# UNIVERSAL OUTPUT FEEDBACK CONTROL OF NONLINEAR SYSTEMS WITH UNKNOWN GROWTH RATE

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Abstract: This paper investigates the problem of global stabilization by output feedback for a family of uncertain nonlinear systems dominated by a triangular system satisfying linear growth condition. In contrast to the previous work in the literature, the growth rate here is a positive constant but *not known a priori,* and therefore the problem cannot be addressed by the existing output feedback control schemes. Using the idea of universal control integrated with the recent output feedback design method that is not based on the separation principle, we construct a universal-type output feedback controller which globally regulates all the states of the uncertain systems without knowing the growth rate. We also discuss briefly how this result can be extended to a larger class of nonlinear systems with unknown parameters. 

Keywords: Nonlinear Systems, Universal Control, Output Feedback, Unknown Parameters.

## 1. INTRODUCTION

We consider the problem of global stabilization by output feedback for a family of single-input single-output, uncertain nonlinear systems described by equations of the form

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi_1(t,x,u)\\ \dot{x}_2 &= x_3 + \phi_2(t,x,u)\\ &\vdots\\ \dot{x}_n &= u + \phi_n(t,x,u)\\ y &= x_1\end{aligned} \tag{1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. The functions $\phi_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, are continuous with respect to all the variables. They represent the system uncertainty and need not to be precisely known. Throughout this paper, we focus our attention on a subfamily of uncertain nonlinear systems (1) characterized by the following linear growth condition.

*Assumption 1.1.* For $i = 1, \cdots, n$, there is an *unknown constant* $c \geq 0$ such that

$$|\phi_i(t,x,u)| \leq c(|x_1| + \cdots + |x_i|). \tag{2}$$

In the case when the growth rate $c$ is known, Assumption 1.1 reduces to the condition introduced in (Tsinias , 1991), where a linear state feedback control law was designed achieving global exponential stabilization of (1). Under the exactly same condition, it was shown in (Qian and Lin, 2002) that global exponential stabilization of a family of uncertain nonlinear systems (1) is also possible by a linear dynamic output compensator. The linear output feedback controller was constructed by using the traditional high-gain observer (Khalil and Saberi, 1987; Gauthier *et al.*, 1992) together with a coupled observer-controller design method (Qian and Lin, 2002), which is not based on the separation principle. A nice feature of such an output feedback design is that no precise knowledge of the system uncertainty is required. What really needed is the information of the bounding system. That is, the growth rate $c$ in Assumption 1.1 must be a known constant.

When the parameter $c$ in (2) is an *unknown constant*, global output feedback control of the uncertain system (1) becomes much more involved due to the lack of effective adaptive observer design techniques. In the existing literature, important questions such as how to design nonlinear adaptive observers and how to achieve global adaptive stabilization via output feedback have been investigated, for

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instance, in (Krstić *et al.*, 1995; Marino and Tomei, 1995) as well as the references therein. Unfortunately, most of the results are only applicable to a class of uncertain nonlinear systems in the parametric output feedback form (Krstić *et al.*, 1995; Marino and Tomei, 1995)(i.e., the system uncertainty $\phi_i(t,x,u)$ in (1) is dominated by $cb(y)$, where $b(y)$ is a known smooth function but $c > 0$ is an unknown constant). They cannot, however, be employed to control nonlinear systems with unknown parameters beyond the parametric output feedback form, such as the uncertain system (1) satisfying Assumption 1.1, in which the unknown parameters appear not only in the front of the system output but also in the front of the unmeasurable states $(x_2, \cdots, x_n)$. The latter prevents one to design a conventional observer (Isidori, 1995; Krener and Respondek, 1985; Xia and Gao, 1989) that is often composed of a copy of the original system. Indeed, such an observer contains the system uncertainty and hence is not implementable. The essential difficulty of this kind makes global adaptive stabilization of the uncertain system (1) by output feedback non-trivial, even under the linear growth condition (2).

To control the uncertain system (1) satisfying (2) by output feedback, we first proposed in the work (Qian and Lin, 2003) a *time-varying* linear output feedback control scheme. The key idea behind (Qian and Lin, 2003) was to employ a linear observer with time-varying gains, integrated with a time-varying state feedback controller, so that the unknown parameter of (1) that is related to the unmeasurable states can be dominated. Moreover, global state regulation of (1) and global boundedness of the closed-loop system can be achieved. Although the result presented in (Qian and Lin, 2003) is mathematically sound, it is, however, not easy to be implemented from a practical point of view, due to the use of time-varying gains in both the observer and controller (Qian and Lin, 2003). Therefore, a theoretically sound and practically feasible output feedback control strategy yet needs to be developed for the uncertain system (1) under Assumption 1.1.

This issue will be addressed in this paper. The main contribution of the paper is to provide a new output feedback control scheme which is *time-invariant in nature.* Instead of using a time-varying strategy (Qian and Lin, 2003) to handle the unknown parameter $c$ in (2), we shall design a universal-type, adaptive output feedback controller which globally regulates the states of the uncertain system (1) while keeping boundedness of all the signals.

To be precise, we shall show that under the linear growth condition (2) with the unknown growth rate $c$, there exists a smooth dynamic output compensator of the form

$$\begin{aligned} \dot{z} &= f(z, L, y) \\ \dot{L} &= h(z, L, y) \\ u &= g(z, L, y) \end{aligned} \tag{3}$$

such that all the solutions of the closed-loop system (1)-(3) are well-defined and globally bounded on $[0, +\infty)$. Moreover,

$$\lim_{t \to +\infty} (x(t), z(t)) = (0, 0), \qquad \lim_{t \to +\infty} L(t) = \bar{L} \in R_+.$$

For the sake of convenience, we refer such an adaptive control problem, with a bit abuse of terminology, as *global adaptive regulation by output feedback.*

In the next section, an adaptive output feedback controller of the form (3) that does the job will be explicitly constructed, using a synthesis of ideas and techniques drawn from the theory of *universal control* (Willems and Byrnes, 1984) and from the non-separation principle based output feedback control scheme (Qian and Lin, 2002; Qian and Lin, 2003). There are two new ingredients in the proposed universal-like output feedback controller. The first one is the use of a dynamic gain, rather than a time-varying gain (Qian and Lin, 2003), in the linear high-gain observer. The observer gain is updated, in a "universal" manner (Willems and Byrnes, 1984), by an error signal between the system output and its estimate. This gain update law bears a strong resemblance to the adapted high-gain controller which is commonly employed in the literature of universal control (Willems and Byrnes, 1984). It turns out, as one might expect, that the introduction of a dynamic gain makes our universal-type output feedback controller capable of handling the entire family of nonlinear systems (1) characterized by Assumption 1.1, without knowing the growth rate $c$. The other ingredient is the development of a simpler output feedback design method than the one in (Qian and Lin, 2002), which needs not to go through the recursive design procedure as suggested in (Qian and Lin, 2002). The new output feedback control algorithm simplifies significantly the analysis and synthesis of the universal adaptive output feedback controller as well as the resulted closed-loop system.

## 2. UNIVERSAL OUTPUT FEEDBACK CONTROL

In this section, we prove that without knowing the growth rate $c$ in (2), it is still possible to globally regulate the whole family of uncertain systems (1) by a universal-type output feedback controller. Formally, the main result of this paper can be summarized in the following statement.

*Theorem 2.1.* Under Assumption 1.1, the problem of global adaptive regulation for the uncertain nonlinear system (1) is solvable by a universal output feedback controller of the form (3).

**Proof.** Motivated by the work (Qian and Lin, 2002; Qian and Lin, 2003), we begin by designing a high-gain observer (Khalil and Saberi, 1987; Gauthier *et al.*, 1992) for the uncertain nonlinear system (1) (regardless of the uncertain terms $\phi_i(t,x,u), 1 \le i \le n$)

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + L^2 a_2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + L^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{4}$$

where $L \ge 1$ is a *dynamic gain* to be determined later on, and $a_i > 0,\ i = 1, \cdots, n$ are the coefficients of the Hurwitz polynomial $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Let $e_i = x_i - \hat{x}_i$ be the estimate error. Then, the error dynamics is given by

$$\begin{aligned} \dot{e}_1 &= e_2 - La_1 e_1 + \phi_1(t, x, u) \\ \dot{e}_2 &= e_3 - L^2 a_2 e_1 + \phi_2(t, x, u) \\ &\vdots \\ \dot{e}_n &= -L^n a_n e_1 + \phi_n(t, x, u). \end{aligned} \tag{5}$$

To simplify the analysis and design, we introduce the following rescaling transformation $(i = 1, \cdots, n)$

$$\varepsilon_i = \frac{e_i}{L^i}, \qquad z_i = \frac{\hat{x}_i}{L^i} \qquad \text{and} \qquad v = \frac{u}{L^{n+1}}. \tag{6}$$

With the help of (6), it is easy to see that the composite system (4)-(5) can be represented in a compact form:

$$\begin{aligned} \dot{\varepsilon} &= LA\varepsilon + \Phi(t, x, u, L) - \frac{\dot{L}}{L} D\varepsilon \\ \dot{z} &= L(A_0 z + b_0 v) + La\varepsilon_1 - \frac{\dot{L}}{L} Dz \end{aligned} \tag{7}$$

where

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}$$

$$\Phi(\cdot) = \begin{bmatrix} \frac{1}{L}\phi_1(t,x,u) \\ \frac{1}{L^2}\phi_2(t,x,u) \\ \vdots \\ \frac{1}{L^n}\phi_n(t,x,u) \end{bmatrix}, \; A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix}$$

and

$$A_0 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \; b_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; a = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}$$

Since the pair $(A_0, b_0)$ is controllable, there exists a real constant matrix $K = -[k_1 \; k_2 \; \cdots \; k_n]$ such that all the eigenvalues of the matrix $B := A_0 + b_0 K$ are located on the open left-half plane.

In view of the discussion above, it is deduced from (7) that the linear feedback control law

$$v = Kz = -[k_1 z_1 + k_2 z_2 + \cdots + k_n z_n]$$

or, equivalently, the controller

$$u = -L^{n+1}[k_1 z_1 + k_2 z_2 + \cdots + k_n z_n] \tag{8}$$

results in the closed-loop system

$$\begin{aligned} \dot{\varepsilon} &= LA\varepsilon + \Phi(t,x,u,L) - \frac{\dot{L}}{L}D\varepsilon \\ \dot{z} &= LBz + La\varepsilon_1 - \frac{\dot{L}}{L}Dz. \end{aligned} \tag{9}$$

By construction, both $A$ and $B$ are Hurwitz matrices. According to (Krishnamurthy and Khorrami, 2002), there exist $P = P^T$ and $Q = Q^T$, which are positive definite, such that

$$\begin{aligned} A^TP + PA \le -I \quad &\text{and} \quad DP + PD \ge 0, \\ B^TQ + QB \le -2I \quad &\text{and} \quad DQ + QD \ge 0. \end{aligned} \tag{10}$$

With this in mind, we choose the Lyapunov function

$$V(\varepsilon, z) = (m_1 + 1)V_1(\varepsilon) + V_2(z) \tag{11}$$

for the closed-loop system (9), where $m_1 = ||Q||^2||a||^2$ and

$$V_1(\varepsilon) = \varepsilon^T P\varepsilon \qquad \text{and} \qquad V_2(z) = z^TQz. \tag{12}$$

Then, a simple calculation yields

$$\begin{aligned} \dot{V} \le& -L(m_1+1)||\varepsilon||^2 - (m_1+1)\frac{\dot{L}}{L}\varepsilon^T(DP+PD)\varepsilon \\ &-2L||z||^2 - \frac{\dot{L}}{L}z^T(DQ+QD)z \\ &+2(m_1+1)\varepsilon^TP\Phi(\cdot) + 2L\varepsilon_1 z^TQa. \end{aligned}$$

Inspired by the design of universal controllers (Willems and Byrnes, 1984), we design the following gain update law

$$\dot{L} = \varepsilon_1^2 = \frac{(x_1 - \hat{x}_1)^2}{L^2} \qquad \text{with} \qquad L(0) = 1. \tag{13}$$

Using the inequality (10) and the fact that $L(t) \ge 1 \; \forall t \ge 0$, we immediately arrive at

$$\begin{aligned} \dot{V} \le& -L(m_1+1)||\varepsilon||^2 - 2L||z||^2 \\ &+2(m_1+1)\varepsilon^TP\Phi(\cdot) + 2L\varepsilon_1 z^TQa. \end{aligned} \tag{14}$$

Next, we estimate the last two terms on the right-hand side of the inequality above. It is not difficult to prove that

$$|2L\varepsilon_1 z^TQa| \le L||z||^2 + L||Q||^2||a||^2\varepsilon_1^2. \tag{15}$$

$$|2(m_1+1)\varepsilon^TP\Phi| \le c\; m(m_1+1)(||z||^2 + ||\varepsilon||^2) \tag{16}$$

where $m = 3n\|P\|$.

Substituting the estimations (15) and (16) into (14), we have

$$\dot{V} \;\le\; -\Big[L - cm(m_1+1)\Big]\Big(||\varepsilon||^2 + ||z||^2\Big). \tag{17}$$

With the aid of (17), it is not difficult to prove that Theorem 2.1 holds. That is, starting from any initial condition $(\varepsilon(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ and $L(0) = 1$, the closed-loop system (9)-(13) has the properties:

- **(i)** All the states of (9)-(13) are well defined and globally bounded on $[0, +\infty)$;
- **(ii)** Moreover,

$$\lim_{t\to+\infty}(z(t), \varepsilon(t)) = (0,0), \quad \lim_{t\to+\infty} L(t) = \bar{L} \in R_+.$$

In what follows, we shall prove the conclusions (i) and (ii) via a contradiction argument.

To begin with, we suppose the closed-loop system (9)-(13) has a solution $(L(t), \varepsilon(t), z(t))$ that is not well defined nor globally bounded on $[0, +\infty)$. Then, there exists a maximal time interval $[0, T)$ on which $(L(t), \varepsilon(t), z(t))$ are well defined. Furthermore,

$$\lim_{t\to T} ||(L(t), \varepsilon(t), z(t))^T|| = +\infty.$$

In other words, $T > 0$ is a finite escape time of the closed-loop system (9)-(13).

We first claim that $L(t)$ cannot escape at $t = T$. To prove this claim, suppose that $\lim_{t\to T} L(t) = +\infty$. Since $\dot{L} = \varepsilon_1^2 \ge 0$, $L(t)$ is a monotone nondecreasing function. Thus, there exists a finite time $t^* > 0$, such that

$$L(t) \ge c\; m(m_1+1) + 1, \quad \text{when} \quad t^* \le t < T.$$

From (17) it follows that

$$\dot{V}(\eta(t)) \le -||\eta(t)||^2, \qquad \forall t \in [t^*, T)$$

where $\eta(t) = (\varepsilon(t), z(t))^T$.

As a consequence,

$$\int_{t^*}^{T} \varepsilon_1^2 dt \le \int_{t^*}^{T} ||\eta(t)||^2 dt \le V(\eta(t^*)) = \text{constant}. \tag{18}$$

Using (18), one has

$$\begin{aligned} +\infty = L(T) - L(t^*) = \int_{t^*}^{T} \dot{L}(t)dt = \int_{t^*}^{T} \varepsilon_1^2(t)dt \\ \le V(\eta(t^*)) = \text{constant}, \end{aligned}$$

which leads to a contradiction. Therefore, the dynamic gain $L(t)$ is well defined and bounded on $[0, T]$. From $\dot{L} = \varepsilon_1^2$, it is concluded that $\int_0^T \varepsilon_1^2 dt$ is bounded as well.

Next, we claim that $z(t)$ is well defined and bounded on the interval $[0, T]$. To see why, consider the Lyapunov function $V_2(z) = z^TQz$ for the $z$-dynamic system of (9). Clearly, a direct computation gives

$$\dot{V}_2(z) \le -L||z||^2 + m_1 L\varepsilon_1^2 \le -||z||^2 + m_1 L\dot{L}$$

This, in turn, leads to

$$\lambda_{\min}(Q)||z(t)||^2 - z(0)^T Qz(0) \le V_2(z(t)) - V_2(z(0))$$
$$\le -\int_0^t ||z(t)||^2 dt + \frac{m_1}{2}[L^2(t) - 1], \quad (19)$$

from which it follows that

$$||z(t)||^2 \le \frac{1}{\lambda_{\min}(Q)}\Big(z(0)^T Qz(0) + \frac{m_1}{2}[L^2(t) - 1]\Big).$$

Since $L(t)$ is bounded on $[0, T]$, the inequality above implies boundedness of $z(t)$ over $[0, T]$. Consequently, $\int_0^t ||z(t)||^2 dt$ is also bounded $\forall t \in [0, T]$.

Finally, we prove that $\varepsilon(t)$ is bounded on $[0, T]$. To this end, we introduce the change of coordinates

$$\xi_i = \frac{e_i}{(L^*)^i}, \qquad i = 1, 2, \cdots, n \quad (20)$$

where $L^*$ is a constant satisfying

$$L^* \ge \max\{L(T), 3cn||P|| + 3\}.$$

Then, the error dynamics (5) is transformed into

$$\dot{\xi}_1 = L^*\xi_2 - L^* a_1\xi_1 + L^* a_1\xi_1 - La_1\xi_1 + \frac{\phi_1(\cdot)}{L^*}$$
$$\dot{\xi}_2 = L^*\xi_3 - L^* a_2\xi_1 + L^* a_2\xi_1 - L\frac{L}{L^*}a_2\xi_1 + \frac{\phi_2(\cdot)}{(L^*)^2}$$
$$\vdots$$
$$\dot{\xi}_n = -L^* a_n\xi_1 + L^* a_n\xi_1 - L(\frac{L}{L^*})^{n-1} a_n\xi_1 + \frac{\phi_n(\cdot)}{(L^*)^n}$$

which can be written in the following compact form

$$\dot{\xi} = L^* A\xi + L^* a\xi_1 - L\Gamma a\xi_1 + \Phi^*(\cdot) \quad (21)$$

where

$$\Gamma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{L}{L^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\frac{L}{L^*})^{n-1} \end{bmatrix}, \quad \Phi^*(\cdot) = \begin{bmatrix} \frac{\phi_1(t,x,u)}{L^*} \\ \frac{\phi_2(t,x,u)}{(L^*)^2} \\ \vdots \\ \frac{\phi_n(t,x,u)}{(L^*)^n} \end{bmatrix}.$$

Now, consider the Lyapunov function $V_3(\xi) = \xi^T P\xi$ for system (21). A straightforward calculation shows that along the trajectories of (21),

$$\dot{V}_3 \le -L^*||\xi||^2 + 2\xi_1 L^* a^T P\xi - 2\xi_1 La^T\Gamma P\xi + 2\Phi^{*T}(\cdot)P\xi.$$

Observe that

$$\Big|2\xi_1 L^* a^T P\xi\Big| \le L^{*2}||a^T P||^2\xi_1^2 + ||\xi||^2$$
$$\Big|2\xi_1 La^T\Gamma P\xi\Big| \le L^2||a^T\Gamma P||^2\xi_1^2 + ||\xi||^2.$$

Moreover, using (6), (20) and Assumption 1.1 yields

$$\Big|\frac{\phi_i(\cdot)}{(L^*)^i}\Big| \le c\sqrt{n}(||z|| + ||\xi||)$$
$$\Big|2\Phi^{*T}(\cdot)P\xi\Big| \le 3cn||P||(||z||^2 + ||\xi||^2).$$

In view of the estimations above, we have

$$\dot{V}_3 \le -(L^* - 3cn||P|| - 2)||\xi||^2 + 3cn||P|| \cdot ||z||^2$$
$$+\Big((L^*)^2||a^T P||^2 + L^2||a^T\Gamma P||^2\Big)\xi_1^2$$
$$\le -||\xi||^2 + \mu||z||^2 + \mu\varepsilon_1^2 \quad (22)$$

where $\mu > 0$ is a suitable constant depending on the unknown parameter $c$.

From (22) it follows that

$$\lambda_{\min}(P)||\xi(t)||^2 - \xi(0)^T P\xi(0)$$
$$\le V_3(\xi(t)) - V_3(\xi(0)) \quad (23)$$
$$\le -\int_0^t ||\xi(t)||^2 dt + \mu\int_0^t ||z||^2 dt + \mu\int_0^t \varepsilon_1^2 dt.$$

Since $\int_0^T ||z||^2 dt$ and $\int_0^T \varepsilon_1^2 dt$ are bounded, it is concluded immediately from (23) that both $\int_0^t ||\xi||^2 dt$ and $\xi(t)$ are bounded on $[0, T]$. This, in view of (20) and (6), results in the boundedness of $\int_0^t ||\varepsilon||^2 dt$ and $\varepsilon(t)$ $\forall t \in [0, T]$.

In summary, we have shown that $L(t)$, $\varepsilon(t)$ and $z(t)$ are well-defined and all bounded on $[0, T]$. The conclusion is certainly contradictory to the assumption that $\lim_{t\to T}||(L(t), \varepsilon(t), z(t))^T|| = +\infty$. Therefore, all the states of the closed-loop system must be well defined on the time interval $[0, +\infty)$ and globally bounded.

Using the boundedness of $\int_0^\infty ||z||^2 dt$, $\int_0^\infty ||\varepsilon||^2 dt$ and $(L(t), \varepsilon(t), z(t))$ on $[0, +\infty)$, it is straightforward to deduce that $\varepsilon \in L_2$, $\dot{\varepsilon} \in L_\infty$ and $z \in L_2$, $\dot{z} \in L_\infty$. By the well-known Barbalat's Lemma,

$$\lim_{t\to+\infty} z(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \varepsilon(t) = 0.$$

We conclude this section with two examples that illustrate how Theorem 2.1 can be employed to solve the problem of global regulation by output feedback for a class of nonlinear systems with the *unknown* linear growth rate.

*Example 2.2.* Consider the SISO nonlinearly parameterized system

$$\dot{x}_1 = x_2 + \frac{x_1}{(1 - c_1x_2)^2 + x_2^2}$$
$$\dot{x}_2 = u + \ln(1 + (x_2^2)^{c_2})$$
$$y = x_1 \quad (24)$$

where $c_1$ and $c_2 \ge 1$ are *unknown* constants.

It is worth pointing out that (24) is neither in a triangular form nor in the parametric output feedback form. Thus, the problem of global adaptive regulation by output feedback is not solvable by existing design methods. However, a simple analysis indicates that the uncertain system (24) does satisfy Assumption 1.1. As a matter of fact, the following estimations

$$\left|\frac{x_1}{(1 - c_1x_2)^2 + x_2^2}\right| \le (1 + c_1^2)|x_1|$$
$$\Big|\ln(1 + (x_2^2)^{c_2})\Big| \le (2c_2 - 1)|x_2|$$

can be easily obtained. Hence, Assumption 1.1 holds with $c = \max\{2c_2 - 1,\ 1 + c_1^2\}$, where $c$ is an unknown constant. By Theorem 2.1, there exists a universal output feedback controller of the form (4)-(6)-(8)-(13) such that all the states of the nonlinearly parameterized system (24) are globally regulated.

Following the design procedure in the last section, one can find that the universal output feedback controller (4), (8) and (13) with $(a_1, a_2) = (1, 1)$, $(k_1, k_2) = (1, 1)$ does the job.

A numerical simulation is given in Fig. 1, illustrating the effectiveness of the universal output feedback controller. The simulation was carried out with the system parameters $c_1 = 1$ and $c_2 = 2$. The initial condition is $(x_1(0), x_2(0)) = (1, 5)$ and $(\hat{x}_1(0), \hat{x}_2(0)) = (-10, -2)$.

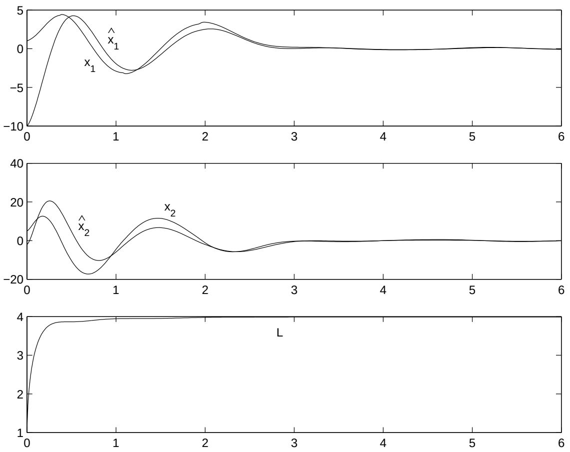


Fig. 1. The transient responses of the closed-loop system (24)-(4)-(8)-(13)

*Example 2.3.* A single-link robot arm system can be modelled by (see, for instance, (Isidori, 1995) and (Marino and Tomei, 1995))

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \frac{K}{J_2 N} z_3 - \frac{F_2(t)}{J_2} z_2 - \frac{K}{J_2} z_1 - \frac{mgd}{J_2} \cos z_1 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= \frac{1}{J_1} u + \frac{K}{J_1 N} z_1 - \frac{K}{J_2 N} z_3 - \frac{F_1(t)}{J_1} z_4 \\
y &= z_1 \qquad (25)
\end{aligned}$$

where $J_1,\ J_2,\ K,\ N,\ m,\ g,\ d$ are known parameters, while $F_1(t)$ and $F_2(t)$ are viscous friction coefficients that are not precisely known. Suppose $F_1(t)$ and $F_2(t)$ are bounded by an *unknown* constant, say $C > 0$. Our control goal is to adaptively stabilize the equilibrium $(z_1, z_2, z_3, z_4) = (0, 0, \frac{mgdN}{K}, 0)$ by output feedback, i.e., using only the output signal $z_1$, which represents the link displacement of system (25).

To design an adaptive output feedback controller, we first introduce the change of coordinates

$$x_1 = z_1,\ x_2 = z_2,\ x_3 = \frac{K}{J_2 N} z_3 - \frac{mgd}{J_2},\ x_4 = \frac{K}{J_2 N} z_4,$$

and the pre-feedback

$$v = \frac{K}{J_2 N}\left(\frac{1}{J_1} u - \frac{mgd}{J_2}\right),$$

which transform (25) into

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 - \frac{F_2(t)}{J_2} x_2 - \frac{K}{J_2} x_1 - \frac{mgd}{J_2}(\cos x_1 - 1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v + \frac{K^2}{J_1 J_2 N^2} x_1 - \frac{K}{J_2 N} x_3 - \frac{F_1(t)}{J_1} x_4 \\
y &= x_1. \qquad (26)
\end{aligned}$$

The robot arm system is in a lower-triangular form but not in the so-called parametric output feedback form. As a result, most of the adaptive output feedback control algorithms, including those proposed in (Krstić *et al.*, 1995; Marino and Tomei, 1995; Krishnamurthy and Khorrami, 2002) cannot be applied to (26).

On the other hand, it is easy to see that

$$|\cos x_1 - 1| \le |x_1|,\ \left|\frac{F_2(t)}{J_2} x_2\right| \le c|x_2|,\ \left|\frac{F_1(t)}{J_1} x_4\right| \le c|x_4|$$

where $c = \max\{\frac{C}{J_2}, \frac{C}{J_1}\}$ is an *unknown* constant. Hence, system (26) satisfies Assumption 1.1. Using Theorem 2.1, one can design a universal output feedback controller of the form (4)-(6)-(8)-(13), achieving global adaptive regulation for the single-link robot arm system (26) in the presence of the *unknown* bound $c$.

The simulation result shown in Fig. 2 for the robot arm system (25) was obtained under the following system parameters: $K/J_2 = 5, mgd/J_2 = 4, K^2/(J_1 J_2 N^2) = 2, K/(J_2 N) = 3$. The initial conditions of the whole system are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-5, -1, 4, 2)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0)) = (5, 3, -1, -4)$. The universal controller used in the simulation is composed of the high-gain observer (4) with $a = [4\ \ 6\ \ 4\ \ 1]^T$, the adaptive updated law (13) and the controller (8) with $K = -[40\ \ 78\ \ 49\ \ 12]$.

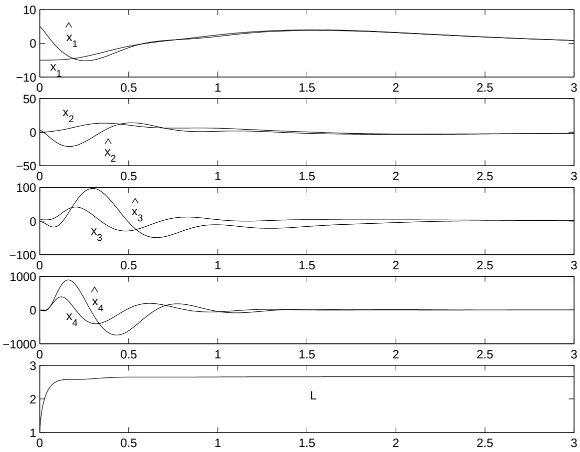


Fig. 2. Transient response of the robot arm system when $c = 10$

## 3. A FURTHER EXTENSION

So far, we have investigated the problem of globally regulating all the states of systems (1) via output feedback under Assumption 1.1. The purpose of this section is to discuss briefly how Theorem 2.1 can be generalized to a wider class of nonlinear systems with unknown parameters under a weaker condition than Assumption 1.1.

In the recent papers (Yang and Lin, 2004; Krishnamurthy and Khorrami, 2004), it has been shown that global stabilization of system (1) by output feedback is still possible even if $c$ in Assumption 1.1 is replaced by a continuous function $C(y) \ge 0$, while in (Praly and Jiang, 2004) it was shown that under an extra requirement that $C(y)$ be a *polynomial* function of $y$ with a fixed order, global stabilization of (1) can even be achieved by a linear-like output feedback controller.

In what follows, we shall show that by taking advantage of the linear structure of the output feedback controller proposed in (Praly and Jiang, 2004), it is possible to extend Theorem 2.1 to a class of nonlinear systems characterized by the following condition.

*Assumption 3.1.* For $i = 1, \cdots, n$, there is an *unknown constant* $c \ge 0$ and a known integer $p$, such that

$$|\phi_i(t, x, u)| \le c(1 + |x_1|^p)(|x_1| + \cdots + |x_i|). \qquad (27)$$

Assumption 3.1 requires that the bounding system of (1) be linear in its unmeasurable states $x_2, \cdots, x_n$ but can be a polynomial function of $y$, whose coefficients are unknown and bounded by an unknown constant $c \ge 0$. The main result of this section is the following theorem that provides an explicit design of universal output feedback controllers for the uncertain system (1).

*Theorem 3.2.* Under Assumption 3.1, the problem of global adaptive stabilization of the uncertain nonlinear system (1) is solvable by a universal output feedback controller of the form

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + LMa_1(x_1 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= \hat{x}_3 + (LM)^2 a_2(x_1 - \hat{x}_1)\\ &\vdots\\ \dot{\hat{x}}_n &= u + (LM)^n a_n(x_1 - \hat{x}_1)\end{aligned} \tag{28}$$

$$u = -[(LM)^n k_1\hat{x}_1 + \cdots + LMk_n\hat{x}_n] \tag{29}$$

with the gain update law

$$\begin{aligned}\dot{M} &= -\alpha M + \Delta(y) \qquad \text{with} \qquad M(0) = 1\\ \dot{L} &= M^{1-2r}\frac{(x_1 - \hat{x}_1)^2}{L^{2r}} \qquad \text{with} \quad L(0) = 1\end{aligned} \tag{30}$$

where $a_i$ and $k_i$, $i = 1, \cdots, n$ are the coefficients of the Hurwitz matrices $A$ and $B$, respectively (defined in section 2), $\alpha > 0$ is a suitable constant, $\Delta(y) \geq \alpha$ is a suitable continuous function that can be explicitly constructed, and $r$ is a constant satisfying $0 < r < \frac{1}{2p}$.

Due to the limit of the space, the proof of Theorem 3.2 is omitted here. The reader is referred to the paper (Lei and Lin, 2005) for further technical details.

We conclude this section by using an example from (Krishnamurthy and Khorrami, 2004), which was listed as an unsolved problem, to demonstrate the significance of Theorem 3.2.

*Example 3.3.* Consider the adaptive output feedback stabilization of the uncertain nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u + \theta x_1^2 x_3\\ y &= x_1.\end{aligned} \tag{31}$$

where $\theta$ is an *unknown* constant.

As mentioned in (Krishnamurthy and Khorrami, 2004), adaptive control of (31) by output feedback cannot be solved by the existing methods in the literature. This problem can now be solved by Theorem 3.2. In fact, it is easy to check that Assumption 3.1 holds with $p = 2$. Thus, a universal output feedback controller of the form (28)-(29)-(30) can be constructed achieving global state regulation of system (31). The details and simulations can be found in (Lei and Lin, 2005).

## 4. CONCLUSION

By integrating the idea of universal control (Willems and Byrnes, 1984) and the robust output feedback design method (Qian and Lin, 2002; Qian and Lin, 2003), we have explicitly constructed a universal-type output feedback controller that achieves global state regulation, for a family of uncertain nonlinear systems whose global adaptive stabilization problem by output feedback has remained unsolved until now. Our universal output feedback control law would simultaneously regulate a whole family of nonlinear systems with unknown parameters, as long as they are dominated by a linearly growing triangular system (Assumption 1.1) or by a bounding system that depends on the unmeasured state linearly and the output polynominally (Assumption 3.1), with an unknown growth rate. It was demonstrated, by means of examples and simulation, that the proposed adaptive output feedback controller can be easily designed and implemented.

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