STABILIZING STATIC OUTPUT FEEDBACK VIA COARSEST QUANTIZERS

Hernan Haimovich

hhaimo@ee.newcastle.edu.au
ARC Centre for Complex Dynamic Systems and Control
The University of Newcastle, Australia

Abstract: This paper deals with quadratic stabilization of linear systems via a static quantized state or output feedback. The measure of quantization density introduced by Elia and Mitter (2001) is considered and used to derive the coarsest state quantizer of a specific form that is able to quadratically stabilize the given system with respect to a given control Lyapunov function. This result is then employed to find the coarsest output quantizer that may be utilized to obtain a stabilizing static output feedback. By optimizing quantization density with respect to a parameter in the specific form of quantizers considered, Theorem 2.1 of Elia and Mitter (2001) is recovered by alternative means.

Keywords: quantized static output feedback, quadratic stabilization, quantization density, coarse quantizers, networked control systems.

1. INTRODUCTION

Systems involving quantization arise naturally in many areas of engineering, especially when digital implementations are involved. In recent years, especially motivated by control of systems over communication networks, different control schemes have been developed where a limited information constraint is imposed (Wong and Brockett, 1999; Brockett and Liberzon, 2000; Nair and Evans, 2000; Elia and Mitter, 2001). Elia and Mitter (2001) utilize a quadratic stabilization approach to design quantizers having minimum quantization density. The results of Elia and Mitter are also obtained in Fu and Xie (2003), via a sector bound approach.

The work in this paper is directly related with that of Elia and Mitter (2001) and Fu and Xie (2003). State quantizers of "parallel-hyperplane" (PH) form are considered. Such a quantizer is shown in Figure 1, where $x \in \mathbb{R}^n$ denotes the system state, $u \in \mathbb{R}$ is the control, $d \in \mathbb{R}^n$ is a direction vector and $\mathring{q}$ is a scalar quantizer. First, the coarsest PH quantizer that, for a *given* direction d and candidate quadratic Lyapunov function is able to stabilize the given system is derived. This result is then employed to find the coarsest output quantizer that may be utilized, for a given candidate quadratic Lyapunov function, to obtain a stabilizing *static* output feedback. By contrast, the output feedback strategies in Elia and Mitter (2001), and Fu and Xie (2003), are *dynamic*. Finally, we find the direction d with respect to which a state quantizer with minimum (more precisely, infimum) quantization density is achieved. This result recovers that in Elia and Mitter (2001, Theorem 2.1) by alternative means and thus may provide new insights into the generalization to multiple-input systems (Kao and Venkatesh, 2002; Elia and Frazzoli, 2002). The results in the current paper are of a theoretical nature and build upon the geometric approach to quadratic stabilization with quantizers, which was developed in Haimovich and Serón (2004).

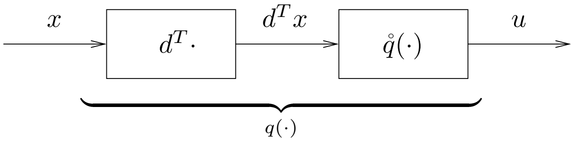

Fig. 1. PH state quantizer.

The remainder of the paper is organized as follows. Section 2 defines the different concepts employed and specifies the adopted approach. Section 3 contains a set of preliminary results that are needed in the subsequent sections. In Section 4, the coarsest PH quantizer that, for a given direction and candidate quadratic Lyapunov function is able to stabilize the given system is derived. Section 5 utilizes the result of Section 4 to obtain a stabilizing static output feedback that involves a quantizer which is coarsest for a given candidate quadratic Lyapunov function. Section 6 recovers the result in Elia and Mitter (2001, Theorem 2.1) and conclusions are drawn in Section 7.

2. PROBLEM STATEMENT

Consider a single-input discrete-time linear time-invariant system, defined by

$$x^+ = Ax + Bu, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and x^+ denotes the successor state. The following is assumed:

A1) The matrix A has at least one eigenvalue outside or on the unit circle.
A2) The pair (A, B) is stabilizable.

Two different scenarios are considered. First, quadratic stabilization of system (1) is analyzed, where the control u is based only on a quantized measurement of the *state*, that is, $u = q(x)$. Second, asymptotic stabilization of system (1) with a single output

$$y = Cx, \tag{2}$$

where $C \in \mathbb{R}^{1\times n}$, is analyzed. In this latter case, the control u can only be based on a quantized measurement of the *output*, that is, $u = \mathring{q}(y)$. The following definition is used throughout the paper.

Definition 1. (Quantizer). A *quantizer* q is a function $q : \mathbb{R}^p \to \mathbb{R}$ of the form

$$q(z) = u_i \text{ if } z \in \mathcal{R}_i, \quad \text{for } i \in \mathbb{Z}. \tag{3}$$

The sets $\mathcal{R}_i$ are called the *quantization regions* of q and u_i is called the *value* of q corresponding to $\mathcal{R}_i$. The sets $\mathcal{R}_i$ satisfy $\bigcup_{i\in\mathbb{Z}} \mathcal{R}_i = \mathbb{R}^p$ and $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ whenever $i \neq j$.

The term *state* quantizer is used to refer to a quantizer $q : \mathbb{R}^n \to \mathbb{R}$, that is, a quantizer that maps the state-space into $\mathbb{R}$. Similarly, the term *output* quantizer is used to refer to quantizers $\mathring{q} : \mathbb{R} \to \mathbb{R}$. As in Elia and Mitter (2001), and without loss of generality, only quantizers that satisfy

$$q(z) = -q(-z), \text{ for all } z \in \mathbb{R}^p \tag{4}$$

are considered. If q is a quantizer, let $\#q[\epsilon]$ denote the number of values that q has in the interval $[\epsilon, 1/\epsilon]$, where $0 < \epsilon < 1$. The density of q is defined as

$$\eta(q) = \limsup_{\epsilon\to 0} \frac{\#q[\epsilon]}{-\ln \epsilon}. \tag{5}$$

Under this definition, a quantizer with a finite number of levels has zero density, a linear quantizer has infinite density and a logarithmic quantizer has a finite nonzero density.

2.1 State Quantizers

In the first scenario, the aim is to obtain a quantizer having minimum density by searching over all state quantizers q that make a quadratic function of the form

$$V(x) \triangleq x^T P x, \text{ where } P = P^T > 0, \tag{6}$$

a Lyapunov function for the closed-loop system $x^+ = Ax + Bq(x)$. That is, the search is performed over state quantizers q that satisfy

$$V(Ax + Bq(x)) - V(x) < 0, \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{7}$$
$$V(Ax + Bq(x)) - V(x) = 0, \text{ if } x = 0. \tag{8}$$

Note that the symmetry condition (4) implies that $q(0) = 0$ and hence (8) is always satisfied. The following definitions are used throughout the paper.

Definition 2. (QAS State Quantizer). A (state) quantizer $q : \mathbb{R}^n \to \mathbb{R}$ that satisfies (7) and such that $q(0) = 0$, with V as in (6), will be called *quadratically asymptotically stabilizing* (QAS) with respect to P. Throughout this paper, P is fixed. Thus, the phrase 'with respect to P' will be omitted.

Definition 3. (QAS Pair). Let $\mathcal{R} \subset \mathbb{R}^n$, let $u \in \mathbb{R}$ and consider V as in (6). The pair $(u, \mathcal{R})$ is said to be QAS (with respect to P) if and only if

$$V(Ax + Bu) - V(x) < 0, \forall x \in \mathcal{R} \setminus \{0\}, \tag{9}$$
$$u = 0 \quad \text{if} \quad 0 \in \mathcal{R}. \tag{10}$$

The following lemma, whose proof is straightforward and can be consulted in Haimovich and Serón (2004), stresses the importance of Definition 3.

Lemma 4. Let $q : \mathbb{R}^n \to \mathbb{R}$ be a state quantizer, let $\mathcal{R}_i$, $i \in \mathbb{Z}$, be its quantization regions and let u_i be the value of q corresponding to $\mathcal{R}_i$, for all $i \in \mathbb{Z}$. Then, q is QAS if and only if $(u_i, \mathcal{R}_i)$ is QAS, for all $i \in \mathbb{Z}$.

Since system (1) is linear, there exist quadratic Lyapunov functions for the closed-loop system resulting from $u = Kx$, that is, $x^+ = (A + BK)x$, provided that $A + BK$ have all its eigenvalues inside the unit circle. By assumption A2), such a K exists and thus the following assumption is made:

A3) The matrix $P = P^T \in \mathbb{R}^{n\times n}$ in (6) is such that there exists $K \in \mathbb{R}^{1\times n}$ satisfying

$$(A + BK)^T P(A + BK) - P < 0. \quad (11)$$

As mentioned in Section 1, quantizers of the form shown in Figure 1, that is, PH quantizers, are considered. These quantizers are next defined without resorting to scalar quantizers.

Definition 5. (PH Region). Let $d \in \mathbb{R}^n$, $d \neq 0$. A parallel-hyperplane (PH) region $\mathcal{R}$ with direction d is a set defined in any of the following alternative ways:

$$\mathcal{R} = \{x \in \mathbb{R}^n : a \dashv_a d^T x \dashv_b b\}, \quad (12)$$

where '$\dashv_a$' and '$\dashv_b$' represent either '$<$' or '$\leq$', and $a \in \mathbb{R}$ or $a = -\infty$, and $b \in \mathbb{R}$ or $b = \infty$.

Remark 6. Given a PH region $\mathcal{R}$, the quantities a, b and d that define it are not unique. It is straightforward to check that if $\mathcal{R}$ has direction d, then it also has direction αd, where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

Definition 7. (PH Quantizer). Fix $d \in \mathbb{R}^n$, $d \neq 0$. A PH quantizer q with direction d is a quantizer whose quantization regions, $\mathcal{R}_i$, for all $i \in \mathbb{Z}$, are PH regions with (the same) direction d.

2.2 Output Quantizers

In the second scenario, the search for an infimum density quantizer is performed over output quantizers $\mathring{q}$ that make the function V defined in (6) a Lyapunov function for the closed loop system $x^+ = Ax + B\mathring{q}(Cx)$.

Definition 8. (QAS Output Quantizer). An (output) quantizer $\mathring{q} : \mathbb{R} \to \mathbb{R}$ that satisfies

$$V(Ax + B\mathring{q}(Cx)) - V(x) < 0, \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \quad (13)$$

with V as defined in (6) and such that $\mathring{q}(0) = 0$, will be called QAS (with respect to P).

3. PRELIMINARY RESULTS

This section presents two technical results that are needed in the sequel. The first result (Lemma 9) is important in the characterization of the set of all states for which the increment of the quadratic function V in (6) is negative for a given control u. The second result (Lemma 10) is related with properties of a PH region $\mathcal{R}$ when the pair $(u, \mathcal{R})$ is QAS. Showing the full implications of these results is beyond the scope of the paper. The interested reader may consult Haimovich and Serón (2004) for explanations and proofs.

Recall that assumptions A1) to A3) have been made.

Lemma 9. Define

$$L \triangleq A^T PA - P, \quad M \triangleq A^T PB, \quad (14)$$

and suppose that L is invertible. Then, the real number H defined by

$$H \triangleq B^T PB - M^T L^{-1} M, \quad (15)$$

satisfies $H < 0$.

Lemma 10. Suppose that L in (14) is invertible. Let $d \in \mathbb{R}^n$, $d \neq 0$, be such that $d^T L^{-1} d \geq 0$ and define

$$\gamma \triangleq \sqrt{-Hd^T L^{-1} d}, \quad \beta \triangleq -d^T L^{-1} M, \quad (16)$$

where H and M were defined in (15) and (14). Then, $|\beta| > \gamma$.

4. COARSE PH STATE QUANTIZERS WITH A GIVEN DIRECTION

The main aim of this section is to solve the problem of finding, by searching over all QAS PH quantizers with a given direction d, one that has infimum density. As is later shown, the solution to this problem allows one to find conditions for solving the problem of *static* stabilizing output feedback using coarse output quantizers (Section 5), and to recover the result in Elia and Mitter (2001, Theorem 2.1) (Section 6).

Problem 11. Let $d \in \mathbb{R}^n$, $d \neq 0$.

$$q_d = \arg\inf \eta(q), \quad \text{subject to} \quad (17)$$
$$q \text{ is QAS PH with direction } d, \quad (18)$$
$$q(x) = -q(-x), \text{ for all } x \in \mathbb{R}^n, \quad (19)$$

where $\eta(q)$ is the density of q, defined in (5).

To solve Problem 11, conditions for a quantizer to satisfy constraint (18) need to be obtained. These conditions are derived by means of the following lemma, whose proof can be consulted in Haimovich and Serón (2004).

Lemma 12. (Characterization of QAS Pairs). Let $\mathcal{R} \triangleq \{x \in \mathbb{R}^n : a \dashv_a d^T x \dashv_b b\}$ be nonempty and let $u \in \mathbb{R}$. Then, $(u, \mathcal{R})$ is QAS if and only if one of the following statements holds:

1) $u = 0,\ d^T L^{-1} d > 0,$ and
$\mathcal{R} = \{x \in \mathbb{R}^n : d^T x = 0\}$,
2) $u \neq 0,\ d^T L^{-1} d = 0,$ and
$\mathcal{R} = \{x \in \mathbb{R}^n : d^T x = \beta u\}$,
3) $u \neq 0,\ d^T L^{-1} d > 0,$
$\beta u - \gamma |u| \dashv_a^T a$ and $b \dashv_b^T \beta u + \gamma |u|$,

where

$$`\dashv^T' = \begin{cases} `\leq' & \text{if } `\dashv' = `<', \\ `<' & \text{if } `\dashv' = `\leq', \end{cases} \tag{20}$$

and L is defined in (14), and γ and β in (16).

From Lemma 12 and Lemma 4, there exists no QAS PH quantizer having a direction d such that $d^T L^{-1} d < 0$. If $d^T L^{-1} d = 0$, then $\gamma = 0$ [see (16)] and from Lemma 10, it follows that $\beta \neq 0$. Then, Lemma 12, item 2) and Lemma 4 imply that no QAS PH quantizer can have such a direction, since none of its quantization regions may contain the origin. Hence, it is only meaningful to solve Problem 11 when the given direction d satisfies $d^T L^{-1} d > 0$. Then, since Lemma 9 insures that $H < 0$, γ, defined in (16), is real and positive. Also, it may be assumed, without loss of generality, that d satisfies $-d^T L^{-1} M = \beta > 0$ [see (16)], since any PH quantizer with direction d also has direction $-d$ (see Remark 6). Therefore, using Lemma 10, it follows that, without loss of generality, it may be assumed that $\beta > \gamma > 0$ and hence $\beta \pm \gamma > 0$ and $0 < \frac{\beta-\gamma}{\beta+\gamma} < 1$. The following result can now be obtained.

Theorem 13. Let $d \in \mathbb{R}^n$ satisfy $d^T L^{-1} d > 0$ and $-d^T L^{-1} M = \beta > 0$. Then, any quantizer q_d defined in the following way, where $u_0^+ > 0$, solves Problem 11:

$$q_d(x) = \begin{cases} 0 & \text{if } x \in \mathcal{R}^0, & \text{(21a)} \\ u_i^+ & \text{if } x \in \mathcal{R}_i^+, & \text{(21b)} \\ -u_i^+ & \text{if } x \in -\mathcal{R}_i^+. & \text{(21c)} \end{cases}$$

$$\mathcal{R}^0 = \{x \in \mathbb{R}^n : d^T x = 0\}, \tag{22}$$

$$u_i^+ = \rho^{-i} u_0^+, \tag{23}$$

$$\mathcal{R}_0^+ = \{x \in \mathbb{R}^n : \sigma_0^+ \dashv_+ d^T x \dashv_+^T \rho^{-1} \sigma_0^+\}, \tag{24}$$

$$\mathcal{R}_i^+ = \rho^{-i} \mathcal{R}_0^+, \tag{25}$$

$$0 < \rho = \frac{\beta - \gamma}{\beta + \gamma} < 1, \tag{26}$$

$$\sigma_0^+ = (\beta - \gamma) u_0^+. \tag{27}$$

PROOF. The proof consists in proving that the density of q_d in (21) to (27) coincides with the infimum of Problem 11. Let $\mathcal{R}_0^\star, \mathcal{R}_1^\star$ be two adjacent PH regions with direction d, that is,

$$\mathcal{R}_0^\star = \{x \in \mathbb{R}^n : a_0 \dashv_{a_0} d^T x \dashv_{b_0} b_0\},$$
$$\mathcal{R}_1^\star = \{x \in \mathbb{R}^n : b_0 \dashv_{b_0}^T d^T x \dashv_{b_1} b_1\}.$$

Let $u_0^\star, u_1^\star > 0$ be such that $(u_0^\star, \mathcal{R}_0^\star)$ and $(u_1^\star, \mathcal{R}_1^\star)$ are QAS. According to the definition of quantization density in (5), any quantizer that solves Problem 11 has its values as far as possible from each other, searching over all quantizers that satisfy (18) and (19). We thus proceed to find the maximum separation between the values $u_0^\star$ and $u_1^\star$. Since $(u_0^\star, \mathcal{R}_0^\star)$ is QAS, Lemma 12, item 3) imposes the following conditions:

$$(\beta - \gamma) u_0^\star \dashv_{a_0}^T a_0, \quad b_0 \dashv_{b_0}^T (\beta + \gamma) u_0^\star. \tag{28}$$

Likewise, since $(u_1^\star, \mathcal{R}_1^\star)$ is QAS:

$$(\beta - \gamma) u_1^\star \dashv_{b_0} b_0, \quad b_1 \dashv_{b_1}^T (\beta + \gamma) u_1^\star. \tag{29}$$

Combining the second condition in (28) with the first one in (29):

$$u_1^\star < \frac{\beta + \gamma}{\beta - \gamma} u_0^\star = \rho^{-1} u_0^\star, \tag{30}$$

where (26) was used. Now, consider $u_0^\star$ to be fixed and find the supremum of $u_1^\star$ subject to (30), to obtain

$$\sup_{u_1^\star < \rho^{-1} u_0^\star} u_1^\star = \rho^{-1} u_0^\star.$$

Note that the development above is independent of the value of $u_0^\star$. Hence, it is now proved that the values of any quantizer that solves Problem 11 must satisfy (23), since in this case u_i^+ is as far as possible from both u_{i-1}^+ and u_{i+1}^+, for all $i \in \mathbb{Z}$. According to Lemma 12, $(u_i^+, \mathcal{R}_i^+)$ as defined in (23) to (27) are not QAS. However, the regions

$$\mathring{\mathcal{R}}_0^+ = \{x \in \mathbb{R}^n : \sigma_0^+ < d^T x < \rho^{-1} \sigma_0^+\},$$
$$\mathring{\mathcal{R}}_i^+ = \rho^{-i} \mathring{\mathcal{R}}_0^+,$$

which differ from $\mathcal{R}_i^+$ only by replacing a nonstrict inequality by a strict one, make $(u_i, \mathring{\mathcal{R}}_i^+)$ QAS, for all $i \in \mathbb{Z}$. Therefore, q_d, as defined in (21) to (27) solves Problem 11 and the proof is concluded. □

Under the definition of quantization density given by (5), it can be verified that $\eta(q_d) = -2/\ln \rho$. Note that q_d is not a QAS quantizer. This is because the constraint set given by (18) and (19) is not closed, and hence the infimizer, that is, q_d, need not belong to the constraint set. However, q_d is in the boundary of the constraint set and, loosely speaking, it is arbitrarily close to a quantizer that satisfies constraints (18) and (19). It can be shown that each quantization region of q_d contains at most one point for which (9) is not satisfied and hence such a quantizer may be called coarse-almost-QAS. Characterization and properties of coarse-almost-QAS quantizers can be found in Haimovich and Serón (2004). For example, a QAS quantizer having a density arbitrarily close to that of q_d (or any coarse-almost-QAS quantizer) can always be constructed.

5. STATIC OUTPUT FEEDBACK VIA COARSEST OUTPUT QUANTIZER

Consider the single-input system (1), having a single output defined by (2). The aim is to find an output quantizer $\mathring{q} : \mathbb{R} \to \mathbb{R}$ that has minimum density over all QAS output quantizers. Consequently, the following problem is formulated.

Problem 14.

$$\mathring{q}_\star = \arg\inf \eta(\mathring{q}), \text{ subject to} \tag{31}$$
$$\mathring{q} \text{ is QAS}, \tag{32}$$
$$\mathring{q}(y) = -\mathring{q}(-y), \text{ for all } y \in \mathbb{R}. \tag{33}$$

Note that the composition of the output equation (2) with the output quantizer $\mathring{q}$ is equivalent to a PH state quantizer with given direction $d = C^T$. That is, $\mathring{q}(Cx) = q(x)$, for all $x \in \mathbb{R}^n$, where q is PH with direction C^T (See also Figure 1). Hence, Problem 14 can equivalently be posed as Problem 11, where $d = C^T$. Then, $\mathring{q}_\star(Cx) = q_d(x)$, for all $x \in \mathbb{R}^n$, where q_d solves Problem 11 with $d = C^T$.

As explained in Section 4, in order that a quantizer q satisfying constraint (18) exist, the direction d must satisfy $d^T L^{-1} d > 0$. Therefore, it follows that in order that $\mathring{q}$ satisfying (32) exist, then C must satisfy $CL^{-1}C^T > 0$. If this is the case, then Theorem 13 may be used to obtain $\mathring{q}_\star$. (If $d = C^T$ does not satisfy $-d^T L^{-1} M = \beta > 0$, then Theorem 13 must be applied with $d = -C^T$ and $\mathring{q}_\star(Cx) = -q_d(x)$.) Thus, we have shown how to obtain a quantizer that solves Problem 14 assuming that a matrix P [see (6), (14)] has been given such that C satisfies $CL^{-1}C^T > 0$. The following theorem shows that if the system is stabilizable via (linear) static output feedback, then a matrix P such that $CL^{-1}C^T > 0$ can be found. The proof of this theorem is a straightforward application of Haimovich and Serón (2004, Lemma 4.5).

Theorem 15. Consider system (1) with the output (2). Suppose that assumption A3) holds with $K = \alpha C \in \mathbb{R}^{1\times n}$, where $\alpha \in \mathbb{R}$, and that L in (14) is invertible. Then, $CL^{-1}C^T > 0$.

Whenever a matrix P exists such that L is invertible and $CL^{-1}C > 0$, a coarsest static output feedback quantizer may be found by means of Theorem 13. To find the matrix P for which a minimum density output quantizer may be used, the sector bound approach in Fu and Xie (2003) may be employed.

6. COARSEST PH QUANTIZER

The first scenario is now reconsidered in order to find the coarsest PH (state) quantizer [with respect to all possible directions but for a given fixed matrix P in (6)]. This recovers the result in Elia and Mitter (2001, Theorem 2.1) when optimization is performed over PH quantizers.

Problem 16.

$$q_{d_\star} = \arg\inf \eta(q), \quad \text{subject to}$$
$$q \text{ is QAS PH}$$
$$q(x) = -q(-x), \text{ for all } x \in \mathbb{R}^n,$$

where $\eta(q)$ is the density of q, defined in (5).

Note that $q_{d_\star}$ satisfies

$$\eta(q_{d_\star}) = \inf_{d\in\mathbb{R}^n,\, d\neq 0} \eta(q_d),$$

where q_d solves Problem 11. As explained in Section 4, it is only meaningful to consider PH quantizers with direction d that satisfies $d^T L^{-1} d > 0$, and there is no loss of generality in only considering d such that $\beta = -d^T L^{-1} M > 0$ and in assuming that $\|d\|$ is some arbitrary positive number (see Remark 6). Since the density of q_d is directly related to the quantity ρ in Theorem 13 [see (26)], the following related problem can be formulated.

$$d_\star = \arg\inf_{d\in\mathbb{R}^n} \frac{\beta-\gamma}{\beta+\gamma}, \text{ subject to} \tag{34}$$
$$d^T L^{-1} d > 0, \tag{35}$$
$$d^T d = M^T M, \tag{36}$$
$$-d^T L^{-1} M > 0, \tag{37}$$

where L and M were defined in (14), and γ and β in (16).

Theorem 17. The direction $d_\star \in \mathbb{R}^n$ in (34) is given by $d_\star = -M$, where M was defined in (14).

PROOF. Since $d^T L^{-1} d > 0$, and, by Lemma 9, $H < 0$ then γ, defined in (16), is real and positive. Therefore,

$$\frac{\beta-\gamma}{\beta+\gamma} = \frac{\beta/\gamma - 1}{\beta/\gamma + 1}. \tag{38}$$

Using Lemma 10, and since $\beta > 0$ is an optimization constraint [see (16) and (37)], then $\beta/\gamma > 1$. The function $g(\beta/\gamma)$ defined by the right-hand side of (38) is strictly increasing and hence the direction $d_\star$ that minimizes (34) also minimizes β/γ, subject to constraints (35) to (37). In turn, $d_\star$ also minimizes β^2/γ^2 subject to the same constraints. From (16), it follows that

$$\frac{\beta^2}{\gamma^2} = \frac{d^T L^{-1} M M^T L^{-1} d}{-H d^T L^{-1} d}. \tag{39}$$

From Lemma 10, we have $|\beta| > \gamma$, even in the case when $d^T L^{-1} d = 0$. Hence,

$$\lim_{d\to\bar{d},\ \text{s.t. } d^T L^{-1} d \to 0^+} \beta > \lim_{d^T L^{-1} d \to 0^+} \gamma = 0, \tag{40}$$

and it follows that $d_\star$ cannot satisfy $d_\star^T L^{-1} d_\star = 0$ and therefore

$$d_\star^T L^{-1} d_\star > 0. \tag{41}$$

Note that $d_\star$ minimizes (39) subject to (35)–(37) if and only if $v \triangleq L^{-1}d_\star$ minimizes the function

$$f(w) \triangleq \frac{w^T MM^T w}{w^T Lw}, \tag{42}$$

subject to $w^T Lw > 0$, $w^T LLw = M^T M$, and $-w^T M > 0$, since $H < 0$ by Lemma 9. Then, from (40) it follows that the minimizer of (39) can only occur at a point $d_\star = Lv$ where $\nabla f(v) = 0$. Thus,

$$\nabla f(v) = 2\frac{(v^T Lv)MM^T v - (v^T MM^T v)Lv}{(v^T Lv)^2} = 0.$$

From (41), $v^T Lv > 0$ and it follows that

$$(v^T Lv)MM^T v - (v^T MM^T v)Lv = \left[(v^T Lv)MM^T L^{-1} - (v^T MM^T v)I\right]Lv = 0. \tag{43}$$

Since L is invertible, (43) implies that [1]

$$\det\left[(v^T Lv)MM^T L^{-1} - (v^T MM^T v)I\right] = (-v^T MM^T v)^n + (-v^T MM^T v)^{n-1}(v^T Lv)M^T L^{-1}M = 0. \tag{44}$$

Note that, since $v = L^{-1}d_\star$, then

$$v^T MM^T v = (d_\star^T L^{-1}M)^2 = \beta^2 > 0.$$

Thus, dividing (44) by $(-v^T MM^T v)^{n-1} \neq 0$ yields $(v^T Lv)(M^T L^{-1}M) = v^T MM^T v$ and substituting into (43), then

$$(v^T Lv)\left[MM^T L^{-1} - (M^T L^{-1}M)I\right]Lv = 0,$$

whence, since $v^T Lv > 0$ and $v = L^{-1}d_\star$,

$$\left[MM^T L^{-1} - (M^T L^{-1}M)I\right]d_\star = 0. \tag{45}$$

Lemma 9 establishes that $H < 0$. Using (15), and since $B^T PB > 0$ because $P > 0$, it follows that $M^T L^{-1}M > 0$. Hence, if $d_\star$ satisfies (45), then $MM^T L^{-1}d_\star \neq 0$. The matrix $MM^T L^{-1}$ has rank one and thus $MM^T L^{-1}d_\star = (M^T L^{-1}d_\star)M = \tilde{\alpha}M$, where $\tilde{\alpha} = M^T L^{-1}d_\star \in \mathbb{R}$. In order that $d_\star$ satisfy (45), it is necessary that $\tilde{\alpha}M = (M^T L^{-1}M)d_\star$, with $M^T L^{-1}M \in \mathbb{R}$, $M^T L^{-1}M > 0$. Hence, $d_\star = \alpha M$, for some $\alpha \in \mathbb{R}$. Then, constraint (36) implies that $\alpha = \pm 1$, and since $M^T L^{-1}M > 0$, constraint (37) may be used to obtain $\alpha = -1$. Hence, $d_\star = -M$, concluding the proof. □

It is straightforward to check that $q_{d_\star}$, which solves Problem 16, coincides with the quantizer in Elia and Mitter (2001, Theorem 2.1). The result in Elia and Mitter (2001) is more general than the one derived here in the sense that Elia and Mitter do not search only over PH quantizers. Then, Elia and Mitter directly prove that $q_{d_\star}$ achieves infimum density over all QAS quantizers. Though less general, imposing the PH quantizer constraint has the advantage of making the formulation of Problem 11 possible, where the direction d is a given parameter. This, in turn, allowed the derivation of the static output feedback strategy that employs a coarsest quantizer.

[1] $\det(xy^T + kI) = k^n + k^{n-1}y^T x,\ x, y \in \mathbb{R}^{n\times 1},\ k \in \mathbb{R}$.

7. CONCLUSIONS

The problem of finding a quantizer having minimum (infimum) density, by searching over all PH quantizers (with a given direction) that are able to quadratically stabilize a system, was addressed and solved. This result was used to find the coarsest output quantizer that may be utilized in a static output feedback scenario. Finally, the result in Elia and Mitter (2001, Theorem 2.1) was recovered by alternative means. The results in the current paper were based on the geometric approach to quadratic stabilization with quantizers developed in Haimovich and Serón (2004).

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