ell}_{i_j} = \{i_1, \ldots, i_{j-1}\}$ rows of $\mathbf{C}$. In order to reduce the notational burden and ease understanding of the following developments, consider independently a given column j and a predetermined index i_j associated with it. Replace the notation $\bar{\ell}_{i_j}$ with $\bar{\ell}$ and keep in mind that, for each choice of i_j, there is a uniquely defined $\bar{\ell}$. By rewriting $\mathbf{C}_{\bar{\ell}_{i_j}}$ as $\mathbf{C}_{\bar{\ell}}$, one can now define the partial discrete-time controllability and observability grammians as:

$$\mathbf{W}^2_{\mathrm{do}_{\bar{\ell}}} = \sum_{\kappa=0}^{\infty} \left(\mathbf{C}_{\bar{\ell}}\mathbf{A}_{\mathrm{d}}^{\kappa}\right)'\left(\mathbf{C}_{\bar{\ell}}\mathbf{A}_{\mathrm{d}}^{\kappa}\right) \quad (3)$$

$$\mathbf{W}^2_{\mathrm{dc}_{i_j}} = \sum_{\kappa=0}^{\infty} \left(\mathbf{A}_{\mathrm{d}}^{\kappa}\mathbf{b}_{\mathrm{d}_{i_j}}\right)\left(\mathbf{A}_{\mathrm{d}}^{\kappa}\mathbf{b}_{\mathrm{d}_{i_j}}\right)' \quad (4)$$

respectively.

Solving the interconnection weakening problem requires the definition of a quantitative measure regarding the strength of the interconnections of the system. At this point, the notions of Hankel-norm and observability and controllability grammians are very useful because of their explicit relationship. The Hankel-norm of a stable system is given by the square root of the maximum eigenvalue of the product of the observability and controllability grammians (Glover, 1984). Since the magnitude of an eigenvalue is always less than or equal to the norm of the corresponding matrix, it turns out that the Hankel-norm of a transfer matrix is less than or equal to the square root of the norm of the product of the associated grammians. This means that the norm of the product of the grammians provides a measure of the gain, or strength, of the transfer matrix. One can then define a measure for the strength of a set of interconnections that links the set of inputs $\bar{\imath} \subset \bar{\mathrm{m}}$ to the set of outputs $\bar{\mathrm{o}} \subset \bar{\mathrm{m}}$ as:

$$\mathcal{S}_{\bar{\mathrm{o}},\bar{\imath}}(\chi_{i_j}) = \|\mathbf{W}_{\mathrm{do}_{\bar{\mathrm{o}}}}\mathbf{W}_{\mathrm{dc}_{\bar{\imath}}}(\chi_{i_j})\|_F^2, \tag{5}$$

where, from (2) and (4), $\mathbf{W}_{\mathrm{dc}_{\bar{\imath}}}(\chi_{i_j})$ is given by:

$$\mathbf{W}^2_{\mathrm{dc}_{\bar{\imath}}}(\chi_{i_j}) = \sum_{\kappa=0}^{\infty}\left(\mathbf{A}_{\mathrm{d}}^{\kappa}\mathbf{W}_{\mathrm{cc}_{\bar{\imath}}}\chi_{i_j}\right)\left(\mathbf{A}_{\mathrm{d}}^{\kappa}\mathbf{W}_{\mathrm{cc}_{\bar{\imath}}}\chi_{i_j}\right)',$$

and $\|\cdot\|_F$ denotes the Frobenius norm. Such a norm is defined for a given matrix $\mathbf{X}$ as $\|\mathbf{X}\|_F = \sqrt{\mathrm{Tr}(\mathbf{X}\mathbf{X}')}$, where $\mathrm{Tr}(\cdot)$ represents the trace operator. Then, in the particular case when $\bar{\imath} = i_j$ and $\bar{\mathrm{o}} = \bar{\ell} = \{i_1, \ldots, i_{j-1}\}$, the interconnection strength is given by:

$$\mathcal{S}_{\bar{\ell},i_j}(\chi_{i_j}) = \|\mathbf{W}_{\mathrm{d}_{\bar{\ell}},i_j}(\chi_{i_j})\|_F^2 \tag{6}$$

and $\mathbf{W}_{\mathrm{d}_{\bar{\ell}},i_j}(\chi_{i_j}) := \mathbf{W}_{\mathrm{do}_{\bar{\ell}}}\mathbf{W}_{\mathrm{dc}_{i_j}}(\chi_{i_j})$. Hence, the problem is to find vectors χ_{i_j}, $j = 2, \ldots, m$ that solve:

$$\min_{\chi'_{i_j}\chi_{i_j}=1} \mathcal{S}_{\bar{\ell},i_j}(\chi_{i_j}), \quad \chi_{i_j} \in \mathbb{R}^n. \tag{7}$$

The problem (7) has a bounded solution only if a vector $\chi_{i_j} \in \mathbb{R}^n$ exists such that $\mathrm{d}\mathcal{S}_{\bar{\ell},i_j}/\mathrm{d}\chi_{i_j} = 0$ and

$$\frac{\mathrm{d}^2\mathcal{S}_{\bar{\ell},i_j}(\chi_{i_j})}{\mathrm{d}^2\chi_{i_j}} = 2\left(\sum_{\kappa=0}^{\infty}\bar{\mathbf{A}}'_{\bar{\ell},\kappa}\bar{\mathbf{A}}_{\bar{\ell},\kappa}\right) \triangleq \mathbf{P}_{\bar{\ell},i_j}$$

is positive definite with each constant matrix $\bar{\mathbf{A}}_{\bar{\ell},\kappa}$ defined as $\bar{\mathbf{A}}_{\bar{\ell},\kappa} = \mathbf{W}_{\mathrm{do}_{\bar{\ell}}}\mathbf{A}_{\mathrm{d}}^{\kappa}\mathbf{W}_{\mathrm{cc}_{i_j}}$.

Given that the trivial solutions $\chi_{i_j} = 0$, $j \in \{2, \ldots, m\}$ must be excluded to preserve controllability, it turns out that the two conditions given above are equivalent to those stated in (Aghdam and Davison, 2002). This implies that the strength of certain interconnections cannot be minimized without affecting other interconnections in the same column of the transfer function matrix. Nevertheless, it is possible to minimize the influence of a set of interconnections while preserving that of the others. This new problem can then be formulated as a constrained optimization problem given by:

$$\begin{aligned}&\min_{\chi_{i_j}\in\mathbb{R}^n} \mathcal{S}_{\bar{\ell},i_j}(\chi_{i_j}),\\ &\text{s.t. } \mathcal{S}_{\bar{\ell}^*,i_j}(\chi_{i_j}) = \rho_{i_j},\end{aligned} \tag{8}$$

where $\rho_{i_j} \in \mathbb{R}^+$ is a design parameter, $\bar{\ell}^* = \bar{\mathrm{m}} - \bar{\ell}$, and the constraint function is given by:

$$\mathcal{S}_{\bar{\ell}^*,i_j}(\chi_{i_j}) = \|\mathbf{W}_{\mathrm{d}_{\bar{\ell}^*},i_j}\|_F^2 \tag{9}$$

and $\mathbf{W}_{\mathrm{d}_{\bar{\ell}^*},i_j} = \mathbf{W}_{\mathrm{do}_{\bar{\ell}^*}}\mathbf{W}_{\mathrm{dc}_{i_j}}(\chi_{i_j})$. In analogy to $\mathbf{C}_{\bar{\ell}}$ and (3), $\mathbf{W}_{\mathrm{do}_{\bar{\ell}^*}}$ is given by

$$\mathbf{W}^2_{\mathrm{do}_{\bar{\ell}^*}} = \sum_{\kappa=0}^{\infty}\left(\mathbf{C}_{\bar{\ell}^*}\mathbf{A}_{\mathrm{d}}^{\kappa}\right)'\left(\mathbf{C}_{\bar{\ell}^*}\mathbf{A}_{\mathrm{d}}^{\kappa}\right)$$

with $\mathbf{C}_{\bar{\ell}^*} = [\ \mathbf{c}'_{i_j}\ \ldots\ \mathbf{c}'_{i_m}\ ]'$ for $j = 2, \ldots, m$.

The existence of solutions to (8) follows from the fact that the matrix-norms $\|\mathbf{W}_{\mathrm{d}_{\bar{\mathrm{q}}},i_j}(\chi_{i_j})\|^2$, $\bar{\mathrm{q}} = \bar{\ell}, \bar{\ell}^*$ are quadratic functions and can be rewritten in the standard form $\chi'\mathbf{P}_{\bar{\mathrm{q}},i_j}\chi$ with $\mathbf{P}_{\bar{\mathrm{q}},i_j}$ square and symmetric. Therefore, they define a nonempty feasible set for any $\rho_{i_j} > 0$. The problem (8) can then be expressed as:

$$\begin{aligned}&\min_{\chi_{i_j}\in\mathbb{R}^n} \chi'_{i_j}\mathbf{P}_{\bar{\ell},i_j}\chi_{i_j}\\ &\text{s.t. } \chi'_{i_j}\mathbf{P}_{\bar{\ell}^*,i_j}\chi_{i_j} = \rho_{i_j}\end{aligned} \tag{10}$$

where $\mathbf{P}_{\bar{\ell},i_j}$ and $\mathbf{P}_{\bar{\ell}^*,i_j}$ are positive semidefinite symmetric matrices that satisfy $\ker(\mathbf{W}_{\mathrm{cc}_{i_j}}) \subset \ker(\mathbf{P}_{\bar{\ell},i_j})$ and $\ker(\mathbf{W}_{\mathrm{cc}_{i_j}}) \subset \ker(\mathbf{P}_{\bar{\ell}^*,i_j})$.

The optimization problem (10) falls into the terrain of quadratically constrained quadratic programming (QQP) and the existence of its global solution has been thoroughly studied (Moré and Sorensen, 1983; Fortin and Wolkowicz, 2003).

In the case in which $\mathbf{P}_{\bar{\ell}^*,i_j}$ is full rank and ρ_{i_j} is positive, the optimization constraint in (10) describes a nonempty ellipsoid in the χ_{i_j} coordinates and a unitary sphere in the normalized coordinates. A basic optimization result (Fraleigh and Beauregard, 1995) states that the maximum and the minimum of a quadratic function over the surface of the unitary sphere are given by the largest and the smallest eigenvalues of the objective function matrix, respectively. Therefore, the minimum of the objective function can be obtained by computing the eigenvalues of the matrix equivalent to $\mathbf{P}_{\bar{\ell},i_j}$ in the normalized coordinates.

In the case in which $\mathbf{P}_{\bar{\ell}^*,i_j}$ is not full rank, results from operations research can be used to determine the existence of a global minimizer. Accordingly, one must rewrite (8) in the standard form:

$$\begin{aligned}&\min_{\chi_{i_j}\in\mathbb{R}^n} \phi_{\bar{\ell},i_j}(\chi_{i_j}),\\ &\text{s.t. } \varphi_{\bar{\ell}^*,i_j}(\chi_{i_j}) = 0\end{aligned} \tag{11}$$

where $\phi_{\bar{\ell},i_j}(\chi_{i_j}) = \chi'_{i_j}\mathbf{P}_{\bar{\ell},i_j}\chi_{i_j}$ and $\varphi_{\bar{\ell}^*,i_j}(\chi_{i_j}) = \chi'_{i_j}\mathbf{P}_{\bar{\ell}^*,i_j}\chi_{i_j} - \rho_{i_j}$ are semi-convex because both $\mathbf{P}_{\bar{\ell},i_j}$ and $\mathbf{P}_{\bar{\ell}^*,i_j}$ are positive semidefinite. Their associated hessians,

$$\begin{aligned}\mathbf{Q}_{\bar{\ell},i_j} &= \nabla^2\phi_{\bar{\ell},i_j}(\chi_{i_j}) = 2\mathbf{P}_{\bar{\ell},i_j}\\ \mathbf{Q}_{\bar{\ell}^*,i_j} &= \nabla^2\varphi_{\bar{\ell}^*,i_j}(\chi_{i_j}) = 2\mathbf{P}_{\bar{\ell}^*,i_j},\end{aligned}$$

yield the condition:

$$w'\mathbf{Q}_{\bar{\ell}^*,i_j}w = 0 \quad \Rightarrow \quad w'\mathbf{Q}_{\bar{\ell},i_j}w > 0 \tag{12}$$

which, when satisfied for any $w \neq 0$, ensures that a global minimizer exists (Moré, 1993). An equivalent condition similar to

$$\mathrm{rank}\left[\ \mathbf{P}'_{\bar{\ell},i_j}\ \ \mathbf{P}'_{\bar{\ell}^*,i_j}\ \right]' = n \tag{13}$$

appears in (Gander, 1981). Both (12) and (13) are too restrictive since they require that $\ker(\mathbf{P}_{\bar{\ell},i_j}) \cap \ker(\mathbf{P}_{\bar{\ell}^*,i_j}) = \{0\}$, which implies that the discrete-time equivalent system is completely controllable from each input i_j ($\mathrm{rank}(\mathbf{W}_{\mathrm{dc}_{i_j}}) = n$) and that $\ker(\mathbf{W}_{\mathrm{do}_{\bar{\ell}}}) \cap$

$\ker(\mathbf{W}_{\mathrm{do}_{\bar{\ell}^*}}) = \{0\}$. The next theorem, also extracted from (Moré, 1993), provides more complex but less conservative optimality conditions.

Theorem 1. **(Moré, 1993)** Let $\phi_{\bar{\ell},i_j} : \mathbb{R}^n \to \mathbb{R}$ and $\varphi_{\bar{\ell}^*,i_j} : \mathbb{R}^n \to \mathbb{R}$ be quadratic functions defined on $\mathbb{R}^n$. Assume that the condition:

$$\min_{\chi_{i_j}\in\mathbb{R}^n} \varphi_{\bar{\ell}^*,i_j}(\chi_{i_j}) < 0 < \max_{\chi_{i_j}\in\mathbb{R}^n} \varphi_{\bar{\ell}^*,i_j}(\chi_{i_j}) \qquad (14)$$

holds and that $\nabla^2\varphi_{\bar{\ell}^*,i_j} \neq 0$. A vector $\hat{\chi}_{i_j}$ is a global minimizer of problem (11) if and only if $\varphi_{\bar{\ell}^*,i_j}(\hat{\chi}_{i_j}) = 0$ and there is a multiplier $\hat{\lambda}_j \in \mathbb{R}$ such that the Kuhn-Tucker condition

$$\nabla\phi_{\bar{\ell},i_j}(\hat{\chi}_{i_j}) + \hat{\lambda}_j \nabla\varphi_{\bar{\ell}^*,i_j}(\hat{\chi}_{i_j}) = 0 \qquad (15)$$

is satisfied with

$$\nabla^2\phi_{\bar{\ell},i_j}(\hat{\chi}_{i_j}) + \hat{\lambda}_j \nabla^2\varphi_{\bar{\ell}^*,i_j}(\hat{\chi}_{i_j}) \qquad (16)$$

positive semidefinite.

The minimum of $\varphi_{\bar{\ell}^*,i_j}(\chi_{i_j})$ is given by $-\rho_{i_j}$, which is negative by definition. Its maximum value is positive and unbounded; therefore, (14) is satisfied. In turn, the conditions (15) and (16) can be rewritten as follows:

$$\left(\mathbf{P}_{\bar{\ell},i_j} + \hat{\lambda}_j \mathbf{P}_{\bar{\ell}^*,i_j}\right)\hat{\chi}_{i_j} = 0 \qquad (17a)$$

$$\mathbf{P}_{\bar{\ell},i_j} + \hat{\lambda}_j \mathbf{P}_{\bar{\ell}^*,i_j} \geq 0 \qquad (17b)$$

$$\mathbf{W}_{\mathrm{cc}_{i_j}}\hat{\chi}_{i_j} \neq 0 \qquad (17c)$$

$$\mathbf{W}_{\mathrm{do}_{\bar{\ell}^*_j}}\hat{\chi}_{i_j} \neq 0, \qquad (17d)$$

where the last two inequalities must be imposed to prevent loss of controllability and observability. Thus, if a solution to (17) exists, a global minimizer $\hat{\chi}_{i_j}$ to (8) also exists such that no column vector $\mathbf{b}_{\mathrm{d}_j}$ is zeroed and such that the conditions of Theorems 1 and 2 in (Aghdam and Davison, 2002) are not satisfied.

Remark 1. The norm operator is defined only for stable systems and therefore the measure of strength $\mathcal{S}_{\bar{o},\bar{\imath}}(\chi_{i_j})$ is meaningful only for stable interconnections.

Notice that weak interconnections are desirable when, for example, an already stable large-scale system does not satisfy the required performance measures. If the interconnections are strong, then each control agent would affect the performance of its neighbors. In contrast, if the interconnections are weak, each local controller may not significantly affect the operation of the other agents. The system can then be discretized with a set of GSHF's that minimizes the strength of the desired interconnections in order to isolate the subsystems; thus, centralized control techniques can be applied to each control agent. This is possible because discretization does not affect the stability of the plant.

2.2 *Optimal reordering of the control agents*

The extent to which the sampled system will resemble a hierarchical system depends on the numerical properties of the plant parameters and on the way the control agents are ordered. Thus, the design procedure must identify, among the $n!$ different orders of the control agents, the most convenient one by first defining a quantitative measure of "hierarchicalness". Such a quantitative metric will then be a function of the possible orders of the control agents $\bar{\mathrm{r}} \in \mathcal{P}(\bar{\mathrm{m}})$, where $\mathcal{P}(\cdot)$ represents all the possible permutations of its argument set. For instance, the possible orders $\bar{\mathrm{r}}$ for a 3-input system are $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, and $\{3,2,1\}$. Each index i_j, $j \in \bar{\mathrm{m}}$ takes the value of the j-th element for each of these ordered sets (for example, $i_1 = 3$, $i_2 = 2$, and $i_3 = 1$ for the order $\{3,2,1\}$).

For a given order $\bar{\mathrm{r}}$, define the degree of hierarchicalness as:

$$\Psi(\bar{\mathrm{r}}) = \min_{j=2,\ldots,m} \frac{\mathcal{S}_{\bar{\ell},i_j}(\hat{\chi}_{i_j})}{\mathcal{S}_{\bar{\ell}^*,i_j}(\hat{\chi}_{i_j})}, \qquad (18)$$

where $\mathcal{S}_{\bar{\ell},i_j}(\chi_{i_j})$ and $\mathcal{S}_{\bar{\ell}^*,i_j}(\chi_{i_j})$ are defined in (6) and (9) respectively, $\ell = \{i_1,\ldots,i_{j-1}\}$, $\bar{\ell}^* = \bar{\mathrm{m}} - \bar{\ell}$, and $\hat{\chi}_{i_j}, i_j \in \bar{\mathrm{m}}$, are the solutions to the minimization problem (8) for each of the columns of the transfer function matrix. According to this metric, if a continuous-time system has a perfectly hierarchical discrete-time equivalent, its degree of hierarchicalness would be $\Psi(\bar{\mathrm{r}}) = \infty$. Otherwise, the larger $\Psi(\bar{\mathrm{r}})$ is, the closer the discrete-time equivalent system is to a hierarchical structure. The definition (18) provides clear guidelines for a control design procedure, whose first step should consist of finding the maximal degree of hierarchicalness by solving the optimization problem:

$$\hat{\Psi}(\bar{\mathrm{r}}) = \max_{\bar{\mathrm{r}}\in\mathcal{P}(\bar{\mathrm{m}})} \Psi(\bar{\mathrm{r}}). \qquad (19)$$

The best $\bar{\mathrm{r}}$ can then be obtained according to the choice of the $m-1$ constants ρ_{i_j}. Different criteria can be defined to choose the constants; for example, one can find measures of strength of the interconnections under the diagonal in each column of the transfer function matrix as if a ZOH was applied. These measures can then define values of the associated ρ_{i_j} so as to preserve the strength of the interconnections under the main diagonal of the transfer function matrix. Many other criteria can be proposed and the properties of the plant as well as the control objective will determine which one is the most suitable.

3. NUMERICAL EXAMPLES

Example 1. Consider the continuous-time system represented by the following matrices:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -0.3 \\ 0 & 1 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \quad \mathbf{C}' = \begin{bmatrix} 1 & 0 \\ -5 & 0 \\ 1 & 2 \end{bmatrix}$$

This system is stable and the roots of its characteristic polynomial are located at $\{-1 \pm 0.5477\mathrm{i}, -1\}$. Assume that the settling time must be reduced to 3sec. This can be achieved if the all the poles of the system are placed at $s = -1.5$. To this end, one can minimize $\mathcal{S}_{1,2}(\chi_{i_2})$ and place the system poles by means of centralized control techniques applied to each individual subsystem.

Let $\mathbf{W}_{\mathrm{dc}_{z1}}$ be the discrete-time controllability grammian of the pair $(\mathbf{A}, \mathbf{b}_1)$, when a ZOH is used to sample its control channel. The control channel #2 is to be sampled using a GSHF so as to weaken the interconnection between input #2 and output #1. A solution to the optimization problem (8) with $T = 0.05$sec and $\rho_2 = \|\mathbf{W}_{\mathrm{do}_2}\mathbf{W}_{\mathrm{dc}_{z1}}\|_F$ is given by $\chi_2 = [0.0277 \quad 0.0816 \quad 0.0528]'$.

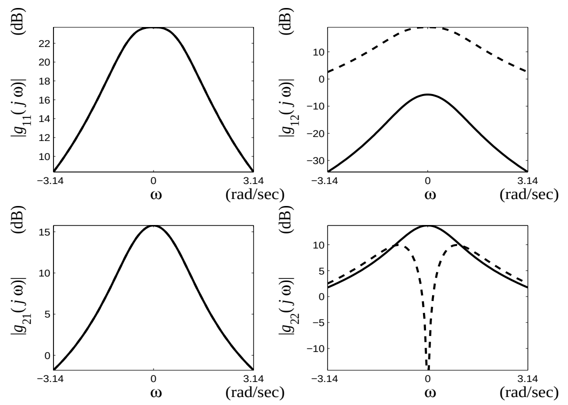

Fig. 1. Example 1: Magnitude response for each interconnection when the plant is sampled with ZOH (dashed lines) and GSHF (solid lines).

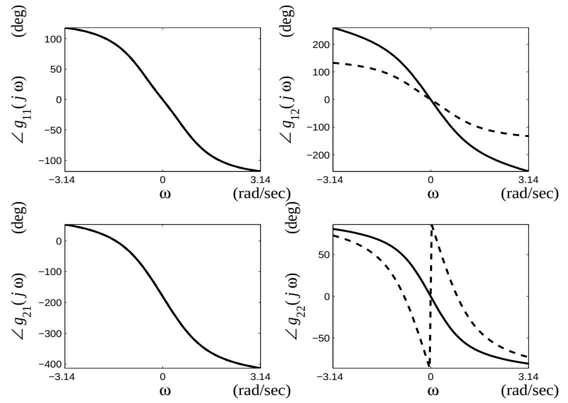

Fig. 2. Example 1: Phase response for each interconnection when the plant is sampled with ZOH (dashed lines) and GSHF's (solid lines).

The frequency response of the resulting discrete-time model appears in Figs. 1 and 2, which show that the GSHF reduces the magnitude of the response of the transfer function $g_{12}(z)$ by at least 25dB compared to the ZOH. The transfer functions of the sampled system are given by:

$$g_{11}(z) = \frac{0.2198z^2 - 0.3710z + 0.1536}{z^3 - 2.8530z^2 + 2.7138z - 0.8607}$$
$$g_{12}(z) = \frac{0.0005z^2 - 0.0013z + 0.0009}{z^3 - 2.8530z^2 + 2.7138z - 0.8607}$$
$$g_{21}(z) = \frac{0.0950z^2 - 0.1862z + 0.0903}{z^3 - 2.8530z^2 + 2.7138z - 0.8607}$$
$$g_{22}(z) = \frac{0.1938z^2 - 0.3664z + 0.1733}{z^3 - 2.8530z^2 + 2.7138z - 0.8607}.$$

Consider now the decentralized dynamic feedback:

$$\frac{\tilde{u}_1(z)}{y_1(z)} = \frac{-7.6634z^2 + 12.3234z - 5.1337}{z^3 - 0.1002z^2 - 1.4990z + 0.9154} \quad (20a)$$
$$\frac{\tilde{u}_2(z)}{y_2(z)} = \frac{-9.8539z^2 + 16.4820z - 7.1037}{z^3 - 0.1886z^2 - 1.8060z + 1.4296}, \quad (20b)$$

which was obtained through Kalman filter designs applied independently to the subsystems $g_{11}(z)$ and $g_{22}(z)$ to place their poles at $z = \mathrm{e}^{-1.5T}$. The poles of the overall discrete-time equivalent closed-loop system are given by $\{0.4661, 0.7077 \pm 0.2807\mathrm{i}, 0.8986, 0.9173 \pm 0.0317\mathrm{i}, 0.9526, 0.9400 \pm 0.0111\mathrm{i}\}$ and the time response to initial conditions $x(0) = [1 \ 1 \ 1]'$ appears in Fig. 3 (DT control), together with the response of the open-loop system. Fig. 3 also shows the response of the non-discretized system when controlled with a continuous-time design (CT control) which is equivalent to the previous one in the sense that the same weighting matrices were used for the Kalman filters and that the subsystems' poles were placed at $s = -1.5$. It is clear that the discrete-time decentralized controllers (20a) reduce the settling time to approximately 3sec while the continuous-time design failed to achieve the same results.

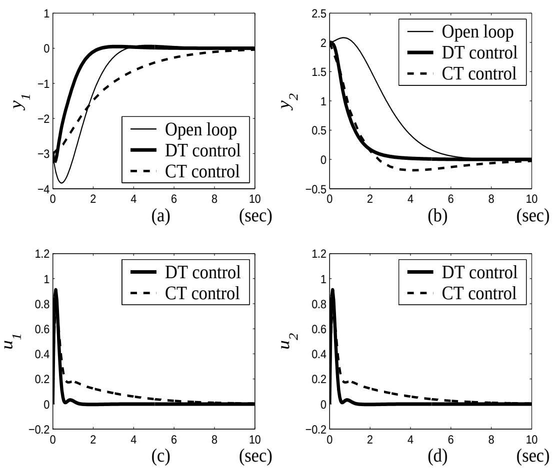

Fig. 3. Example 1: (a) output response in control agent #1; (b) output response in control agent #2; (c) control signal in control agent #1; (d) control signal in control agent #2.

Example 2. Consider a system with the following matrices:

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 0 & 0 \\ -1 & -4 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & -3 \end{bmatrix}$$

$$\mathbf{B}' = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \; \mathbf{C} = \begin{bmatrix} -4 & 0 & 2 & 0 \\ 0 & -2 & 11 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

The solution to (19) for this plant is given by the reordering sequence $i_1 = 3$, $i_2 = 1$, and $i_3 = 2$. The associated maximum degree of hierarchicalness is $\hat{\Psi}(\bar{r}) = 5.9470 \times 10^6$ and the minimizers are given by $\chi_1 = [0.0087 \quad 0.0488 \quad 0.0901 \quad -0.0269]'$ and $\chi_2 = [0.4570 \quad 0.0068 \quad 0.4700 \quad 0.1819]'$.

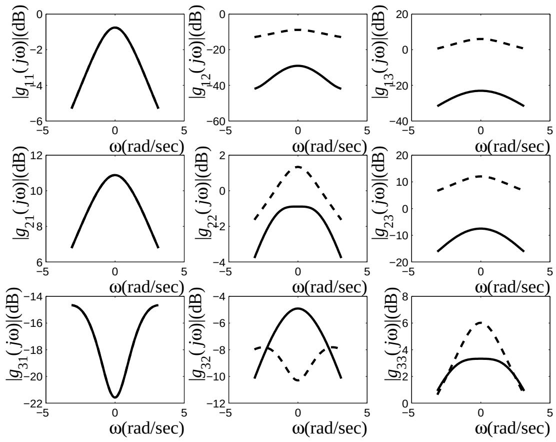

Fig. 4. Example 2: Magnitude response of each interconnection when the plant is sampled with ZOH (dashed lines) and GSHF's (solid lines).

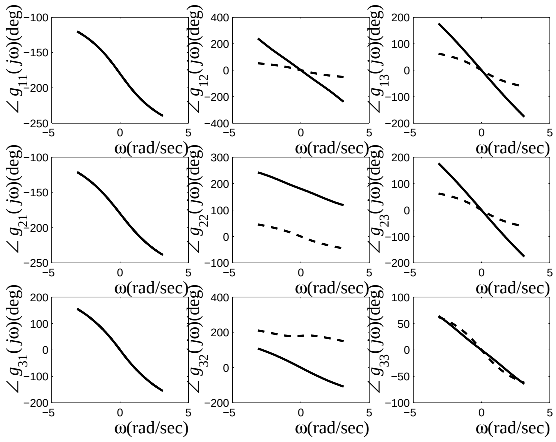

Fig. 5. Example 2: Phase response of each interconnection when the plant is sampled with ZOH (dashed lines) and GSHF's (solid lines).

The frequency response of the discrete-time transfer functions of the reordered system appears in Figs. 4 and 5. It is noticeable that the gains of the transfer functions $g_{12}(z)$, $g_{13}(z)$, and $g_{23}(z)$ of the reordered system are reduced by at least 20dB.

4. CONCLUSIONS

This paper provides conditions under which the interconnections of a large-scale system can be weakened so as to obtain a discrete-time model with a structure that approaches a hierarchical one. An application to inteconnection weakening is presented with simulations that show that such a procedure can simplify the decentralized control design of stable systems. In general, the results show that GSHF's can be used to simplify the design of decentralized controllers for multivariable systems.

REFERENCES

Aghdam, A. G. and E. J. Davison (1999*a*). Decentralized control of systems, using generalized sampled-data hold functions. In: *Proceedings of the 38'th IEEE Conference on Decision and Control*. pp. 3912–3913.

Aghdam, A. G. and E. J. Davison (1999*b*). Decentralized switching control using a family of controllers approach. In: *Proceedings of the 1999 American Control Conference*. pp. 47–52.

Aghdam, A. G. and E. J. Davison (2002). Application of generalized sampled-data hold functions to decentralized control structure modification. In: *Proceedings of 15th IFAC Triennial World Congress*. Barcelona, Spain.

Chammas, A. B. and C. T. Leondes (1979). On the finite time control of linear systems by piecewise constant output feedback. *International Journal of Control* **30**(2), 227–234.

Fortin, C. and H. Wolkowicz (2003). The trust region subproblem and semidefinite programming. *Optimization Methods and Software*. To appear in the special issue dedicated to Jochem Zowes 60th birthday, Taylor and Francis Publisher.

Fraleigh, J. B. and R. A. Beauregard (1995). *Linear Algebra*. Addison-Wesley.

Gander, Walter (1981). Least squares with a quadratic constraint. *Numerische Mathematik* **36**(3), 291–307.

Glover, K. D. (1984). All optimal Hankel-norm approximation of linear multivariable systems and their L^∞-error bounds. *Int. J. Control* **39**(6), 1115–1193.

Kabamba, P. T. (1987). Control of linear systems using generalized sampled-data hold functions. *IEEE Trans. on Automatic Control* **AC-32**(9), 772–783.

Ü. Özgüner and E. J. Davison (1985). Sampling and decentralized fixed modes. In: *Proceedings of the 1985 American Control Conference*. pp. 257–262.

Moré, J. J. (1993). Generalization of the trust region problem. *Optimization Methods and Software* **2**, 189–209.

Moré, J. J. and D. C. Sorensen (1983). Computing a trust region step. *SIAM J. Sci. Statist. Comput.* **4**, 445–572.