FIRST AND SECOND ORDER SLIDING MODE REGULARIZATION TECHNIQUES: THE APPROXIMABILITY PROPERTY

G. Bartolini * **E. Punta** ** **T. Zolezzi** ***

* *DIEE - University of Cagliari*
Piazza d'Armi - 09123 Cagliari, Italy
e-mail: giob@diee.unica.it
** *ISSIA CNR - National Research Council of Italy*
Via De Marini, 6 - 16149 Genova, Italy
e-mail: punta@ge.issia.cnr.it
*** *DIMA - University of Genova*
Via Dodecaneso, 35 - 16146 Genova, Italy
e-mail: zolezzi@dima.unige.it

Abstract: New definitions of approximability are presented for nonlinear second order sliding mode control systems. Such robustness properties are compared with those already known for first order methods. Sufficient conditions are obtained for second order regularization and a relevant example is discussed.

Keywords: Approximability, Sliding Mode Control Systems, Second Order Sliding Mode Control.

1. INTRODUCTION

Sliding mode control can be viewed as a method to artificially enforce a constrained motion characterized by the fact that some functions of the state vector, after a finite time, are steered to zero; these functions are often called sliding outputs. The constraint equations define suitable surfaces in the state space, such that any motion constrained on them, is characterized by some good property with respect to a specified control aim (often the state space is the error state space, when the control aim is expressed in the form of a model tracking). The role of the control law is twofold, since it has to fulfill the constraint equations and then ensure that the motion, projected on the relevant surfaces, is characterized by the desired behaviours. Once the constraint equations are defined, the problem arises of forcing the system's state to reach in a finite time the relevant surfaces and to remain there.

In many practical situations the system equations, which are available to the designer, are imprecise models of the real phenomena or the model itself, even if it is accurate, is so complex that it is impractical to take into account this knowledge in the controller design. The problem can be represented in terms of uncertain systems and the controller has to guarantee the finite time fulfillment of the constraint despite the uncertainties. The aim of this paper is that of providing a theoretical framework to analyse the behaviour of the systems controlled by second order sliding mode (Bartolini *et al.*, 1998), (Fridman and Levant, 2002), (Bartolini *et al.*, 2003) when, due to nonidealities of unspecified nature, the chosen constraints are only approximatively satisfied.

A class of regular perturbations need to be identified, so that all the corresponding real state trajectories converge to the unique solution (if it exists) of the differential algebraic equations representing the exact fulfillment of the chosen constraints. This property, called approximability, has been analysed in (Bartolini and Zolezzi, 1986), (Zolezzi, to appear) for sliding mode control of first order. This approach, called regularization in (Utkin, 1992), is a basic procedure in the mathematical analysis and validation of sliding mode control techniques. In this paper we obtain, for the first time, second order regularization results of sliding mode control systems. A key property of sliding mode control systems deals with the behavior of real states, which fulfill only approximately the sliding conditions due to non idealities (of any nature), as compared with the properties of the ideal states, which fulfill exactly the prescribed sliding condition. The given system fulfills the approximability property whenever all real states converge (on the fixed bounded time interval we consider) to the ideal state as the non idealities disappear. This property, discussed in (Utkin, 1992), was mathematically formalized in (Bartolini and Zolezzi, 1986) and (Bartolini and Zolezzi, 1993) for nonlinear control systems. There, sufficient conditions for approximability were found, generalizing the regularization results of (Utkin, 1992), about first order sliding mode control methods.

In this work we introduce new definitions of second order approximability and we compare them with the one related to first order methods. Moreover we obtain sufficient conditions for second order approximability. Finally we discuss an example of theoretical as well as practical significance.

2. PROBLEM STATEMENT

We consider sliding mode control systems

$$\dot{x} = f(t, x, u), \quad u \in U; \quad s(t, x) = 0, \quad 0 \leq t \leq T, \quad (1)$$

on a fixed bounded time interval $[0, T]$, where $x \in I\!R^N$, $u \in I\!R^K$, and $s \in I\!R^M$, $T > 0$. We assume $f \in C^2([0, T] \times I\!R^N \times W)$ with W an open set containing U, and $s \in C^2([0, T] \times I\!R^N)$.

Following the terminology of (Utkin, 1992), page 13, and the approach of (Bartolini and Zolezzi, 1986), (Bartolini and Zolezzi, 1993), we deal with *ideal* states which fulfill exactly the sliding condition $s[t, x(t)] = 0$ for every t, as opposite to *real* states, fulfilling only approximately such a condition.

A physically relevant property of (1), essential to successfully implement sliding mode control strategies, is the following. All real states of (1) converge toward a well defined ideal state as the non-idealities disappear. This property rules out any ambiguous behaviour (according to the terminology of (Utkin, 1992)), and constitutes the basis for the regularization and validation of sliding mode techniques. See (Utkin, 1992), Chapter 2 for a discussion of this topic.

Such a property, called *approximability*, was mathematically defined for the first time in (Bartolini and Zolezzi, 1986) assuming the existence of the equivalent control, and generalized in (Bartolini and Zolezzi, 1993). For a different generalization see (Nistri, 1989). Sufficient conditions for approximability are given in (Bartolini and Zolezzi, 1986), (Bartolini and Zolezzi, 1993) and (Zolezzi, to appear). We refer to (Utkin, 1992) for a discussion of the physical and control theoretic significance of this property.

We refer to (Bartolini *et al.*, 1998), (Fridman and Levant, 2002), and (Bartolini *et al.*, 2003) for overviews of second order sliding modes and applications. Real second order states and sliding accuracy are considered in (Levant, 1993).

In this section we propose a new framework to define rigorously these properties within second order sliding mode methods. We present new definitions of second order approximability, and compare them with the corresponding first order property.

A parameter ε, belonging to a metric space with a fixed element conventionally denoted by 0, will be used to represent non idealities of any nature in the real sliding. We write $x_\varepsilon \to x$ to denote convergence of x_{ε_k} towards x for each sequence $\varepsilon_k \to 0$. The notation $x_\varepsilon \rightrightarrows x$ means uniform convergence on $[0, T]$.

The reaching phase is not at issue in our discussion of approximability properties, only the behaviour near the sliding manifold matters. First order sliding mode control techniques employ discontinuous feedback control laws in order to reach in a finite time the sliding manifold. Then the appropriate meaning of solution to (1) corresponding to discontinuous feedback is that of Filippov, see (Utkin, 1992) and (Filippov, 1988).

Given $\overline{x} \in I\!R^N$, we consider as admissible control laws in (1) all functions $u : [0, T] \times I\!R^N \to U$ which are $L \otimes B -$ measurable, i.e. measurable with respect to the $\sigma -$ algebra generated by the products of the Lebesgue measurable subset of $[0, T]$ and the Borel measurable subset of $I\!R^N$, and which fulfill the following property. For any such u, if the differential system in (1) has a classical (i.e. almost everywhere), or a Filippov solution x on $[0, T]$ for any $\overline{x}$, then $u[\cdot, x(\cdot)] \in L^\infty(0, T)$. We denote by U_∞ the set of all admissible controls.

To simplify the notations, we shall write $\dot{x} = f(t,x,u)$ on $[0,T]$ meaning that $u \in U_\infty$ and x is either a classical of Filippov solution on $[0,T]$.

The sliding mode control system (1) fulfills the *first order approximability* property whenever the following is true. For every $x_0 \in I\!R^N$ such that $s(0,x_0) = 0$ there exists a unique sliding state y, i.e. for some control $u^* \in U_\infty$ we have

$$\dot{y} = f(t,y,u^*) \quad \text{on } [0,T], \quad y(0) = x_0, \tag{2}$$

$$s[t,y(t)] = 0, \qquad 0 \le t \le T. \tag{3}$$

Moreover for every sequence $(u_\varepsilon, x_\varepsilon)$ such that $\dot{x}_\varepsilon = f(t,x_\varepsilon,u_\varepsilon)$ on $[0,T]$ and $s(t,x_\varepsilon)\rightrightarrows 0$ we have $x_\varepsilon \rightrightarrows y$ provided $x_\varepsilon(0) \to y(0)$.

The above definition, compared with that presented in (Bartolini and Zolezzi, 1986), does not require either uniqueness of the sliding control law u^* or existence of the equivalent control, moreover s is allowed to depend on t as well.

Uniqueness of the sliding state is of course fulfilled whenever the equivalent control is available, see (Utkin, 1992), (Bartolini and Zolezzi, 1986).

We consider the *first order ideal* system made up of control-state pairs (v,y) such that

$$\begin{cases} \dot{y} = f(t,y,v) \quad \text{on } [0,T], \quad v \in U_\infty; \\ s_t(t,y) + s_x(t,y) f(t,y,v) = 0, \end{cases}$$

and the *second order ideal* system of control-state pairs (u,z), both absolutely continuous in $[0,T]$, such that for a.e. $t \in (0,T)$

$$\begin{cases} \dot{z} = f(t,z,u), \quad u(t) \in U; \\ P(t,z,u) + Q(t,z,u)\dot{u} = 0 \end{cases} \tag{4}$$

where $P = s_{tt} + 2s_{tx}f + f' s_{xx} f + s_x f_t + s_x f_x f$, $Q = s_x f_u$, and $f' s_{xx} f$ denotes the vector of components $f' s_{jxx} f$, $j = 1, \ldots, M$.

We model the non idealities acting on the second order ideal system (4) by using two different terms. The first, denoted by $b_\varepsilon = b_\varepsilon(t) \in I\!R^M$, takes into account second order sliding non idealities, so that in the real second order system we have $\ddot{s} = b_\varepsilon$. The second, denoted by $c_\varepsilon = c_\varepsilon(t) \in I\!R^K$, takes into account non idealities in obtaining $\dot{u}_\varepsilon$, so that in the real second order system we work with $w_\varepsilon = \dot{u}_\varepsilon + c_\varepsilon$ instead of $w_\varepsilon = \dot{u}_\varepsilon$.

The *second order real* control-state pairs are thereby given by pairs $(u_\varepsilon, x_\varepsilon)$, both absolutely continuous in $[0,T]$, such that for a.e. $t \in (0,T)$

$$\begin{cases} \dot{x}_\varepsilon = f(t,x_\varepsilon,u_\varepsilon), \quad u_\varepsilon(t) \in U; \\ P(t,x_\varepsilon,u_\varepsilon) + Q(t,x_\varepsilon,u_\varepsilon) w_\varepsilon = b_\varepsilon(t); \\ w_\varepsilon(t) = \dot{u}_\varepsilon(t) + c_\varepsilon(t). \end{cases} \tag{5}$$

About the non idealities c_ε we shall employ the condition

$$c_\varepsilon \to 0 \ \text{ in } \ W^{-1,\infty}(0,T) \ \text{ as } \ \varepsilon \to 0, \tag{6}$$

which means that $c_\varepsilon \in L^1(0,T)$ and $\sup\{|\int_0^t c_\varepsilon(r)dr| : 0 \le t \le T\} \to 0$ as $\varepsilon \to 0$.

About b_ε we consider two different ways b_ε can vanish as $\varepsilon \to 0$, namely

$$\text{either} \quad b_\varepsilon \to 0 \ \text{ in } \ W^{-2,\infty}(0,T), \tag{7}$$

$$\text{or} \quad b_\varepsilon \to 0 \ \text{ in } \ W^{-1,\infty}(0,T). \tag{8}$$

By (7) we mean that $b_\varepsilon \in L^1(0,T)$ and $\sup\{|\theta_\varepsilon(t)| : 0 \le t \le T\} \to 0$ where $\ddot{\theta}_\varepsilon = b_\varepsilon$ a.e. in $(0,T)$, $\theta_\varepsilon(0) = \dot{\theta}_\varepsilon(0) = 0$. Accordingly, we formulate two definitions of second order approximability of (1).

Definition 1. The control system (1) fulfills second order approximability of the *first kind* if

Condition A: For every x_0 such that $s(0,x_0) = 0$ there exists a unique sliding state y issued from x_0 corresponding to some continuous control $u^*(t) \in U$.

Condition B: Given any sequences b_ε, c_ε satisfying (6) and (8), we have $x_\varepsilon \rightrightarrows y$ provided $x_\varepsilon(0) \to y(0)$, $u_\varepsilon(0) \to u^*(0)$ for every sequence x_ε fulfilling (5).

Definition 2. The control system (1) fulfills second order approximability of the *second kind* if (A) and (B) are fulfilled except that (8) is replaced by (7).

Since strong convergence in $W^{-1,\infty}(0,T)$ implies the same in $W^{-2,\infty}(0,T)$, we have that second kind implies first kind approximability. However, if $\sup_\varepsilon |b_\varepsilon| \in L^1(0,T)$, a stronger condition than (9), convergence of b_ε in $W^{-2,\infty}(0,T)$ implies the same in $W^{-1,\infty}(0,T)$.

Roughly speaking, first kind approximability means that such a robust behaviour of the control system is guaranteed whenever $\dot{s}$ is uniformly small. Second kind approximability means that the same behaviour is present whenever s is small. This property is similar to that required in the definition of first order approximability (by the quite different first order methods).

3. APPROXIMABILITY RESULTS

The main result of this section is the following.

Theorem 1. (Bartolini et al.*, 2004)*

Let

$$\sup_\varepsilon \int_0^T (|b_\varepsilon(t)| + |c_\varepsilon(t)|) dt < +\infty, \tag{9}$$

$$\sup_\varepsilon \int_0^T |\dot{Q}[t, x_\varepsilon(t), u_\varepsilon(t)]| dt < +\infty, \qquad (10)$$

If system (1) fulfills first order approximability and, for all x_0 with $s(0, x_0) = 0$, the corresponding sliding state can be generated by a continuous control, then (1) fulfills both second order approximability properties.

Continuity of a sliding control is not a very restrictive assumption, being fulfilled whenever the equivalent control is available, as it often happens in sliding mode control applications, see (Utkin, 1992).

According to Theorem 1, if first order sliding mode control methods cannot give rise to ambiguous motions, no further ambiguous behaviour can be induced by any second order control algorithm. Hence the validation of second order methods relies on checking known criteria yielding first order approximability, as those known from (Bartolini and Zolezzi, 1986), (Bartolini and Zolezzi, 1993) and (Zolezzi, to appear).

Examples show that the converse to Theorem 1 fails. A sliding mode control system may fulfill second kind approximability of second order, and may fail to possess first order approximability.

We end this section obtaining sufficient conditions for the convergence of second order real states to ideal ones.

Let $(u_\varepsilon, x_\varepsilon)$ fulfill (5) and put $\alpha_\varepsilon(t) = u_\varepsilon(0) + \int_0^t w_\varepsilon dr$.

Let Q be everywhere nonsingular. Then, a.e. in $(0, T)$

$$\dot{x}_\varepsilon = f_\varepsilon(t, x_\varepsilon, \alpha_\varepsilon), \qquad \dot{\alpha}_\varepsilon = h_\varepsilon(t, x_\varepsilon, \alpha_\varepsilon) \qquad (11)$$

where $f_\varepsilon(t, x, u) = f[t, x, u - \gamma_\varepsilon(t)]$, $\gamma_\varepsilon(t) = \int_0^t c_\varepsilon dr$, $h_\varepsilon(t, x, u) = Q^{-1}[t, x, u - \gamma_\varepsilon(t)](b_\varepsilon(t) - P[t, x, u - \gamma_\varepsilon(t)])$.

Let (u, z) fulfill (4). Thus x_ε are second order real states, and z is any second order ideal state. In stating the convergence result we employ the following terminology. A function $g = g(t, x, u) : [0, T] \times \Omega \times U \to I\!R^N$ is called:

locally p-integrably Lipschitz, if for every compact $L \subset I\!R^N \times U$ there exists $c \in L^p(0, T)$ such that

$$|g(t, x', u') - g(t, x'', u'')| \le c(t)\,(|x' - x''| + |u' - u''|)$$

for a.e. t and every (x', u'), (x'', u'') in L;

locally p-integrably bounded if for every compact $L \subset I\!R^N \times U$ there exists $c \in L^p(0, T)$ such that $|g(t, x, u)| \le c(t)$ for a.e. t and every $(x, u) \in L$.

Theorem 2. (Bartolini et al.*, 2004)*

The pair $(u_\varepsilon, x_\varepsilon) \rightrightarrows (u, z)$ provided $(u_\varepsilon(0), x_\varepsilon(0)) \rightrightarrows (u(0), z(0))$, Q is everywhere nonsingular and the following hold:

- *$b = \sup_\varepsilon |b_\varepsilon| \in L^p(0, T)$ for some $p > 1$ and $(b_\varepsilon, c_\varepsilon) \to 0$ in $W^{-1,\infty}(0, T)$;*
- *Q^{-1} is locally q-integrably Lipschitz, $\frac{1}{p} + \frac{1}{q} = 1$; $Q^{-1}P$ is locally 1-integrably Lipschitz and locally r-integrably bounded for some $r \in (1, p)$;*
- *$|Q^{-1}(t, x, u)| \le A_0(t) + B_0(t)(|x| + |u|)$ with A_0, $B_0 \in L^\alpha(0, T)$ some $\alpha > \frac{pr}{p-r}$;*
- *$|Q^{-1}(t, x, u)P(t, x, u)| + |f(t, x, u)| \le A(t) + B(t)(|x| + |u|)$, with A, $B \in L^\beta(0, T)$, some $\beta > 1$.*

Corollary 1. (Bartolini et al.*, 2004)*

Let us assume the hypotheses of Theorem 2. If there exists a unique sliding state y such that $x_\varepsilon(0) \to y(0) = z(0)$, $\dot{s}[0, z(0)] = 0 = s[0, z(0)]$ then $x_\varepsilon \rightrightarrows y$.

4. EXAMPLE

In this section we present an example related to second order approximability.

Let $y_d(t) \in C^2([0, T])$ be an available signal such that $|\ddot{y}_d(t)| \le L$ for every t. Consider the sliding mode control system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $|u| \le L$; $s(t, x_1) = x_1 - y_d$, $0 \le t \le T$. Here $N = 2$ and $K = M = 1$. Since $Q = s_x f_u = 0$ everywhere, the approximability criteria developed in (Bartolini and Zolezzi, 1986) do not apply. We check first order approximability via Corollary 4.1 of (Zolezzi, to appear). The required properties of linear growth and local Lipschitz continuity of the dynamics are obviously fulfilled, as well as convexity of $f(t, x, U)$. Given $x_0 \in I\!R^2$ such that $s(t, x_0) = 0$, i.e. $x_{10} = y_d(0)$, if $x_{20} = \dot{y}_d(0)$ there exists a unique sliding state y issued from x_0, namely $y = (y_d, \dot{y}_d)'$, which corresponds to the continuous control $u = \ddot{y}_d$ (in the a.e. sense). Hence, by Theorem 1, second order approximability holds. Then, in a finite time, x_1 copies y_d and x_2 copies $\dot{y}_d$. This happens for every non idealities b_ε, c_ε acting on the system and fulfilling (6), (7) or (8), (9), independently of the particular second order sliding mode algorithm employed to control the system.

5. CONCLUSIONS

We introduced new definitions of approximability for nonlinear second order sliding mode control systems. We compared such robustness properties with those already known for first order methods. Sufficient conditions were obtained for second order regularization.

REFERENCES

Bartolini, G., A. Ferrara and E. Usai (1998). Chattering avoidance by second order sliding modes control. *IEEE Trans. Automatic Control* **43**(2), 241–247.

Bartolini, G., A. Pisano, E. Punta and E. Usai (2003). A survey of applications of second-order sliding mode control to mechanical systems. *Int. J. Control* **76**, 875–892.

Bartolini, G. and T. Zolezzi (1986). Control of nonlinear variable structure systems. *J. Math. Anal. and Appl.* **118**, 42–62.

Bartolini, G. and T. Zolezzi (1993). Behaviour of variable-structure control systems near the sliding manifold. *Systems and Control Letters* **21**, 43–48.

Bartolini, G., E. Punta and T. Zolezzi (2004). *Approximability Properties for Second Order Sliding Mode Control Systems.* in preparation.

Filippov, A.F. (1988). *Differential equations with discontinuous right-hand sides.* Kluwer.

Fridman, L. and A. Levant (2002). Higher order sliding modes. In: *Sliding Mode Control in Engineering* (W. Perruquetti and J.P. Barbot, Eds.). pp. 53–96. Dekker.

Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control. *Int. J. Control* **58**, 1247–1263.

Nistri, P. (1989). A note on the approximability property of nonlinear variable structure systems. *The IEEE* 28^{th} *CDC* pp. 815–818.

Utkin, V.I. (1992). *Sliding Modes in Control and Optimization.* Springer Verlag.

Zolezzi, T. (to appear). Well-posedness and sliding mode control. *ESAIM Control Optim. Calc. Var.*