EFFICIENT COMPUTATION OF A GUARANTEED ROBUSTNESS MARGIN

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Abstract: A computationally very efficient method is proposed to compute a guaranteed robustness margin, by combining methods which compute a guaranteed μ upper bound over a frequency interval and those which eliminate frequency intervals inside which μ is less than a treshold. The μ upper bound is computed at several frequency points simultaneously either using the LMI Control Toolbox or the μ Analysis and Synthesis Toolbox. The method compares very favorably with existing μ methods. The algorithm is available on the web as part of the new version of the Skew Mu Toolbox (SMT).

Keywords: robustness analysis,structured singular value (μ),repeated parametric uncertainties,flexible structures.

1. INTRODUCTION

Consider a closed loop subject to neglected dynamics and parametric uncertainties. The issue is to compute (an estimate of) the robustness margin, i.e. the maximal size of model uncertainties for which closed loop stability or performance is still guaranteed. This robustness margin is obtained as the inverse of the maximal structured singular value (s.s.v.) $\mu(\omega)$ over the frequency domain $[0,\ +\infty)$.
Since computing the exact value of μ is known to be NP hard, a guaranteed robustness margin is usually obtained by computing an upper bound of $\mu(\omega)$ on $[0,\ +\infty)$. The best known upper bound is the polynomial-time one proposed in Fan et al. (1991) and Young et al. (1995). Nevertheless two difficulties remain:

- Computing the optimal value of this upper bound, which is obtained as the solution of an LMI problem, can become cumbersome in the case of largely repeated parametric uncertainties. As a consequence the routine *mu.m* of the μ Analysis and Synthesis Toolbox proposes to compute a suboptimal value of this upper bound with a reasonable computational requirement. An option enables to tune the trade-off between this requirement and the accuracy of the upper bound (see Young et al. (1995)).
- Nevertheless the routine *mu.m* is to be called at each point of a frequency gridding, and more generally the μ upper bound is usually computed on a frequency gridding of $[0,\ +\infty)$. This solution may not be reliable, especially in the case of flexible systems, since it is impossible to guarantee that the critical μ peak was not missed between two points of the frequency gridding. Moreover computing the μ upper bound on a fine frequency gridding, so as to reduce the probability to miss a peak, can be time-consuming. In this context the solution is either to compute a guaranteed μ upper bound β_i over a union of frequency intervals $[\omega_i,\ \omega_{i+1}]$, i.e. $\mu(\omega) \leq \beta_i\ \ \forall\omega \in [\omega_i,\ \omega_{i+1}]$, or to eliminate frequency intervals inside which it can be guaranteed that $\mu(\omega)$ is less than a given threshold (see

Sideris (1992); Magni et al. (1999); Ferreres (1999); Ferreres and Biannic (2001); Ferreres et al. (2003); Ly et al. (1998)).

In the spirit of Ferreres et al. (2003) a computationally very efficient method is described here to compute a guaranteed robustness margin: methods for computing a guaranteed μ upper bound over a frequency interval and for eliminating frequency intervals inside which μ is less than a treshold are merged into a single algorithm which is available on the web as a part of the free Skew Mu Toolbox (see Ferreres and Biannic (2004)).
The paper is organised as follows. Two preliminary technical results are presented in section 2, which form the basis of the algorithm described in section 3. In section 4, different ways of using the algorithm are first presented on a simple system. Then, a challenging example is proposed to compare computational-time and accuracy with other μ methods. Concluding remarks end the paper.

2. TWO PRELIMINARY TECHNICAL RESULTS

In this section, the two main technical results which form the basis of our proposed algorithm are presented. We first consider the problem of computing a μ upper-bound which is simultaneously valid at several frequency points. Then, as already proposed in Ferreres et al. (2003), we briefly describe a simple test to compute a set of frequency intervals inside which the μ upper-bound is guaranteed to remain below a prescribed level.

2.1 A fast μ upper-bound computation at several frequency points

In Ferreres et al. (2003), this problem is solved by an LMI approach. Here, to reduce the computational-time, a specific approach is developed using the routine *mu.m* of the μ Analysis and Synthesis Toolbox.

Let us consider the standard interconnection structure $M(s) - \Delta$ where $M(s)$ is a stable LTI system and Δ a mixed model perturbation containing real scalars (associated to parametric uncertainties) and full complex blocks (which represent neglected dynamics). The mixed-μ upper-bound proposed in Young et al. (1995) is summarized in the following proposition :

Proposition 1. If there exist scaling matrices $D \in \mathcal{D}$ and $G \in \mathcal{G}$ (see Young et al. (1995) for a definition of these two sets) and a positive scalar β such that :

$$\overline{\sigma}\left(F^{-1/4}\left(\frac{DM(j\omega)D^{-1}}{\beta} - jG\right)F^{-1/4}\right) \leq 1 \quad (1)$$

with $F = I + G^2$, then $\mu(M(j\omega)) \leq \beta$.

For any given frequency point, the problem of minimizing β with respect to scaling variables D and G is solved (possibly sub-optimally) quite efficiently with the routine *mu.m*. However, by such an approach, the scalings might not be "robust" versus frequency variations, especially near flexible modes of the system. To avoid this problem and thus improve the performance of the frequency elimination technique (see the next subsection), we propose to compute an upper-bound β and associated scalings which are simultaneously valid for two frequency points ω_1 and ω_2.

The proposed approach is simple. Let us first denote $M_i = M(j\omega_i)$ and $M_0 = (M_1 + M_2)/2$. Then, β_0, D and G can be obtained by a standard use of the routine *mu.m*. Finally, it remains to increase β_0 until the singular value constraint (1) is satisfied for $M = M_1$ and $M = M_2$. As stated by the following proposition (whose proof is given in appendix), the determination of β reduces to generalized eigenvalues computation :

Proposition 2. With the notations of proposition 1 in mind, let us denote $U_i = F^{-1/4}DM_iD^{-1}F^{-1/4}$ and $V = -jG(I + G^2)^{-1/2}$. Let us further define $\hat{\beta}$ as the largest positive scalar such that :

$$det\left(\begin{bmatrix} 0 & U_i \\ U_i^* & 0 \end{bmatrix} + \hat{\beta}\begin{bmatrix} I & V \\ V^* & I \end{bmatrix}\right) = 0 \quad (2)$$

then, for all $\beta \geq \hat{\beta}$,

$$\overline{\sigma}\left(F^{-1/4}\left(\frac{DM_iD^{-1}}{\beta} - jG\right)F^{-1/4}\right) \leq 1 \quad (3)$$

Remark 3. As already pointed out, all solutions of equation (2) can be easily obtained as the solutions of a generalized eigenvalue problem. Furthermore, note that these eigenvalues are all real.

Remark 4. By a continuity argument, if ω_1 and ω_2 are sufficiently close, then $\hat{\beta}$ also remains close to β_0. Conversely, a significant gap between $\hat{\beta}$ and β_0 means that ω_1 and ω_2 are not close enough, which might lead to a conservative result.

2.2 Frequency elimination technique

The frequency elimination technique which we propose here is essentially based on an extension of the scaling validation method depicted in Ferreres et al. (2003). Following the above procedure, suppose β, D and G have been computed and are

simultaneously valid for ω_1 and ω_2. It now remains to check wether the scalings are valid on the whole segment $[\omega_1\ ,\ \omega_2]$. The following technical result, adapted from Ferreres et al. (2003), provides an exact answer to the above question. Moreover, without any further significant computations, it also enables to detect additional intervals, outside $[\omega_1\ ,\ \omega_2]$ in which scaling matrices are valid as well.

Lemma 5. Given ω_1, ω_2, β, D and G as introduced above, set $\omega_0 = (\omega_1 + \omega_2)/2$ and compute the augmented matrix $\mathcal{H}(\omega_0)$ as detailed in proposition 2.5 of Ferreres et al. (2003). Denote $\eta_1, \ldots, \eta_q$ the **real** eigenvalues of $\mathcal{H}(\omega_0)$, then : $\forall \hat{\omega}_k = \omega_0 + \frac{1}{\eta_k}\ ,\ \ k = 1 \ldots q\ ,\ \exists i$ such that :

$$\sigma_i \left(F^{-1/4} \left(\frac{D\hat{M}D^{-1}}{\beta} - jG \right) F^{-1/4} \right) = 1 \quad (4)$$

with $\hat{M} = M(j\hat{\omega}_k)$

The above lemma can be easily used to compute a set of frequency intervals inside which scaling matrices are valid. Suppose that the critical frequency points $\hat{w}_k$ are sorted in ascending order and consider the frequency segment $I_k = [\hat{w}_k\ ,\ \hat{w}_{k+1}]$. Select a frequency point inside I_k. According to the lemma it is then readily checked that if the scaling matrices are valid for this point, they necessarily remain valid on the whole segment. Several intervals may thus be obtained at a very low cost, by iteratively considering all segments I_k.

Remark 6. In Ferreres et al. (2003), a similar approach is used to check the validity of the scaling matrices. However, only one segment is evaluated at each iteration. Regarding computational-time, our approach will then be more efficient, since a larger number of frequency intervals may be eliminated at each iteration.

3. DESCRIPTION OF THE ALGORITHM

We are now ready to describe the main steps of our proposed algorithm which essentially consists in iterating upper-bound computations (using sub-optimal scaling matrices which are simultaneously valid at two frequency points) and a powerful frequency-elimination technique. The basic algorithm is presented first, then some possible extensions are described to improve the accuracy.

3.1 Basic algorithm

Define an initial frequency gridding $\{\omega_1, \omega_2, \ldots, \omega_N\}$ *and generate the associated set of intervals :*

$$\mathcal{I} = \{I_k = [\omega_k\ ,\ \omega_{k+1}]\ , k = 1, \ldots, N-1\}$$

then repeat the following steps while $\mathcal{I}$ *is not empty :*

(1) Consider an interval I_k of $\mathcal{I}$ and denote $\overline{\omega}_k = (\omega_k + \omega_{k+1})/2$,
(2) Using the approach of subsection 2.1, compute β_k, D_k and G_k which are simultaneously valid for ω_k and ω_{k+1},
(3) If the above computation reveals that ω_k and ω_{k+1} were not sufficiently close (according to Remark 4), add the central point $\overline{\omega}_k$ to the gridding, update $\mathcal{I}$ and go back to step (1). Otherwise, update β_{max} :

$$\beta_{max} = max(\beta_{max}, \beta_k)$$

(4) Using method of subsection 2.2, compute the set $\mathcal{J}$ of frequency intervals inside which β_{max}, D_k and G_k are valid,
(5) Update $\mathcal{I}$ by removing all intervals of $\mathcal{J}$:

$$\mathcal{I} \leftarrow \mathcal{I} - \mathcal{J}$$

Remark 7. From the above description of the algorithm, it clearly appears that the initial frequency gridding is automatically refined if required. Consequently, it is useless to initialize the procedure with a fine gridding.

Remark 8. The accuracy of the algorithm can be easily tuned by adapting a tolerance parameter in step 3. In that step indeed, it is decided wether ω_k and ω_{k+1} are sufficiently close or not. As already detailed, to ensure the validity of the scaling matrices at two frequency points simultaneously, the upper-bound β_k is obtained by increasing the nominal value (see subsection 2.1). When the expansion factor remains close to 1 this clearly means that the method is not hardly more conservative than a standard gridding-based approach. Then, a natural way to control the accuracy of the μ upper-bound consists in observing the expansion factor and adding a frequency point (to reduce the size of the segments) when it gets too large.

Remark 9. This algorithm significantly differs from the previous one proposed in Ferreres et al. (2003), since :

- the computation of the scaling matrices is no longer based on LMI-optimization, which permits to consider more challenging problems with highly repeated uncertainties,
- the frequency elimination technique is more efficient as it now fully exploits the result of lemma 5. This is especially useful for high-order systems where the number of eliminated frequency segments at each iteration may be significant.

Nevertheless, the overall architecture is not so different. As an important point, it should be emphasized that in both algorithms, sub-optimal scaling matrices are used. This property plays a key-role in the convergence proof of the algorithm proposed in Ferreres et al. (2003). Consequently, the convergence of this new algorithm can also be established using the same guidelines.

Remark 10. The proposed algorithm is essentially devoted to the computation of a guaranteed maximum value of a μ upper-bound over a frequency range. It is not supposed to provide any accurate information regarding secondary peaks of the μ plot. This information can be obtained by computing more accurately a shape of the μ upper-bound as a function of frequency. This implies slight modifications of step 4. First, β_{max} should be replaced by β_k. Then, as proposed in Ferreres et al. (2003), only the detection of the main interval (containing $\overline{\omega}_k$) should be activated. Note that such modifications might significantly increase the number of iterations and then the computational burden.

3.2 Further improving accuracy

The proposed method to compute a μ upper-bound and associated scaling matrices which are simultaneously valid at two frequency points might give pessimistic results mainly because the computation - based on $mu.m$ - is performed in two steps. As detailed in Ferreres et al. (2003), the problem is clearly convex and can be solved (at least theoretically!) - more optimally - using an LMI solver. But remember that the computations can become cumbersome for too largely repeated uncertainties.

In order to optimize the tradeoff between accuracy and computational-time, a combined approach could be proposed. This one consists in applying first the standard algorithm described above and then in refining computations by an LMI approach on the most critical frequency segments.

4. APPLICATIONS

The above algorithm has been implemented in a Matlab routine **mu_max_3.m** and is available as part of the new version of SMT Toolbox (see Ferreres and Biannic (2004)). This routine implements the basic algorithm with several options in order to improve either the computational-time or the accuracy. Optionally, it also permits to perform computations on the borderline of a truncated sector instead of considering the classical imaginary axis. Thus, the robustness of a modal performance can be evaluated. For more details, the reader is referred to the help file of this function and the pdf document to be downloaded with the Toolbox.

The first subsection illustrates the different ways of using the algorithm on a very simple example, while the second one compares the computational-time and accuracy with other existing μ methods on a challenging example. Both examples are available with the Toolbox.

4.1 An illustrative example

In this subsection, we consider a very simple fourth-order system initially proposed in DeGaston and Safonov (1988) with three non-repeated **real** uncertainties. The purpose here is just to illustrate four different ways of parametrizing the algorithm according to the objectives to be reached :

- **Parametrization 1:** This is the most standard way to parametrize the algorithm. The objective here is to make the frequency elimination technique as efficient as possible to reduce computational time. This means, that all possible intervals are eliminated at each iteration. Moreover, to improve the accuracy, after a preliminary solution is given by fast iterations (using $mu.m$), the algorithm switches to slower (and potentially more accurate) LMI-based iterations only on critical segments,
- **Parametrization 2:** This parametrization is the same as the first one, but now a μ lower-bound is provided, which permits to further improve the efficiency of frequency elimination,
- **Parametrization 3:** Here, we are interested in computing a precise shape of the upper-bound as a function of frequency. For this purpose the parametrization is adapted to restrict frequency elimination only inside the frequency segment for which an upper-bound has been computed. Moreover the upper-bound is evaluated by LMI optimization,
- **Parametrization 4:** In this last case, the objective is totally different. Let be given a test value μ_T, we would like to prove that $\overline{\mu} \leq \mu_T$. The parametrization of the algorithm will be the same as the second one. Yet the iterations here will progress differently, and the switch to LMI optimization might not be necessary.

For the ease of comparison between the above parametrizations the same initial frequency gridding was chosen in each case. We considered 25 linearly-spaced points between 0 and 100 rad/s. A μ lower-bound has been computed with the help

of a specialized routine provided in the Toolbox (see Ferreres and Biannic (2001)). We obtained : $\underline{\mu} = 0.2926$. The test value in the fourth call of the routine was $\mu_T = 0.31$.

Table 1 and figure 1 summarize the results which were obtained by the different parametrizations. All computations were performed on a SunBlade 1500 Workstation. As expected the most time-consuming approach is the third, while the last one is very fast since no LMI step is required here. The third column contains two numbers separated by a '+'. The first one corresponds to the number of fast iterations (based on $mu.m$) while the second is associated to LMI iterations. In the fifth column we have indicated the relative gap between the upper and lower bounds, which thus permits to estimate the conservatism.

Table 1. Comparison of the 4 parametrizations

Par.	CPU time	Iter.	$\overline{\mu}$	Gap
1	$1.54\,sec$	$20+3$	0.2934	0.27%
2	$0.77\,sec$	$7+3$	0.2930	0.14%
3	$6.52\,sec$	$108+3$	0.2936	0.34%
4	$0.43\,sec$	$7+0$	0.3094	5.74%

Interestingly, when a μ lower-bound is provided the number of iterations can be significantly reduced without affecting the precision of the result. On this example, the fifth column shows that the precision is even higher. As expected, in the fourth case, this gap is larger since the algorithm stops as soon as an upper-bound less than μ_T is found.

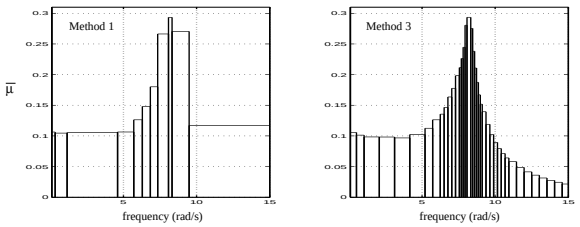

Fig. 1. μ Upper-bounds vs. frequency

4.2 Further evaluation on a challenging example

In this subsection, a challenging example is considered. The system to be analysed exhibits 30 highly flexible modes ranging from 1 to $30\,rad/s$, whose damping ratio lies between 10^{-5} and 10^{-1}. Furthermore, the uncertainty block contains 4, but highly repeated parametric uncertainties :

$$\Delta = \mathbf{blkdiag}\left(\delta_1 I_{10}\,,\ \delta_2 I_{10},\ \delta_3 I_8,\ \delta_4 I_2\right)$$

On such a high-order system, methods which consist in extracting frequency and considering it as an additional real uncertain parameter cannot be applied. The repetition of frequency would indeed be much too high. Moreover, faster approaches combining LMI-based computations of sub-optimal scaling matrices and a frequency elimination technique, still not work here because of high repetitions of the parametric uncertainties. On this example, for each frequency segment, the computation of *D-G* matrices involves the resolution of LMIs with 269 variables, which might take more than $20\,min$. Consequently, the approximate global computational-time (30 iterations at least) would certainly exceed 10 hours.

Comparisons on a truncated system For the ease of comparison, a truncated system is first considered. The simplification consists in removing the first two uncertainties, so that the new system still has 60 states, but a reduced-size uncertainty block :

$$\Delta_r = \mathbf{blkdiag}\left(\delta_3 I_8,\ \delta_4 I_2\right)$$

A μ lower-bound (using the approach proposed in Ferreres and Biannic (2001)) is first computed in order to evaluate the conservatism of the techniques : $\underline{\mu} = 37.7278$. Then, four techniques will be compared :

- **Technique 1 :** Standard application of the proposed algorithm without any LMI step,
- **Technique 2 :** Standard application of the proposed algorithm with LMI steps on the most critical segments,
- **Technique 3 :** LMI-based *D-G* computations combined with a "local" frequency-elimination technique. This method was proposed in Ferreres et al. (2003) and is also implemented in the Toolbox (**mu_max_1.m**),
- **Technique 4 :** Classical griddind-based approach. A very fine frequency gridding containing 1000 linearly-spaced points between 1 and $20\,rad/s$ is generated.

Table 2. Comparisons of techniques

Tec.	CPU time	Freq.	$\overline{\mu}$
1	$6\,sec$	[12.0017 , 12.0020]	39.25
2	$69\,sec$	[12.0017 , 12.0017]	37.79
3	$997\,sec$	[12.0017 , 12.0017]	37.73
4	$70\,sec$	1.82	0.159 !

As expected (see table 2) the first technique requires a very low computational-time, without yet providing a too conservative upper-bound. The gap between upper and lower bounds remains indeed below 4%. As expected again, this gap may be further reduced using LMI steps on the critical frequency segments. By this approach, the computational-time is higher but remains reasonable. Note here that the precision on the frequency segment where the maximum was reached is excellent. The third technique reveals even more accu-

rate (upper and lower bounds are now extremely close!) but also much more demanding. Finally, the fourth technique clearly failed. The proposed frequency gridding (containing 1000 points) was not fine enough for this system including numerous and poorly damped flexible modes. More precisely, let us compute (using $mu.m$) a μ upper-bound for $\omega_1 = 12.00164$ and $\omega_2 = 12.00165$. We respectively obtain $\overline{\mu}_1 = 3.24$ and $\overline{\mu}_1 = 38.6$. This means that a uniform frequency gridding between 1 and $100\,rad/s$, enabling the detection of a more reliable (but not guaranteed!) upper-bound should contain 10^7 points! The corresponding computational-time would then be approximately 200 hours!

Application on the full-complexity system Let us now come back to the initial system, on which our proposed algorithm (technique 1) is still applicable. The following results were obtained :

CPU time	Freq.	$\overline{\mu}$	$\underline{\mu}$
$20\,sec$	[12.0017 , 12.0038]	62.29	61.06

The gap between upper and lower bounds is around 2 %. Furthermore, the computational-time remains quite low despite the complexity of the problem.

5. CONCLUSION AND FUTURE WORK

In this paper, a new algorithm has been developed to compute a guaranteed μ upper-bound on a frequency range. Its efficiency (accuracy and computational-time) has been demonstrated on a challenging example. Such a technique could be very useful for example to evaluate the robustness of flight control systems on large flexible transport aircrafts. In a future work, the method should be adapted in order to handle time-varying uncertainties as well.

Appendix A. PROOF OF PROPOSITION 2

By a Shur complement, and noting that β is strictly positive, it is easily seen that the following matrix inequality :

$$\begin{bmatrix} 0 & U_i \\ U_i^* & 0 \end{bmatrix} + \beta \begin{bmatrix} I & V \\ V^* & I \end{bmatrix} > 0 \qquad \text{(A.1)}$$

is equivalent to :

$$\left(U_i/\beta + V\right)^* \left(U_i/\beta + V\right) < 1 \qquad \text{(A.2)}$$

By definition of the singular value, the above inequality may be rewritten :

$$\overline{\sigma}\left(U_i/\beta + V\right) < 1 \qquad \text{(A.3)}$$

which corresponds to inequality (3).

Noting that $\overline{\sigma}(V) < 1$, we observe that for sufficiently high values of β the inequality (A.1) is always satisfied. Let us now introduce β^*, as the largest real value of β for which the hermitian matrix in (A.1) becomes singular. The existence of a real β^* (to be obtained by a generalized eigenvalue computation) is guaranteed since the matrix in (A.1) is hermitian and moreover it is singular for $\beta = 0$. Thus, for all $\beta > \beta^*$, the inequality (A.1) is strictly satisfied. Recalling that (A.1) and (A.3) are equivalent, and finally invoking a continuity argument (to transform strict inequalities into non-strict ones) permit to conclude the proof. ■

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