CONTROLLABILITY OF GOURSAT-DARBOUX SYSTEMS – SOME NUMERICAL RESULTS [1]

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Abstract: In the paper, we consider a linear nonautonomous Goursat-Darboux system which is a continuous version of discrete Fornasini-Marchesini system. We give some general algorithm for construction a piecewise constant control related to some density result for attainable sets connected with integrable controls and piecewise constant ones.

Keywords: Goursat-Darboux system, controllability, piecewise constant controls.

1. INTRODUCTION

In the paper, we consider the following linear Goursat-Darboux problem

$$\begin{aligned} z_{xy}(x,y) = A(x,y)z(x,y) + A_1(x,y)z_x(x,y) \\ + A_2(x,y)z_y(x,y) + B(x,y)u(x,y) \end{aligned} \quad (1)$$

for almost all (a.a.) $(x,y) \in P := [0,1] \times [0,1] \subset \mathbb{R}^2$,

$$z(x,0) = z(0,y) = 0, \quad (2)$$

for $x, y \in [0,1]$, where $z(x,y) \in \mathbb{R}^n$, $u(x,y) \in \mathbb{R}^m$, $A(x,y)$, $A_1(x,y)$, $A_2(x,y) \in \mathbb{R}^{n\times n}$, $B(x,y) \in \mathbb{R}^{n\times m}$ and z_x, z_y, z_{xy} are the appropriate partial derivatives of z. This problem is a continuous version of the Fornasini-Marchesini problem

$$\begin{aligned} z(i+1,j+1) = A(i,j)z(i,j) + A_1(i,j)z(i+1,j) \\ + A_2(i,j)z(i,j+1) + B(i,j)u(i,j) \end{aligned}$$

for $i, j = 0, 1, ...,$

$$z(i,0) = z(0,j) = 0$$

for $i, j = 0, 1, ...,$ which plays an important role in the theory of automatic control (cf. (Fornasini, Marchesini, 1976), (Kaczorek, 2000)). Continuous system (1) - (2) can be used for modellnig of a gas absorbtion process (cf. (Tikhonov, Samarski, 1990), (Idczak *et al.*, 1994), (Rehbock *et al.*, 1998)).

By a solution of the system (1) - (2), corresponding to the function (control) $u : P \to \mathbb{R}^m$, we mean an absolutely continuous function $z : P \to \mathbb{R}^n$, which satisfies system (1) almost everywhere (a.e.) on the set P and boundary conditions (2). Let us recall (cf. (Walczak, 1987)) that function $z : P \to \mathbb{R}^n$ is called *absolutely continuous* if there exist the functions $l \in L^1(P, \mathbb{R}^n)$ (the space of Lebesgue integrable on P functions with values in $\mathbb{R}^n$), l^1, $l^2 \in L^1([0,1], \mathbb{R}^n)$ and a constant $c \in \mathbb{R}^n$ such that

$$z(x,y) = \int_0^x \int_0^y l(s,t)dtds + \int_0^x l^1(s)ds + \int_0^y l^2(t)dt + c$$

for all $(x,y) \in P$. Such a function possesses the partial derivatives $z_{xy}(x,y) = l(x,y)$, $z_x(x,y) = l^1(x) + \int_0^y l(x,t)dt$, $z_y(x,y) = l^2(y) + \int_0^x l(s,y)ds$ for a.a. $(x,y) \in P$. If an absolutely continuous function $z : P \to \mathbb{R}^n$ satisfies boundary conditions

[1] The preparation of this paper was supported by the State Committee for Scientific Research, Poland, Grant 7 T11A 004 21.

(2), then l^1, $l^2 \equiv 0$, $c = 0$ and, consequently, $z_x(x,y) = \int_0^y l(x,t)dt$, $z_y(x,y) = \int_0^x l(s,y)ds$.

The set of all absolutely continuous functions $z : P \to \mathbb{R}^n$ satisfying boundary conditions (2) will be denoted as AC_0. It is easy to see that AC_0 with the norm $\|z\|_{AC_0} = \iint_P |z_{xy}(x,y)|\, dxdy$ is complete.

One can show (cf. (Idczak *et al.*, 1994)) that for any control $u \in L^1(P, \mathbb{R}^m)$ there exists a unique solution $z^u \in AC_0$ of system (1) provided A, A_1, $A_2 \in L^\infty(P, \mathbb{R}^{n\times n})$ (the space of essentially bounded on P functions with values in $\mathbb{R}^{n\times n}$), $B \in L^\infty(P, \mathbb{R}^{n\times m})$.

Now, let us fix a set $M \subset \mathbb{R}^m$ and consider the set U_M of controls $u \in L^1(P, \mathbb{R}^m)$ such that $u(x,y) \in M$ for a.a. $(x,y) \in P$. By $\mathcal{A}_M(1,1)$ we denote the set $\{z^u(1,1) \in \mathbb{R}^n : u \in U_M\}$ called *an attainable set* for problem (1) - (2), corresponding to the set M. In (Idczak, Walczak, 2004) it was shown that if the set M is compact in $\mathbb{R}^m$, then $\mathcal{A}_M(1,1)$ is convex compact in $\mathbb{R}^n$ and $\mathcal{A}_M(1,1) = \mathcal{A}_{coM}(1,1)$ where coM is the convex hull of M (*bang-bang principle*). Moreover, if $\mathcal{A}_M^{PC}(1,1)$ denotes an attainable set for problem (1) - (2), corresponding to the controls $u \in U_M$ (with M not necessarily compact) that are piecewise constant on P [2], then $\mathcal{A}_M(1,1) \subset \overline{\mathcal{A}_M^{PC}(1,1)}$ [3]. In other words, for any control $u \in U_M$ and $\varepsilon > 0$ there exists a control $v \in U_M^{PC}$ such that $|z^u(1,1) - z^v(1,1)| < \varepsilon$.

Our aim is to show how one can determine the mentioned control v. More precisely, first we shall show that the dependence of $z^u(1,1)$ on the control $u \in L^1(P, \mathbb{R}^m)$ is lipschitzian and Lipschitz constant can be calculated. Next, using the proof of a lemma on the density of piecewise constant functions in the space of integrable ones, we shall give some general algorithm for obtaining the control v.

2. LIPSCHITZIAN DEPENDENCE OF $Z^U(1,1)$ ON U

In (Idczak *et al.*, 1994), the following nonlinear Goursat-Darboux system

$$z_{xy}(x,y) = f(x,y,z(x,y),z_x(x,y), z_y(x,y),u(x,y)),$$

for $(x,y) \in P$ a.e., with boundary conditions (2), where $f = (f^1,...,f^n) : P \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, was considered. It was shown (cf. (Idczak *et al.*, 1994, theorems 1,2)) that for any control $u \in L^1(P, \mathbb{R}^m)$ there exists a unique solution $z^u \in AC_0$ of the above system and the mapping

$$L^1(P, \mathbb{R}^m) \ni u \longmapsto z^u \in AC_0 \qquad (3)$$

is continuous provided

- f is lipschitzian in $(z, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, i.e. there exists a constant $L > 0$ such that
$$|f(x,y,z,z_1,z_2,u) - f(x,y,w,w_1,w_2,u)| \le L(|z-w| + |z_1 - w_1| + |z_2 - w_2|)$$
for a.a. $(x,y) \in P$, $z, z_1, z_2, w, w_1, w_2 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$,
- f is measurable in $(x,y) \in P$ and continuous in $u \in \mathbb{R}^m$,
- for any control $u \in L^1(P, \mathbb{R}^m)$ there exists a point $(\overline{z}, \overline{z}_1, \overline{z}_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that the function
$$P \ni (x,y) \longmapsto f(x,y,\overline{z},\overline{z}_1,\overline{z}_2,u(x,y)) \in \mathbb{R}^n$$
belongs to $L^1(P, \mathbb{R}^n)$

(if the controls take their values in a fixed set $M \subset \mathbb{R}^m$, then it suffices to assume the fulfilment of the above conditions for points $u \in M$ and for controls belonging to U_M).

From the proof of this result it follows that in our (linear) case

$$\|z^u - z^v\|_{AC_0} \le \frac{e^{2k}}{1-\alpha} \|Bu - Bv\|_{L^1(P,\mathbb{R}^n)}$$

where $k \in \mathbb{N}$ is such that $2nL(\frac{1}{k^2} + \frac{1}{k}) < 1$ and $\alpha = 2nL(\frac{1}{k^2} + \frac{1}{k})$. Let us point that in linear case a constant L can be calculated (provided the L^∞-norms of A, A_1, A_2, B are known). Consequently,

$$\|z^u - z^v\|_{AC_0} \le \frac{e^{2k}}{1-\alpha} \|Bu - v\|_{L^1(P,\mathbb{R}^n)} \le \le C\,\|u - v\|_{L^1(P,\mathbb{R}^m)}$$

with constant C that can be calculated. Moreover,

$$|z^u(1,1) - z^v(1,1)| \le \iint_P \left|z^u_{xy}(x,y) - z^v_{xy}(x,y)\right| dxdy = \|z^u - z^v\|_{AC_0}.$$

So,

$$|z^u(1,1) - z^v(1,1)| \le C\,\|u(\cdot,\cdot) - v(\cdot,\cdot)\|_{L^1(P,\mathbb{R}^m)}.$$

Remark 1. From the proof of (Idczak *et al.*, 1994, theorem 2) it follows that the constant C can be calculated also in the case when the function f is lipschitzian in u.

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[2] We say that a function $u : P \to \mathbb{R}^m$ is piecewise constant if there exists a partition $0 = x_0 < x_1 < ... < x_n = 1$ ($n \in \mathbb{N}$) of an interval $[0,1]$ such that the function u is constant on each interval (two-dimensional) $(x_{i-1}, x_i) \times (x_{j-1}, x_j) \subset P$, i, $j = 1,...,n$. The set of all piecewise constant functions $u : P \to \mathbb{R}^m$ such that $u(x,y) \in M$ for a.a. $(x,y) \in P$ is denoted as U_M^{PC}.

[3] If M is compact, then we have the equality
$$\mathcal{A}_M(1,1) = \overline{\mathcal{A}_M^{PC}(1,1)}.$$

3. BIACCIOTTI-SENTIS LEMMA FOR FUNCTIONS OF TWO VARIABLES

In (Idczak, 2002), the following generalization of Biacciotti-Sentis lemma on the density of the piecewise constant functions (of one variable) in the space of integrable ones, to the case of functions of two variables, has been proved (see (Idczak, Walczak, 2004) for the case of n variables).

Lemma 1. If $M \subset \mathbb{R}^m$, then for every integrable function $u : P \to M$ and every $\varepsilon > 0$ there exists a piecewise constant function $v : P \to M$ such that $\iint_P |u(x,y) - v(x,y)|\, dxdy \le \varepsilon$.

We quote below the proof of this lemma because it will play a fundamental role in the next considerations.

Proof of lemma 1. Let us fix an integrable function $u : P \to M$, $\varepsilon > 0$, $v_0 \in M$ and consider a function

$$b_0 : P \ni (x,y) \longmapsto u(x,y) - v_0 \in \mathbb{R}^m.$$

The Tchebychev's inequality (cf. (Kolmogorov, Fomin, 1976)) implies that for any $k \in \mathbb{N}$

$$\mu(\{(x,y) \in P;\ |b_0(x,y)| \ge k\}) \le \frac{1}{k} \int\int_P |b_0(x,y)|\, dxdy$$

(μ denotes the Lebesgue measure in P). From the absolute continuity of the integral it follows that for any $\eta > 0$ there exists $\delta > 0$ such that $\int\int_R |b_0(x,y)|\, dxdy < \eta$ provided $\mu(R) < \delta$. Consequently, there exists $k_0 \in \mathbb{N}$ such that $\int\int_{R_0} |b_0(x,y)|\, dxdy < \frac{\varepsilon}{2}$ where $R_0 = \{(x,y) \in P;\ |b_0(x,y)| \ge k_0\}$. If we put

$$b(x,y) = \begin{cases} v_0 & ;\ (x,y) \in R_0 \\ u(x,y) & ;\ (x,y) \in P \setminus R_0 \end{cases}$$

for $(x,y) \in P$, then we have

$$\int\int_P |u(x,y) - b(x,y)|\, dxdy = \int\int_{R_0} |u(x,y) - b(x,y)|\, dxdy = \int\int_{R_0} |b_0(x,y)|\, dxdy < \frac{\varepsilon}{2}.$$

Let us denote by γ a finite number which bounds the function b on P, i.e. $|b(x,y)| \le \gamma$ for $(x,y) \in P$. The Lusin's theorem (cf. (Lojasiewicz, 1988, theorem 2, §5, chapter V)) implies that there exists a compact set $H \subset P$ such that $\mu(H) > 1 - \frac{\varepsilon}{8\gamma}$ and the function $b\,|_H$ is uniformly continuous. In particular, there exists $\sigma > 0$ such that

$$|b(x,y) - b(\overline{x},\overline{y})| < \frac{\varepsilon}{4}$$

for $(x,y), (\overline{x},\overline{y}) \in H, |(x,y) - (\overline{x},\overline{y})| < \sigma$. Let us fix a number $r \in \mathbb{N}$ such that $\frac{\sqrt{2}}{r} < \sigma$ and consider a partition $P_{ij} = [\frac{i}{r}, \frac{i+1}{r}] \times [\frac{j}{r}, \frac{j+1}{r}]$, $i,j = 0,...,r-1$, of the interval P. Let us also define a function $v : P \to \mathbb{R}^m$,

$$v(x,y) = \begin{cases} b(\widetilde{x}_i, \widetilde{y}_j) & ;\ \begin{array}{l}(x,y) \in IntP_{ij} \text{ and} \\ (IntP_{ij}) \cap H \ne \emptyset\end{array} \\ v_0 & ;\ \text{otherwise} \end{cases},$$

where $(\widetilde{x}_i, \widetilde{y}_j)$ is an arbitrary fixed point of $(IntP_{ij}) \cap H$ for any i,j such that $(IntP_{ij}) \cap H \ne \emptyset$. Of course, v is a piecewise constant function and $v(x,y) \in M$ for $(x,y) \in P$ a.e. Moreover, we have

$$\begin{aligned} &\int\int_P |b(x,y) - v(x,y)|\, dxdy \\ &= \int\int_H |b(x,y) - v(x,y)|\, dxdy + \int\int_{P\setminus H} |b(x,y) - v(x,y)|\, dxdy \\ &\le \sum_{i,j=0}^{r-1} \int\int_{(IntP_{ij})\cap H} |b(x,y) - v(x,y)|\, dxdy + \mu(P \setminus H) 2\gamma \\ &\le r^2 \frac{1}{r^2}\frac{\varepsilon}{4} + \frac{\varepsilon}{8\gamma} 2\gamma = \frac{\varepsilon}{2}. \end{aligned}$$

Finaly, $v : P \to M$ is the piecewise constant function and

$$\begin{aligned} &\int\int_P |u(x,y) - v(x,y)|\, dxdy \\ &\le \int\int_P |u(x,y) - b(x,y)|\, dxdy + \int\int_P |b(x,y) - v(x,y)|\, dxdy \le \varepsilon \end{aligned}$$

which completes the proof. ■

4. DETERMINING OF A CONTROL V

Let us fix a set $M \subset \mathbb{R}^m$, a number $\varepsilon > 0$ and a control $u : P \to M$. We shall give some general algorithm for the construction of a piecewise constant function $v : P \to M$ such that $|z^u(1,1) - z^v(1,1)| < \varepsilon$.

Algorithm
Step 1. We calculate a constant C satisfying (4), i.e. $C = \frac{e^{2k}}{1-\alpha} \operatorname*{esssup}_{(x,y)\in P} |B(x,y)|$ where $|B(x,y)| = \sqrt{\sum_{i=1}^n \sum_{j=1}^m |b_{ij}(x,y)|^2}$, $k \in \mathbb{N}$ is such that $2nL(\frac{1}{k^2} + \frac{1}{k}) < 1$ and $\alpha = 2nL(\frac{1}{k^2} + \frac{1}{k})$ (L is a Lipschitz constant for the right-hand side of the system, with respect to $(z, z_1, z_2) \in (\mathbb{R}^n)^3$, i.e. a constant satisfying the condition

$$\begin{aligned} &|\ A(x,y)(z - w) + A_1(x,y)(z_1 - w_1) + A_2(x,y)(z_2 - w_2)\ | \\ &\le L(|z - w| + |z_1 - w_1| + |z_2 - w_2|) \end{aligned}$$

for a.a. $(x,y) \in P$, z, z_1, z_2, w, w_1, $w_2 \in \mathbb{R}^n$).

Step 2. We choose a number $\varepsilon_1 > 0$ such that $C\varepsilon_1 < \varepsilon$.

Step 3. We fix any point $v_0 \in M$ and consider the function $b_0 : P \ni (x,y) \mapsto u(x,y) - v_0 \in \mathbb{R}^m$.

Step 4. We determine the set $R_0 = \{(x,y) \in P : |b_0(x,y)| \geq k_0\}$, where $k_0 \in \mathbb{N}$ is such that $\iint_{R_0} |b_0(x,y)|\, dxdy < \frac{\varepsilon_1}{2}$. Such k_0 exists by the absolute continuity of the integral (cf. remark 2).

Step 5. We define the function $b : P \to \mathbb{R}^m$,

$$b(x,y) = \begin{cases} v_0 & ;\ (x,y) \in R_0 \\ u(x,y) & ;\ (x,y) \in P \setminus R_0 \end{cases}.$$

This function satisfies the inequality

$$\iint_P |u(x,y) - b(x,y)|\, dxdy < \frac{\varepsilon_1}{2}.$$

Step 6. We put $\gamma = k_0 + |v_0|$ (cf. remark 3).

Step 7. We determine a set $H \subset P$ such $\mu(H) > 1 - \frac{\varepsilon_1}{8\gamma}$ and the function $b\,|_H$ is uniformly continuous (cf. remark 4).

Step 8. We determine a constant $\sigma > 0$ such that

$$|b(x,y) - b(\overline{x},\overline{y})| < \frac{\varepsilon_1}{4}$$

for $(x,y), (\overline{x},\overline{y}) \in H$ provided $|(x,y) - (\overline{x},\overline{y})| < \sigma$.

Step 9. We fix a number $r \in \mathbb{N}$ such that $\frac{\sqrt{2}}{r} < \sigma$.

Step 10. We consider a partition $P_{ij} = \left[\frac{i}{r}, \frac{i+1}{r}\right] \times \left[\frac{j}{r}, \frac{j+1}{r}\right]$, $i,j = 0,...,r-1$, of the interval P.

Step 11. We define a function $v : P \to \mathbb{R}^m$,

$$v(x,y) = \begin{cases} b(\widetilde{x}_i, \widetilde{y}_j) & ;\ \begin{array}{l}(x,y) \in \operatorname{Int} P_{ij} \\ \text{and } (\operatorname{Int} P_{ij}) \cap H \neq \emptyset\end{array} \\ v_0 & ;\ \text{otherwise} \end{cases}$$

where $(\widetilde{x}_i, \widetilde{y}_j)$ is an arbitrary fixed point of $(\operatorname{Int} P_{ij}) \cap H$ for any $i,j = 0,...,r-1$ such that $(\operatorname{Int} P_{ij}) \cap H \neq \emptyset$. This is the searched function.

Remark 2. (ad step 4). In the case when $b_0 \in L^2(P, \mathbb{R}^m)$ and there is a problem to calculate integral $\iint_{R_0} |b_0(x,y)|^2\, dxdy$, but the integral $I = \iint_P |b_0(x,y)|^2\, dxdy$ is known, we have that

$$\iint_{R_0} |b_0(x,y)|\, dxdy = \iint_P \chi_{R_0}(x,y)\, |b_0(x,y)|\, dxdy \leq \sqrt{\mu(R_0)} I;$$

so, in such a case it suffices to choose $\delta > 0$ such that $\sqrt{\delta} I < \frac{\varepsilon_1}{2}$ and define the set $R_0 = \{(x,y) \in P : |b_0(x,y)| \geq k_0\}$ where $k_0 \in \mathbb{N}$ is such that $\mu(R_0) < \delta$.

Remark 3. (ad step 6). Then $|b(x,y)| \leq \gamma$ for $(x,y) \in P$.

Remark 4. (ad step 7). First, we construct a sequence of simple functions $g_n : P \to \mathbb{R}^m$, which is uniformly convergent on P to the function $g(x,y) = \arctan b(x,y)$. It suffices to put (cf. (Lojasiewicz, 1988, proof of theorem 10, §4, chapter IV))

$$g_n(x,y) = \begin{cases} n & ;\ g(x,y) \geq n \\ \dfrac{k}{2^n} & ;\ \dfrac{k}{2^n} \leq g(x,y) < \dfrac{k+1}{2^n} \\ -n & ;\ g(x,y) < n \end{cases}$$

where $k = -n2^n, -n2^n + 1, ..., n2^n - 1$. Let us denote

$$E^n_{n2^n} = \{(x,y) \in P : g(x,y) \geq n\},$$

$$E^n_k = \{(x,y) \in P : \ \frac{k}{2^n} \leq g(x,y) < \frac{k+1}{2^n} \leq n\}$$

for $k = -n2^n, -n2^n + 1, ..., n2^n - 1$,

$$E^n_{-n2^n-1} = \{(x,y) \in P : g(x,y) < n\}.$$

Of course, the sets E^n_k, $k = -n2^n - 1, ..., n2^n$ are disjoint and $\bigcup_{k=-n2^n-1}^{n2^n} E^n_k = P$.

Next, for any $n \in \mathbb{N}$, $k = -n2^n - 1, ..., n2^n$, we construct (cf. (Lojasiewicz, 1988, theorem 2' and proof of theorem 2, §4, chapter V)) a closed set $H^n_k \subset E^n_k$ such that

$$\mu(E^n_k \setminus H^n_k) < \frac{\frac{\varepsilon_1}{8\gamma}}{2^n(n2^{n+1} + 2)}.$$

The sets

$$H^n = \bigcup_{k=-n2^n-1}^{n2^n} H^n_k, \ n \in \mathbb{N},$$

are closed, $\mu(P \setminus H^n) < \frac{\frac{\varepsilon_1}{8\gamma}}{2^n}$ and each of the functions $g_n\,|_{H^n}$ is continuous (cf. (Lojasiewicz, 1988, theorem 1, §1, chapter III)). Now, we consider the set $H = \bigcap_{n=1}^{\infty} H^n$ which is closed, $\mu(P \setminus H) < \frac{\varepsilon_1}{8\gamma}$ and each of the functions $g_n\,|_H$ is continuous. So, from the uniform convergence of $(g_n)_{n\in\mathbb{N}}$ to g on P it follows that $g\,|_H$ is continuous. Consequently, $b\,|_H$ (as a superposition $\tan \circ (g\,|_H)$) is continuous (in fact, it is uniformly continuous because of the compactness of H).

If the control u is bounded and continuous on P then (cf. remark 6 below) $b(x,y) = u(x,y)$ for $(x,y) \in P$ and one can put $H = P$.

Remark 5. (ad step 8). A constant σ can be easily determined when $b\,|_H$ satisfies the Lipschitz condition and the Lipschitz constant is known.

Remark 6. (ad steps 4 and 5). It is easy to see that if the function b_0 is bounded on P, then one can choose k_0 in such a way that the set R_0 is empty and, in consequence, $b(x,y) = u(x,y)$ for $(x,y) \in P$ and the inequality mentioned in step 7 is obviously satisfied.

In the above, we gave some general algorithm for approximation (by v) of a given control u in such a way that $|z^u(1,1) - z^v(1,1)| < \varepsilon$ for any fixed number $\varepsilon > 0$. Step 7 is the most complicated and step 8 seems to be realized individually in considered cases (cf. remark 5).

Example 1. Let us consider the following Goursat-Darboux problem

$$z_{xy}(x,y) = z(x,y) + z_x(x,y) + z_y(x,y) + u(x,y)$$

for $(x,y) \in P$ *a.e.*,

$$z(x,0) = z(0,y) = 0, \ x,y \in [0,1],$$

in the case when $z, z_x, z_y \in \mathbb{R}$, $u \in M = [-1,1] \subset \mathbb{R}$. We know that $\mathcal{A}_{[-1,1]}(1,1) = \overline{\mathcal{A}^{PC}_{[-1,1]}(1,1)}$. So, in particular, for any control $u : P \to [-1,1]$ and $\varepsilon > 0$ there exists a piecewise constant control $v : P \to [-1,1]$ such that $|z^u(1,1) - z^v(1,1)| < \varepsilon$. Let us fix a number $\varepsilon = 1$ and a control

$$u : P \ni (x,y) \mapsto \sin x \in [-1,1].$$

Step 1. It is easy to see that $L = 1$. Moreover, since in our case $n = 1$, $m = 1$, therefore $k = 3$, $\alpha = \frac{8}{9}$, $\operatorname{esssup}_{(x,y)\in P} |B(x,y)| = 1$. In consequence, $C = 9e^6$.

Step 2. It is sufficient to choose $\varepsilon_1 = \frac{1}{9e^6+1}$.

Step 3. We fix a point $v_0 = 0 \in [-1,1]$. So, $b_0(x,y) = u(x,y) = \sin x$ for $(x,y) \in P$.

Steps 4 and 5. Since our control $u(x,y) = \sin x$ is bounded by 1 on P, therefore (cf. remark 6) it is sufficient to put $k_0 = 2$ and then the inequality from step 5 is satisfied.

Step 6. We put $\gamma = k_0 + |v_0| = 2 + 0 = 2$.

Step 7. We put $H = P$ (cf. the last sentence in remark 4).

Step 8. Since our control $u(x,y) = \sin x$ satisfies the Lipschitz condition

$$|u(x,y) - u(\overline{x},\overline{y})| = |\sin x - \sin \overline{x}| \le |x - \overline{x}|$$

for $(x,y), (\overline{x},\overline{y}) \in P$, therefore it suffices to put $\sigma = \frac{1}{4(9e^6+1)}$.

Step 9. We put $r = 20545$ (we use computer algebra program - MAPLE) to calculate $\sqrt{2} \cdot 4(9e^6 + 1) \approx 20544.\,897\,82$).

Step 10. We consider the partition $P_{ij} = \left[\frac{i}{20545}, \frac{i+1}{20545}\right] \times \left[\frac{j}{20545}, \frac{j+1}{20545}\right]$, $i,j = 0,...,20544$, of the interval P.

Step 11. We define the searched piecewise constant function $v : P \to [-1,1]$ in the following way

$$v(x,y) = \begin{cases} \sin\left(\dfrac{i}{20545} + \dfrac{1}{2} \cdot \dfrac{1}{20545}\right) & ; \ x \in I_i \\ 0 & ; \ \text{otherwise} \end{cases}$$

where $I_i = \left(\frac{i}{20545}, \frac{i+1}{20545}\right)$.

Example 2. Let us consider Goursat-Darboux problem from example 1, but with the set $M = [0,\infty] \subset \mathbb{R}$. In this case we have $\mathcal{A}_{[0,\infty]}(1,1) \subset \overline{\mathcal{A}^{PC}_{[0,\infty]}(1,1)}$. Consider the control $u : P \to M$,

$$u(x,y) = \begin{cases} \dfrac{1}{\sqrt{xy}} & ; \ x \ne 0 \wedge y \ne 0 \\ 0 & ; \ x = 0 \vee y = 0 \end{cases}.$$

We will find a piecewise constant control $v : P \to M$ such that $|z^u(1,1) - z^v(1,1)| < \varepsilon$ where $\varepsilon = \frac{1}{10}$. We shall use computer algebra program – MAPLE to make some calculations.

Step 1. As it was in the previous example we have $C = 9e^6$.

Step 2. We choose $\varepsilon_1 = \frac{1}{90e^6+1}$.

Step 3. We fix $v_0 = 0$ *and put* $b_0(x,y) = u(x,y)$ *for* $(x,y) \in P$.

Step 4. We have

$$R_0 = \{(x,y) \in P : |b_0(x,y)| \ge k_0\} = \left\{(x,y) \in P : xy \le \frac{1}{k_0^2}\right\},$$

hence

$$\iint_{R_0} |b_0(x,y)|\, dxdy = \int_0^{k_0^{-2}} \left(\int_0^1 \frac{1}{\sqrt{xy}} dy\right) dx + \int_{k_0^{-2}}^1 \left(\int_0^{k_0^{-2}x^{-1}} \frac{1}{\sqrt{xy}} dy\right) dx = \frac{4}{k_0}\left(1 + \ln(k_0)\right).$$

One can check that for $k_0 = 5 \cdot 10^6$ we have

$$\iint_{R_0} |b_0(x,y)|\, dxdy \approx 1.313\,995\,878 \cdot 10^{-5} < \frac{\varepsilon_1}{2} \approx 1.377\,046\,616 \cdot 10^{-5}.$$

Thus $R_0 = \left\{(x,y) \in P : xy \le 2.5 \cdot 10^{13}\right\}$.

Step 5. We define the function

$$b(x,y) = \begin{cases} 0 & ; \ (x,y) \in R_0 \\ \dfrac{1}{\sqrt{xy}} & ; \ (x,y) \in P \setminus R_0 \end{cases}.$$

For this function we have $\iint_P |u(x,y) - b(x,y)|\, dxdy < \frac{\varepsilon_1}{2}$.

Step 6. We put $\gamma = 5 \cdot 10^6$.

Step 7. We will find the set H *such that* $\mu(P \setminus H) < \frac{\varepsilon_1}{8\gamma} \approx 6.885\,233\,082 \times 10^{-13}$. For a fixed $\alpha > 0$ consider the set

$$P \setminus H_\alpha = \{(x,y) \in P : \frac{1}{k_0^2(x+\alpha)} - \alpha < y < \frac{1}{k_0^2(x-\alpha)} + \alpha\}.$$

Then

$$\mu\left(P\setminus H_\alpha\right)=\int_{x_1}^{x_2}\left(\int_{y_1}^{1}dy\right)dx+\int_{x_2}^{1}\left(\int_{y_1}^{y_2}dy\right)dx,$$

where $x_1=\frac{1}{k_0^2(1+\alpha)}-\alpha$, $x_2=\frac{1}{k_0^2(1-\alpha)}+\alpha$, $y_1=\frac{1}{k_0^2(x+\alpha)}-\alpha$, $y_2=\frac{1}{k_0^2(x-\alpha)}+\alpha$ $\left(k_0=5\cdot 10^6\right)$. Using computer algebra program – MAPLE one can check that for $\alpha=10^{-13}$ we have that $\mu\left(P\setminus H_\alpha\right)=4\cdot 10^{-13}<\frac{\varepsilon_1}{8\gamma}$. Thus $H=H_\alpha$, where $\alpha=10^{-13}$ and the function $b|_H$ is uniformly continuous.

Step 8. We have that $\left|\nabla b\left(x,y\right)\right|=\frac{1}{2}\sqrt{\left(\frac{1}{x^3y}+\frac{1}{y^3x}\right)}$ for $(x,y)\in P\setminus R_0$. Moreover, it is easy to see that $\sup_{(x,y)\in P\setminus R_0}\left|\nabla b\left(x,y\right)\right|\leq\left|\nabla b\left(\frac{1}{k_0},\frac{1}{k_0}\right)\right|=\frac{\sqrt{2}}{2}k_0^2\approx 1.767\,766\,953\cdot 10^{13}$, thus

$$\left|b\left(x,y\right)-b\left(\bar{x},\bar{y}\right)\right|\leq\frac{\sqrt{2}}{2}k_0^2\left|(x,y)-(\bar{x},\bar{y})\right|$$

for $(x,y),(\bar{x},\bar{y})\in H$. Consequently, for $\frac{\varepsilon_1}{4}\approx 6.885\,233\,082\cdot 10^{-6}$ there exists $\sigma=\frac{\varepsilon_1}{2\sqrt{2}k_0^2}\approx 3.894\,876\,002\cdot 10^{-19}$ such that for $(x,y),(\bar{x},\bar{y})\in H$

$$\left|b\left(x,y\right)-b\left(\bar{x},\bar{y}\right)\right|<\frac{\varepsilon_1}{4}$$

provided $\left|(x,y)-(\bar{x},\bar{y})\right|<\sigma$.

Step 9. We fix a number $r=3\,630\,959\,141\,434\,616\,103$. Then $\frac{\sqrt{2}}{r}<\sigma$.

Step 10. We consider a partition

$$P_{ij}=\left[\frac{i}{r},\frac{i+1}{r}\right]\times\left[\frac{j}{r},\frac{j+1}{r}\right],\ i,j=0,...,r-1,$$

of the interval P.

Step 11. We define a function $v:P\to\mathbb{R}^m$,

$$v(x,y)=\begin{cases}b(\widetilde{x}_i,\widetilde{y}_j)\ ; & (x,y)\in\operatorname{Int}P_{ij}\\ & \text{and }(\operatorname{Int}P_{ij})\cap H\neq\emptyset\\ v_0 & ;\ \text{otherwise}\end{cases}$$

$$=\begin{cases}\dfrac{1}{\sqrt{(i+1)(j+1)}} & ;\ \dfrac{1}{k_0\left(x-\alpha\right)}+\alpha>y,\\ & \text{where } i=\left[\dfrac{x}{r}\right],j=\left[\dfrac{y}{r}\right]\\ 0 & ;\ \text{otherwise}\end{cases}$$

$$=\begin{cases}\dfrac{1}{\sqrt{(i+1)(j+1)}} & ;\ \dfrac{2\cdot 10^{-7}}{x-10^{-13}}+10^{-13}>y,\\ & \text{where } i=\left[\dfrac{x}{r}\right],j=\left[\dfrac{y}{r}\right]\\ 0 & ;\ \text{otherwise}\end{cases}$$

which is the searched function ([4]).

5. CONCLUDING REMARKS

In the paper, an algorithm for construction of a piecewise constant control approximating the control moving the hyperbolic system (1) - (2) to a given end-point, is proposed. Such an algorithm can be described (in an analogous way) for an ordinary linear control system (1-D) and for a continuous Roesser system (2-D), too. An open question is the possibility of applying the obtained results to a discrete linear Fornasini-Marchesini system (via a discretization process).

[4] The symbol $[x]$ denotes the greatest integer less or equal to x.

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