NONLINEAR CONTROLLER DESIGN FOR ACTIVE SUSPENSION SYSTEMS USING THE IMMERSION AND INVARIANCE METHOD

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Abstract: We present a controller design based on the immersion and invariance method for an active suspension system, and compare the result with a back-stepping control law. Simulation results show that the immersion and invariance controller can stabilize the full-order system as well as the backstepping controller in the nominal case, but is more robust to some parameter changes in the system. Moreover, when there is an unknown parameter, the adaptive immersion and invariance controller also gives closer response to the known parameter case than the adaptive backstepping controller.

Keywords: immersion and invariance method, backstepping control, active suspension system

1. INTRODUCTION

(Astolfi and Ortega, 2003) propose the immersion and invariance (I&I) method as a new tool to design a controller for nonlinear systems. This method is particularly useful when we know a stabilizing controller of a nominal reduced-order model and we would like to robustify it with respect to higher-order dynamics. A control law could be designed so that the full system dynamics is asymptotically immersed into the reduced-order one (the target system). They apply the technique to design a stabilizing controller for a magnetic levitation system, a global tracking controller for a flexible joint robot and an adaptive controller for a visual servoing system.

In this paper, we present a controller design based on the immersion and invariance method for an active suspension system.

2. THE IMMERSION AND INVARIANCE METHOD

Main results about the immersion and invariance technique can be summarized in the following theorem (Astolfi and Ortega, 2003).

Theorem Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let $x_* \in \mathbb{R}^n$ be the equilibrium point to be stabilized and let $p < n$.

Suppose we can find mappings

$$\alpha(\cdot) : \mathbb{R}^p \to \mathbb{R}^p \quad \pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n \quad c(\cdot) : \mathbb{R}^p \to \mathbb{R}^m$$
$$\phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-p} \quad \psi(\cdot,\cdot) : \mathbb{R}^{n\times(n-p)} \to \mathbb{R}^m$$

such that the following conditions hold.

(A1) *(Target system)* The system

$$\dot{\xi} = \alpha(\xi) \tag{2}$$

with state $\xi \in \mathbb{R}^p$, has a globally asymptotically stable equilibrium at $\xi_* \in \mathbb{R}^p$ and $x_* = \pi(\xi_*)$.

(A2) *(Immersion condition)* For all $\xi \in \mathbb{R}^p$,

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \frac{\partial \pi}{\partial \xi}\alpha(\xi) \tag{3}$$

(A3) *(Implicit manifold)* The following set identity holds

$$\begin{aligned}&\{x \in \mathbb{R}^n | \phi(x) = 0\}\\ &= \{x \in \mathbb{R}^n | x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^p\}\end{aligned} \tag{4}$$

(A4) *(Manifold attractivity and trajectory boundedness)* All trajectories of the system

$$\dot{z} = \frac{\partial \phi}{\partial x}[f(x) + g(x)\psi(x,z)] \quad (5)$$

$$\dot{x} = f(x) + g(x)\psi(x,z) \quad (6)$$

are bounded and satisfy

$$\lim_{t\to\infty} z(t) = 0 \quad (7)$$

Then, x_* is a globally asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(x) + g(x)\psi(x,\phi(x))$$

In this case, we say that the system (1) is *I&I stabilizable* with respect to the target dynamics (2)

The immersion and invariance method can be extended to the problem of adaptive stabilization of nonlinear systems under the following assumption.

(A5) *(Stabilizability)* There exists a parameterized function $\Psi(x,\theta)$, where $\theta \in \mathbb{R}^q$, such that for some unknown $\theta_* \in \mathbb{R}^q$, the system

$$\dot{x} = f_*(x) := f(x) + g(x)\Psi(x,\theta_*) \quad (8)$$

has a globally asymptotically stable equilibrium at $x = x_*$.

The system (1) under the assumption (A5) is said to be *adaptively I & I stabilizable* if the system

$$\dot{x} = f(x) + g(x)\Psi(x,\hat{\theta} + \beta_1(x)) \quad (9)$$

$$\dot{\hat{\theta}} = \beta_2(x,\hat{\theta})$$

with extended state $(x,\hat{\theta})$ and the functions β_1 and β_2, is I & I stabilizable with target dynamics

$$\dot{\xi} = f_*(\xi). \quad (10)$$

Theorem Consider the system (1) with assumptions (A5) and **(A6)** *(Linearly parameterized control)* the function $\Psi(x,\theta)$ may be written as

$$\Psi(x,\theta) = \Psi_0(x) + \Psi_1(x)\theta \quad (11)$$

for some known functions $\Psi_0(x)$ and $\Psi_1(x)$.

Assume that there exists a function $\beta_1 : \mathbb{R}^n \to \mathbb{R}^m$ such that **(A7)** *(Realizability)* $(\partial\beta_1/\partial x)f_*(x)$ is independent of the unknown parameters.

(A8) *(Manifold attractivity and trajectory boundedness)* All trajectories of the error system

$$\dot{x} = f_*(x) + g(x)\Psi_1(x)z \quad (12)$$

$$\dot{z} = \left[\frac{\partial\beta_1}{\partial x} g(x)\Psi_1(x)\right] z \quad (13)$$

$$(14)$$

are bounded and satisfy

$$\lim_{t\to\infty} g(x(t))\Psi_1(x(t))z(t) = 0.$$

Then, (1) is adaptively I & I stabilizable with the parameter update law given by

$$\beta_2(x) = -\frac{\partial\beta_1}{\partial x} f_*(x). \quad (15)$$

3. ACTIVE SUSPENSION SYSTEMS

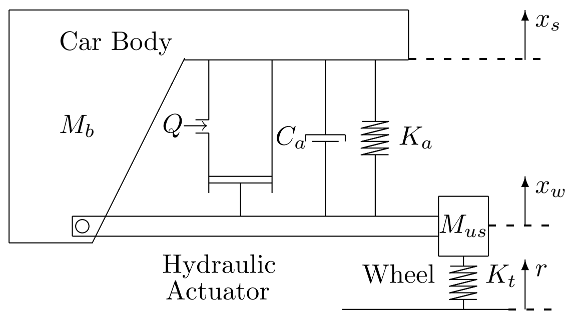

Fig. 1. A quarter-car model for active suspension design

Consider a quarter-car model of an active suspension system consisting of a single wheel and axle connected to the quarter portion of the car body through a passive spring-damper and a hydraulic actuator, as shown in Figure 1. The equations of motion of the system are

$$M_b\ddot{x}_s + K_a(x_s - x_w) + C_a(\dot{x}_s - \dot{x}_w) - u_a = 0$$
$$M_{us}\ddot{x}_w + K_a(x_w - x_s) + C_a(\dot{x}_w - \dot{x}_s) + K_t(x_w - r) + u_a = 0 \quad (16)$$

where M_b and M_{us} are the masses of car body and wheel, x_s and x_w are the displacements of car body and wheel, K_a and K_t are the spring coefficients, C_a is the damper coefficient, r is the road disturbance and u_a is the control force from the hydraulic actuator, which is given by

$$u_a = AP_L$$

where A is the piston area and P_L is the pressure drop across the piston.

The pressure drop P_L is related to the hydraulic load flow Q and the spool valve displacement x_v according to the equations (Merritt, 1967)

$$\frac{V_t}{4\beta_e}\dot{P}_L = Q - C_{tp}P_L - A(\dot{x}_s - \dot{x}_w) \quad (17)$$

$$Q = \mathrm{sgn}\,[P_s - \mathrm{sgn}(x_v)P_L] \times C_d w x_v \sqrt{\frac{1}{\rho}\,[P_s - \mathrm{sgn}(x_v)P_L]} \quad (18)$$

where V_t is the total actuator volume, β_e is the effective bulk modulus, C_{tp} is the total leakage coefficient of the piston, C_d is the discharge coefficient, w is the spool valve area gradient, ρ is the hydraulic fluid density and P_s is the supply pressure.

The spool valve displacement x_v is controlled by the voltage u to the servovalve, which is modeled by a first-order differential equation

$$\dot{x}_v = \frac{1}{\tau}(-x_v + u) \qquad (\tau > 0) \tag{19}$$

Let $x_1 = x_s$, $x_2 = \dot{x_s}$, $x_3 = x_w$, $x_4 = \dot{x}_w$, $x_5 = P_L$ and $x_6 = x_v$, we obtain the state equations of the system as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{M_b}[K_a(x_1 - x_3) + C_a(x_2 - x_4) - Ax_5] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{M_{us}}[K_a(x_1 - x_3) + C_a(x_2 - x_4) \\
&\quad -K_t(x_3 - r) - Ax_5] \\
\dot{x}_5 &= -\beta x_5 - \alpha A(x_2 - x_4) + \gamma x_6 w_3 \\
\dot{x}_6 &= \frac{1}{\tau}(-x_6 + u)
\end{aligned} \tag{20}$$

where $\alpha = \dfrac{4\beta_e}{V_t}$, $\beta = \alpha C_{tp}$, $\gamma = \alpha C_d w \sqrt{\frac{1}{\rho}}$ and

$$w_3 = \text{sgn}\,[P_s - \text{sgn}(x_6)x_5]\,\sqrt{[P_s - \text{sgn}(x_6)x_5]} \tag{21}$$

The essential objectives of the active suspension design are to reduce vertical car body acceleration for passenger comfort and to increase the tire-to-road contact for handling and safety. Other considerations include suspension travel and power consumption. In (Lin and Kanellakopoulos, 1997) the regulated variable is proposed as

$$z_1 = x_1 - \bar{x}_3 \tag{22}$$

where $\overline{x}_3$ is the output of the nonlinear filter

$$\dot{\bar{x}}_3 = -(\varepsilon + \kappa_1\varphi(\zeta))(\bar{x}_3 - x_3) \tag{23}$$

This nonlinearity is intentionally introduced so that the system can emphasize different objectives under different operating conditions.

In (23), $\varepsilon > 0$ and $\kappa_1 \geq 0$ are constants, $\zeta = x_1 - x_3$ is the suspension travel, and the nonlinear function $\varphi(\zeta)$ is defined as

$$\varphi(\zeta) = \begin{cases} \left(\dfrac{\zeta - m_1}{m_2}\right)^4 & \text{if } \zeta > m_1 \\ 0 & \text{if } |\zeta| \leq m_1 \\ \left(\dfrac{\zeta + m_1}{m_2}\right)^4 & \text{if } \zeta < -m_1 \end{cases} \tag{24}$$

where $m_1 \geq 0$ and $m_2 > 0$.

For comparison to our design, we shall consider the following control laws:

- the backstepping control law in (Lin and Kanellakopoulos, 1997), which is computed in four steps resulting in

$$u = \frac{\tau}{w_3}\alpha_4 \tag{25}$$

where the fourth stabilizing function α_4 is

$$\begin{aligned}
\alpha_4 = &-c_4 z_4 - \mu\gamma z_3 - b_4 h_4^2 z_4 + \frac{1}{\tau}x_6 w_3 \\
&+\frac{1}{2|w_3|}|x_6|w_2 - g_4
\end{aligned} \tag{26}$$

- the adaptive backstepping control law in (Lin and Kanellakopoulos, 1996), in which $\theta = \alpha A$ is taken to be the unknown parameter. The update law is given by

$$\mu\dot{\hat{\theta}} = \Gamma\tau_4, \tag{27}$$

where $\Gamma > 0$ is the adaptation gain and τ_4 is the tuning function. The control law is obtained by substituting $\hat{\theta}$ for the unknown parameter θ.

For more details about various terms in (25)-(26) and (27), see (Lin and Kanellakopoulos, 1997) and (Lin and Kanellakopoulos, 1996), respectively.

4. I & I CONTROLLER DESIGN

The immersion and invariance design is performed in two steps. In the first step, we choose a target system and design a stabilizing controller for this reduced-order model. In the second step, we modify the control law obtained in the first step to get the immersion and invariance controller for the full-order model (20).

The target system is chosen as

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -\frac{1}{M_b}[K_a(\xi_1 - \xi_3) + C_a(\xi_2 - \xi_4) - u_a] \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= \frac{1}{M_{us}}[K_a(\xi_1 - \xi_3) + C_a(\xi_2 - \xi_4) - K_t\xi_3 - u_a] \\
\dot{\bar{\xi}}_3 &= -(\varepsilon_0 + \kappa_1\varphi(\zeta))(\bar{\xi}_3 - \xi_3)
\end{aligned} \tag{28}$$

We use the backstepping technique to design a stabilizing control u_a for the target system as

$$\begin{aligned}
u_a = M_b\Big\{ &-(c_1 + c_2)z_2 - (\varepsilon_0 + \kappa_1\varphi(\zeta))z_1 \\
&+(c_1^2 - 1 + c_1(\varepsilon_0 + \kappa_1\varphi(\zeta)))z_1 \\
&-\kappa_1\frac{d\varphi}{d\zeta}(\xi_2 - \xi_4)\zeta\Big\} + K_a(\xi_1 - \xi_3) + C_a(\xi_2 - \xi_4)
\end{aligned}$$

where c_1, c_2 are positive constants and $z_1 = \xi_1 - \bar{\xi}_3$, $z_2 = \xi_2 - \alpha_1$, and

$$\alpha_1 = -c_1 z_1 - (\varepsilon_0 + \kappa_1\varphi(\zeta))\zeta \tag{29}$$

Next, the mapping $x = \pi(\xi)$ is chosen to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ Ax_5 \\ x_6 w_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ u_a(\xi_1, \xi_2, \xi_3, \xi_4) \\ u_b(\xi_1, \xi_2, \xi_3, \xi_4) \end{bmatrix} \quad (30)$$

where u_b is to be determined shortly. The manifold $x = \pi(\xi)$ can be implicitly described by

$$\phi(x) = \begin{bmatrix} Ax_5 - u_a \\ x_6 w_3 - u_b \end{bmatrix} = 0 \quad (31)$$

Define

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} Ax_5 - u_a \\ x_6 w_3 - u_b \end{bmatrix} \quad (32)$$

and choose

$$u_b = \frac{A\alpha}{\gamma}(x_2 - x_4) + \frac{1}{\gamma A}(\beta u_a + \dot{u}_a) \quad (33)$$

we obtain

$$\ddot{\eta}_1 + \beta\dot{\eta}_1 - \gamma A\dot{\eta}_2 = 0$$

Now, let

$$\dot{\eta}_2 = -\frac{1}{\gamma A}(k_1\dot{\eta}_1 + k_2\eta_1) \quad (34)$$

we get

$$\ddot{\eta}_1 + (\beta + k_1)\dot{\eta}_1 + k_2\dot{\eta}_1 = 0 \quad (35)$$

If we want to place the poles of the off-the-manifold dynamics (35) at $-p_1$ and $-p_2$, we can solve for k_1 and k_2 from

$$p_1 + p_2 = -(\beta + k_1), \quad p_1 p_2 = k_2$$

Finally, the control law is given by

$$u = \left(\frac{\tau}{w_3}\right)\left\{\frac{x_6 w_3}{\tau} + \frac{1}{2|w_3|}|x_6| w_2 - \frac{1}{\gamma A}(k_1\dot{\eta}_1 + k_2\eta_1) + \dot{u}_b\right\} \quad (36)$$

where

$$w_2 = -\beta x_5 - \alpha A(x_2 - x_4) + \gamma x_6 w_3 \quad (37)$$

5. ADAPTIVE I & I CONTROLLER DESIGN

We select $\theta_* = \alpha A$ to be the unknown parameter to be estimated as in (Lin and Kanellakopoulos, 1996). The target dynamics is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{M_b}[K_a(x_1 - x_3) + C_a(x_2 - x_4) - Ax_5] \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{M_{us}}[K_a(x_1 - x_3) + C_a(x_2 - x_4) \\
&\quad - K_t(x_3 - r) - Ax_5] \\
\dot{x}_5 &= -\beta x_5 - \theta_*(x_2 - x_4) + \gamma x_6 w_3 \\
\dot{x}_6 &= \frac{1}{\tau}(-x_6 + u(x, \theta_*))
\end{aligned}$$

where $u(x, \theta_*)$ is the backstepping control law (25) which stabilizes the target system when θ_* is known.

The implicit manifold condition (A3) in this case is

$$\phi(x, \hat{\theta}) = \hat{\theta} - \theta_* + \beta_1(x) = 0$$

and the off-the-manifold coordinate is

$$z = \hat{\theta} - \theta_* + \beta_1(x)$$

Its derivative is

$$\dot{z} = \beta_2(x) + \frac{\partial\beta_1}{\partial x}[f_0(x) + f_1(x)\theta_* + g(x)u]$$

Hence, the parameter update law is chosen as

$$\beta_2(x) = -\frac{\partial\beta_1}{\partial x}\left(f_0(x) + f_1\left[\hat{\theta} + \beta_1(x)\right] + g(x)u\right)$$

where

$$f_0(x) = \begin{bmatrix} x_2 \\ -\frac{1}{M_b}[h(x) - Ax_5] \\ x_4 \\ \frac{1}{M_{us}}[h(x) - K_t(x_3 - r) - Ax_5] \\ -\beta x_5 + \gamma x_6 w_3 \\ -\frac{1}{\tau}x_6 \end{bmatrix}$$

and

$$f_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -(x_2 - x_4) \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix}$$

$$h(x) = K_a(x_1 - x_3) + C_a(x_2 - x_4)$$

If we choose

$$\beta_1(x) = k\,\mathrm{sgn}[-(x_2 - x_4)]x_5$$

where $k > 0$ is a constant, then the parameter update law becomes

$$\dot{\hat{\theta}} = -k\,\mathrm{sgn}[-(x_2 - x_4)]\big(-\beta x_5 - (x_2 - x_4) \times (\hat{\theta} + \beta_1(x)) + \gamma x_6 w_3\big) \quad (38)$$

and the off-the-manifold dynamics is

$$\dot{z} = -[k(x_2 - x_4)\,\mathrm{sgn}(x_2 - x_4)]z \quad (39)$$

From (39), it can be seen that z is bounded and converges to zero as $t \to \infty$.

6. SIMULATION RESULTS

The computer simulations are performed using parameter values given in (Lin and Kanellakopoulos, 1997) as follows:

$$\alpha = 4.515 \times 10^{13} \ \ \mathrm{N/m^5}$$
$$\beta = 1 \ \ \mathrm{s^{-1}}$$
$$\gamma = 1.545 \times 10^{9} \ \ \mathrm{N/(m^{5/2}kg^{1/2})}$$
$$\tau = 1/30 \ \ \mathrm{s}$$
$$P_s = 1500 \ \ \mathrm{psi}$$
$$A = 3.35 \times 10^{-4} \ \ \mathrm{m^2}$$

As in (Lin and Kanellakopoulos, 1997), we use $\mu = 10^{-7}$ to rescale the state x_5, i.e. $\overline{x}_5 = \mu x_5$, to improve numerical accuracy and we modify w_3 in the denominator (only) of the control laws (25) and (36) to be

$$w_3 = \begin{cases} 0.5 \text{ if } & 0 \le w_3 \le 0.5 \\ -0.5 \text{ if } & -0.5 \le w_3 < 0 \end{cases}$$

to avoid division by zero.

We also assume the following limits:

- Suspension travel limits: ± 8 cm.
- Spool valve displacement limits: ± 1 cm.

and let the road disturbance r be

$$r = \begin{cases} a(1 - \cos 8\pi t), & 0.5 \le t \le 0.75 \\ 0, & \text{otherwise} \end{cases}$$

For $a = 0.04$, the height of the bump is equal to 8 cm.

The design parameters are

$$\varepsilon_0 = 1.5,\ m_1 = 0.055,\ m_2 = 0.005, \kappa_1 = 0.0125,$$
$$b_3 = b_4 = 0.01, c_1 = c_2 = c_3 = c_4 = 200,$$
$$p_1 = p_2 = 1200$$

We compare the results between the immersion and invariance control law (36) and the backstepping control law (25) in the following cases:

(1) Fig. 2 is the nominal case.
(2) Fig. 3 is when α is increased by 5%.
(3) Fig. 4 is when α is decreased by 5%.

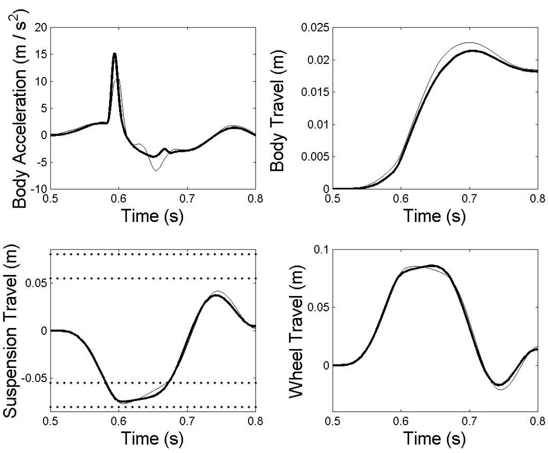

Fig. 2. Comparison of immersion and invariance (thick line) and backstepping (thin line) controllers in the nominal case

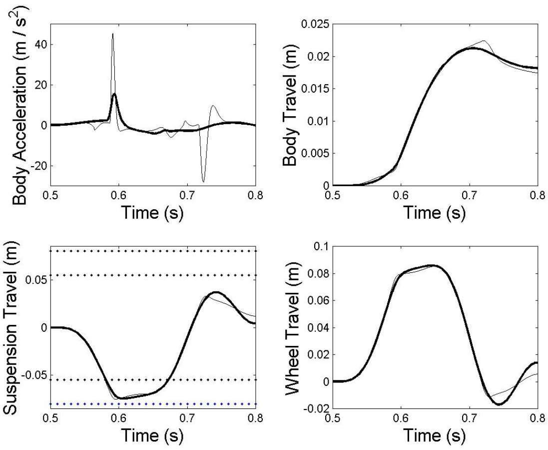

Fig. 3. Comparison of immersion and invariance (thick line) and backstepping (thin line) controllers when α is increased by 5%

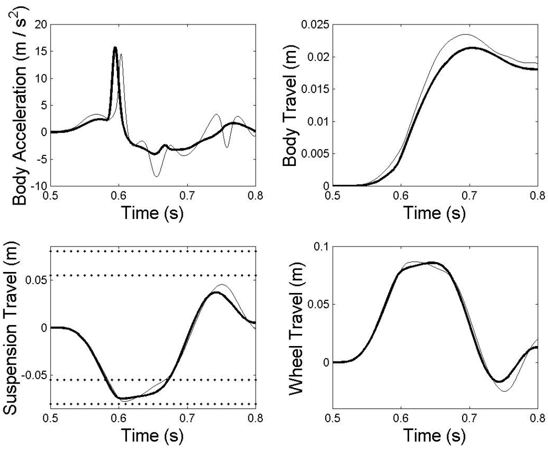

Fig. 4. Comparison of immersion and invariance (thick line) and backstepping (thin line) controllers when α is decreased by 5%

In Figure 2, it is readily seen that the immersion and invariance controller can stabilize the full-order system as well as the backstepping controller, with somewhat larger overshoot but smaller undershoot in the car body acceleration. The body travel, the suspension travel and the wheel travel are similar in both cases.

When α is increased by 5%, the car body acceleration becomes more oscillatory when the backstepping controller is used, but remains essentially the same when the immersion and invariance controller is employed. On the other hand, when α is decreased by 5%, the body acceleration exhibits larger overshoot and undershoot under the backstepping control, but becomes flatter under the immersion and invariance control. Notice that the suspension travel remains within the required limits in all cases considered.

The results show that the immersion and invariance controller can stabilize the full-order system as well as the backstepping controller in the nominal case, but is more robust to some parameter changes in the system.

In the case of adaptive I & I control, we compare the results between the immersion and invariance parameter update law (38) and the tuning function parameter update law (27) in the following cases:

(1) Fig. 5 is when α is unknown and $\hat{\theta}(0)$ is greater than the actual value θ_* by 10 %.
(2) Fig. 6 is when α is unknown and $\hat{\theta}(0)$ is less than the actual value θ_* by 10 %.

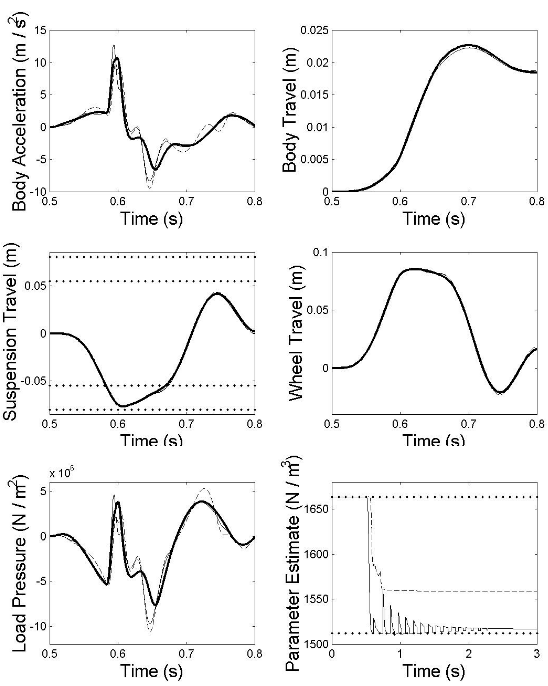

Fig. 5. Comparison of immersion and invariance (thin line) and tuning function (dashed line) update laws with the known parameter case (thick line) when $\hat{\theta}(0)$ is greater than θ_* by 10 %

From Figures 5 and 6, we can see that the car body acceleration in the case of the immersion and invariance update law has smaller overshoots and undershoots than the case of the tuning function update law, while the suspension travel and the car body and wheel positions in both cases are very similar to the known parameter case. Also, the I & I parameter estimator gets closer to the true parameter value than the tuning function parameter estimator. Therefore, it can be concluded that for the active suspension system considered in this work, the immersion and invariance parameter update law performs better than the tuning function update law.

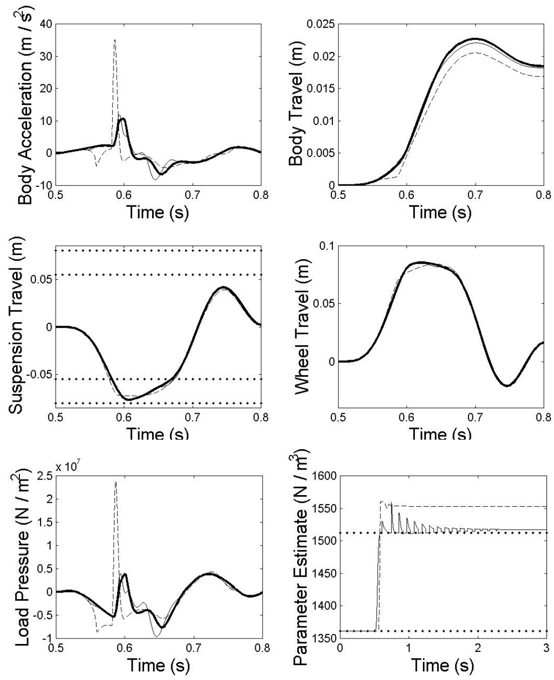

Fig. 6. Comparison of immersion and invariance (thin line) and tuning function (dashed line) update laws with the known parameter case (thick line) when $\hat{\theta}(0)$ is less than θ_* by 10 %

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