FINITE-DIMENSIONAL MODELS IN EVALUATING THE H_2 NORM OF CONTINUOUS-TIME PERIODIC SYSTEMS

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Abstract: Finite-dimensional harmonic models for the H_2 norm evaluation of finite-dimensional linear continuous-time periodic (FDLCP) systems are derived, which are expressed explicitly through finitely many Fourier coefficients of the system matrices, and thus dispense with the transition matrix knowledge of any FDLCP models, as opposed to most existing methods in the literature. This paper also shows that the skew- and square-truncated counterparts to the harmonic state operator are invertible in a class of stable FDLCP systems. This invertibility fact, together with the 2-regularized determinant technique about Hilbert-Schmidt operators, plays a key role in justifying the multiple-step truncation on the unbounded harmonic state operators of FDLCP systems and establishing rigorous convergence arguments for the proposed H_2 norm formulae and the associated finite-dimensional harmonic models.

Keywords: H_2 norm; finite-dimensional model; continuous-time periodic systems.

1. INTRODUCTION

The H_2 norm is a performance index for periodic time-varying control system analysis and synthesis (Bamieh and Pearson, 1992; Green and Limebeer, 1995; Zhang and Zhang, 1997; Zhou, 1998). For instance, helicopter rotors (Dugundji and Wendell, 1983), rolling ships in waves (Allievi and Soudack, 1990), periodic trajectory of robot arms (Jonsson, *et al.,* 2002) are frequently related to periodically time-varying models. The time- and frequency-domain H_2 norms of FDLCP systems have been given by Wereley (1990) and Zhang and Zhang (1997) and their equivalence is established by Zhou and Hagiwara (2002b). The equivalence has also been shown via the lifting technique (Colaneri, 2000). By Zhou, *et al* 2002b), an 'exact' formula for the H_2 norm in FDLCP systems (Zhou, *et al*, 2003) is developed.

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As for the numeric computations of the H_2 norm in FDLCP systems, the lifting (Bamieh and Pearson, 1992; Yamamoto, 1996) may be a useful tool. However, most studies related to the lifting are devoted to sampled-data systems (Bamieh and Pearson, 1992; Chen and Francis, 1995), and the computation formulas developed there are difficult to apply to FDLCP systems in a numeric sense. Recently Cantoni (2002) gave formulas for the H_2 norm computation derived through the lifting. However, these formulas still involve the transition matrix of an augmented equivalent FDLCP model. One can also resort to algorithms based on time-varying Lyapunov differential equations (Green and Limebeer, 1995) in the H_2 norm computation. However, one needs to solve time-varying Lyapunov differential equations and compute relevant integrals. Fast sampling/fast holding (FSFH) approximation (Keller and Anderson, 1992; Yamamoto, *et al*, 1999) is another worthwhile method for the H_2 norm computation. However, due to the unboundedness of the sampling operator (Chen and Francis, 1991; Yamamoto, *et al*, 1999) in the FSFH approximation, the convergence of this approximation has not been verified. There are also efforts to compute the H_2 norm by the parametric transfer functions (PTF) of FDLCP systems (Lampe and Rosenvasser, 2002), by which the norm is defined directly via the transition matrix. A statistical approach leads some closed formulae for the H_2-norm in FDLCP systems by Lampe and Rosenvasser (2003) lately. As a frequency-domain approach in the H_2 norm computation, the 'square' truncation is proposed by Wereley (1990) to overcome the infinite dimensionality of the frequency response operator. As an alternative truncation approach, the skew truncation is introduced to the frequency response operator (Zhou and Hagiwara, 2002b), which leads to 'asymptotic trace formulas'.

There is a common limitation among these methods (except the Lyapunov differential equation method); the transition matrix of the concerned FDLCP system and/or some augmented FDLCP model are needed. To surmount this common limitation, we establish formulas based on finite-dimensional models, which can be represented without the transition matrix of any FDLCP models and expressed explicitly by only finitely many Fourier coefficients of the system matrices. To this end, a multiple-step truncation approach is adopted. This approach, together with the 2-regularized determinant technique, yields a significant contribution if we notices that the approach does provide implementable ways to avoid inverse computations related to infinite-dimensional unbounded operators.

Notations. $\mathbb{Z}$ is the set of all integers. $F(t) \in L_2[0, h]$ means that F is a matrix function, each element of which is h-periodic and belongs to $L_2[0, h]$ when its domain is restricted to $[0, h]$. In the sequel, $L_{\mathrm{CPCD}}[0, h]$ is the set of all continuous functions whose first-order derivatives are piecewise continuous in $[0, h]$, and $L_{\mathrm{CAC}}[0, h]$ is the set of all continuous functions whose Fourier series are absolutely convergent. $C_{\mathcal{HS}}(l_2)$ is the set of all Hilbert-Schmidt operators on l_2.

2. PRELIMINARIES

2.1 FDLCP Systems and H_2 Norm

We consider the FDLCP system

$$\mathcal{G} : \begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x \end{cases} \tag{1}$$

where $A(t), B(t)$ and $C(t)$ are h-periodically time-varying. The transition matrix of (1) is denoted by $\Phi(t, t_0)$ when the initial time is t_0. By the Floquet theorem (Farkas, 1994), if $A(t) \in L_2[0, h]$, then $\Phi(t, t_0)$ possesses a Floquet factorization of $\Phi(t, t_0) = P(t, t_0)e^{Q(t-t_0)}$, where $P(t, t_0)$ is absolutely continuous in t, nonsingular and h-periodic both in t and t_0, and Q is constant. Moreover, the system is asymptotically stable if and only if the eigenvalues of Q lie in the open left-half plane, in which case we simply say that $A(t) \in L_2[0, h]$ is asymptotically stable. In the sequel, we assume $t_0 = 0$.

Let $\sum_m X_m e^{jm\omega_h t}$ be the Fourier series of $X(t)$ with $\omega_h := 2\pi/h$. The Toeplitz transformation on $X(t)$, denoted by $\underline{X}$, maps $X(t)$ into Toeplitz operator of the form

$$\underline{X} := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & X_0 & X_{-1} & X_{-2} & \cdots \\ \cdots & X_1 & X_0 & X_{-1} & \cdots \\ \cdots & X_2 & X_1 & X_0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We further define $\underline{A}$, $\underline{P}$, $\underline{Q}$ to be the Toeplitz transformation of $A(t)$, $P(t, 0)$ and Q.

$$\underline{E}(j\varphi) := \mathrm{diag}[\cdots, j\varphi_{-1}I, j\varphi_0 I, j\varphi_1 I, \cdots] \tag{2}$$

where $\varphi_k := \varphi + k\omega_h, k \in \mathbb{Z}, \varphi \in \mathbb{I}_0 := [-\frac{\omega_h}{2}, \frac{\omega_h}{2})$ and $j\varphi_0 I$ is at the center of $\underline{E}(j\varphi) : l_E \to l_2$ with $l_E := \{\underline{x} \in l_2 : \underline{E}(j0)\underline{x} \in l_2\}$. It follows that $\underline{E}(j\varphi) = \underline{E}(j0) + j\varphi\underline{I}$ with $\underline{I}$ being the Toeplitz transformation of the identity. Clearly, $\underline{E}(j\varphi)$ is unbounded on l_2 and thus restricted to l_E that is a proper subset of l_2 and dense in l_2 (Zhou and Hagiwara, 2002a). In the following, $\underline{E}(j\varphi) - \underline{A} : l_E \to l_2$ will be called the harmonic state operator.

Lemma 1. In the FDLCP system (1), let $A(t) \in L_{\mathrm{CPCD}}[0, h]$. Then, $\underline{P}$ is invertible both on l_2 and l_E. Also, the unbounded operator $\underline{P}(\underline{E}(j0) - \underline{Q})\underline{P}^{-1}$ and $\underline{E}(j0) - \underline{A}$ are densely defined on l_2 and coincide with each other:

$$\underline{P}(\underline{E}(j0) - \underline{Q})\underline{P}^{-1} = \underline{E}(j0) - \underline{A}$$

Furthermore, (1) is asymptotically stable if and only if all eigenvalues of $\underline{Q} - \underline{E}(j0)$ lies in the open left-half plane.

The frequency response operator (Wereley and Hall, 1990; Zhou and Hagiwara, 2002a) of (1) is

$$\underline{G}(j\varphi) := \underline{C}(\underline{E}(j\varphi) - \underline{A})^{-1}\underline{B} : l_2 \to l_2$$

under the stability assumption. Here $\underline{B}$ and $\underline{C}$ are the Toeplitz transformations of $B(t)$ and $C(t)$, respectively, and the inverse $(\underline{E}(j\varphi) - \underline{A})^{-1}$ is well-defined by Lemma 1. It is easy to see that $\underline{G}(j\varphi) \in C_{\mathcal{HS}}(l_2)$ for each $\varphi \in \mathbb{I}_0$. Hence, $\underline{G}(j\varphi)^*\underline{G}(j\varphi)$ is a trace class operator (Gohberg and Goldberg, 1990). The H_2 norm of the system (1) is (Zhang and Zhang, 1997); Zhou and Hagiwara, 2002b)

$$||\mathcal{G}||_2^2 := \frac{1}{2\pi} \int_{-\omega_h/2}^{\omega_h/2} \mathrm{tr}\Big(\underline{G}(j\varphi)^* \underline{G}(j\varphi)\Big) d\varphi \tag{3}$$

2.2 Truncations and Stability

To reduce the definition formula (3) to a finite-dimensional one, we need two kinds of truncations. Firstly, we describe the skew truncation. For example, we skew truncate $\underline{A}$ to $\underline{A}_N := \mathcal{T}\{A_N(t)\}$ where $A_N(t) := \sum_{|m|\le N} A_m e^{jm\omega_h t}$ with $\{A_m\}_{-\infty}^{\infty}$ being the Fourier coefficients sequence of $A(t)$. The skew-truncated version of $\underline{E}(j\varphi) - \underline{A}$, i.e. $\underline{E}(j\varphi) - \underline{A}_N$, will be termed the skew-truncated harmonic state operator. In the sequel, $\underline{B}_N$, $\underline{C}_N$, $B_N(t)$ and $C_N(t)$ are defined similarly to $\underline{A}_N$ and $A_N(t)$ but in terms of the Fourier coefficients of $B(t)$ and $C(t)$.

Secondly, we describe the square truncation; that is, we truncate $\underline{A}_N$ to a square-truncated one $\underline{A}_{NM}$ given by $\underline{A}_{NM} := \mathrm{diag}[\cdots, 0, A_{NM}, 0, \cdots]$ In the above, A_{NM} with $M \ge N+1$ is a finite-dimensional matrix defined by

$$A_{NM} := \underbrace{\begin{bmatrix} A_0 & \cdots & A_{-N} & & 0 \\ \vdots & \ddots & & \ddots & \\ A_N & & \ddots & & A_{-N} \\ & \ddots & & \ddots & \vdots \\ 0 & & A_N & \cdots & A_0 \end{bmatrix}}_{(2M+1)-\text{blocks}}$$

Now, to conform to the block-diagonal form of $\underline{A}_{NM}$, let us also truncate $\underline{E}(j\varphi)$ and $\underline{I}$ into block-diagonal forms accordingly. That is,

$$\underline{E}_M(j\varphi) := \mathrm{diag}[\cdots, 0, E_M(j\varphi), 0, \cdots]$$
$$\underline{I}_M := \mathrm{diag}[\cdots, 0, I_M, 0, \cdots]$$

where I_M is an identity matrix of $(2M+1) \times (2M+1)$ in the blockwise sense and $E_M(j\varphi) = \mathrm{diag}[j\varphi_{-M} I, \cdots, j\varphi_0 I, \cdots, j\varphi_M I]$ with φ_k being defined just below (2). Then, the infinite dimensional matrix expression of the operator $\underline{E}_M(j\varphi) - \underline{A}_{NM}$ is block-diagonal with $E_M(j\varphi) - A_{NM}$ being the only non-zero block matrix at the center. $\underline{E}_M(j\varphi) - \underline{A}_{NM}$ will be termed the square-truncated harmonic state operator of (1).

Now we view $A_N(t)$ as the state matrix of an approximate FDLCP model $(A_N(t), B_N(t), C_N(t))$ of (1). Then, by the Floquet theorem its transition matrix can be expressed as $\Phi_N(t,0) = P_N(t,0)e^{Q_N t}$. We can assert the following results, whose proof is omitted for brevity.

Lemma 2. In the FDLCP system (1), assume that $A(t) \in L_{\mathrm{CPCD}}[0,h]$. Let $A_N(t)$ be the state matrix of an approximate FDLCP model of $A(t)$. Then, we can take a Floquet factorization $\Phi_N(t,0) = P_N(t,0)e^{Q_N t}$ in such a way that $\lim_{N\to\infty} Q_N = Q$ holds. Moreover, if the system (1) is asymptotically stable, then there exist an integer $N_0 > 0$ and a number $K_Q > 0$ independent of $\varphi \in \mathbb{I}_0$ such that for all $N \ge N_0$, all eigenvalues of Q_N have negative real parts and for all $\varphi \in \mathbb{I}_0, N \ge N_0, k \in \mathbb{Z}$,

$$|(j\varphi_k I - Q_N)^{-1}| \le K_Q f(k) \tag{4}$$

where $f(k)$ of an integer k is given by $f(k) = 1$ if $k = 0$ and $f(k) = |k|^{-1}$ if $k \ne 0$.

2.3 Invertibility of Truncated Harmonic State Operators

In the following, we need to use inverses of the truncated counterparts to $\underline{E}(j\varphi) - \underline{A}$, i.e. $\underline{E}(j\varphi) - \underline{A}_N$ and $\underline{E}_M(j\varphi) - \underline{A}_{NM}$. We alert the reader that we use the terminology 'the invertibility' of $\underline{E}_M(j\varphi) - \underline{A}_{NM}$ to imply that of the finite-dimensional matrix $E_M(j\varphi) - A_{NM}$ throughout the paper. In other words, we write

$$\begin{aligned}&(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1} := \\ &\mathrm{diag}[\cdots, 0, (E_M(j\varphi) - A_{NM})^{-1}, 0, \cdots]\end{aligned} \tag{5}$$

whenever $E_M(j\varphi) - A_{NM}$ is invertible. Noting that these truncated counterparts to the harmonic state operator are unbounded since $\underline{E}(j\varphi)$ is, it is hard to ensure their invertibility by directly working on such unbounded operators. Since $\underline{E}(j\varphi + \rho) : l_E \to l_2$ is invertible for each $\varphi \in \mathbb{I}_0$ whenever $\rho > 0$, we have

$$\begin{aligned}&(\underline{E}(j\varphi) - \underline{A})^{-1} = (\underline{E}(j\varphi + \rho) - (\rho \underline{I} + \underline{A}))^{-1} \\ &= (\underline{I} - \underline{E}^{-1}(j\varphi + \rho)(\rho \underline{I} + \underline{A}))^{-1} \underline{E}^{-1}(j\varphi + \rho)\end{aligned}$$

to avoid dealing with unbounded operators directly. In the sequel, we always keep a fixed $\rho > 0$. To understand the following lemmas, we notice (Bottcher and Silbermann, 1990) that for $A \in C_{\mathcal{HS}}(l_2)$, it is justified to define

$$\det{}_2(I + A) := \prod \{1 + \lambda_i(R(A))\} e^{-\lambda_i(R(A))}$$

with $R(A) := (I + A)\exp\{-A\} - I$. $\det_2(I + A)$ is called the 2-regularized determinant of $I + A$.

Proposition 1. Suppose in the FDLCP system (1) that $A(t) \in L_{\mathrm{CPCD}}[0,h]$ is asymptotically stable. Then, $\underline{E}^{-1}(j\varphi + \rho)(\rho \underline{I} + \underline{A}_N) \in C_{\mathcal{HS}}(l_2)$ for any fixed $\rho > 0$, N and any $\varphi \in \mathbb{I}_0$. Moreover, if N is sufficiently large, the operator $\underline{I} - \underline{E}^{-1}(j\varphi + \rho)(\rho \underline{I} + \underline{A}_N)$ is invertible and there exist an integer $N_1 > 0$ and numbers $\gamma > 0$, $K_A > 0$ independent of $\varphi \in \mathbb{I}_0$ and N such that

$$|\det{}_2[\underline{I} - \underline{E}^{-1}(j\varphi + \rho)(\rho \underline{I} + \underline{A}_N)]| \ge \gamma > 0 \tag{6}$$

$$||(\underline{I} - \underline{E}^{-1}(j\varphi+\rho)(\rho\underline{I}+\underline{A}_N))^{-1}||_{l_2/l_2} \le K_A \quad (7)$$

for all $\varphi \in I\!I_0$ and $N \ge N_1$.

Proposition 2. Suppose in the FDCLP system (1) that $A(t) \in L_{\text{CPCD}}[0,h]$ is asymptotically stable. Then for any $\mu > 0$ and any fixed N, there exists an integer $M(N,\mu) > 0$ such that for all $M \ge M(N,\mu)$ and $\varphi \in I\!I_0$

$$\begin{aligned}&|\det{}_2[\underline{I} - \underline{E}^{-1}(j\varphi+\rho)(\rho\underline{I}+\underline{A}_N)]\\ &-\det{}_2[\underline{I} - \underline{E}^{-1}(j\varphi+\rho)(\rho\underline{I}_M+\underline{A}_{NM})]| < \mu\end{aligned}$$

Remark 1. Proposition 1 shows that under the stability assumption of $A(t)$, $\underline{E}(j\varphi) - \underline{A}_N$ is invertible for each $\varphi \in I\!I_0$ and N large enough. This, together with Proposition 2, implies that if N and M are large enough, $\det_2[\underline{I} - \underline{E}^{-1}(j\varphi+\rho)(\rho\underline{I}_M + \underline{A}_{NM})] \ne 0$. Then, by Property 1.8(e) of Bottcher and Silbermann ((, p. 17)), it follows that if N and M are large enough, $\underline{I} - \underline{E}^{-1}(j\varphi+\rho)(\rho\underline{I}_M + \underline{A}_{NM})$ and thus $\underline{E}(j\varphi+\rho) - (\rho\underline{I}_M + \underline{A}_{NM})$ is invertible for each $\varphi \in I\!I_0$. Note that

$$\begin{aligned}&\underline{E}(j\varphi+\rho) - (\rho\underline{I}_M + \underline{A}_{NM})\\ &= \text{diag}[\cdots, (j\varphi_{M+1}+\rho)I, E_M(j\varphi) - A_{NM},\\ &(j\varphi_{-M-1}+\rho)I, \cdots]\end{aligned}$$

It follows that $E_M(j\varphi) - A_{NM}$ and $\underline{E}_M(j\varphi) - \underline{A}_{NM}$ are invertible for each $\varphi \in I\!I_0$. Hence, we can say that whenever N and M are sufficiently large, it is justified to truncate $(\underline{E}(j\varphi) - \underline{A})^{-1}$ into $(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1}$, where the latter can be constructed with only finitely many Fourier coefficients of the state matrix $A(t)$ of the FDLCP system (1). This invertibility fact about $E_M(j\varphi) - A_{NM}$ has been employed implicitly by Zhang and Zhang (1997) for the H_2 performance controller analysis in a slightly different way but no proof is provided there, up to the best understanding of the authors.

3. MULTIPLE-STEP TRUNCATIONS AND H_2 NORM COMPUTATION

The main difficulty in implementing (3) is that $\underline{G}(j\varphi)$ is infinite-dimensional, and it is natural to truncate $\underline{G}(j\varphi)$ to reduce the trace computation in (3) to a finite-dimensional one. Truncating $\underline{G}(j\varphi)$ directly will bring us problems: first, what kind of truncations should be adopted on $(\underline{E}(j\varphi) - \underline{A})^{-1}$, which cannot be explicitly expressed in a general FDLCP system; second, the truncation convergence problem. We will adopt a multiple-step truncation approach instead.

3.1 Truncating $\underline{G}(j\varphi)$ to $\underline{G}^{(\text{sk})}_{NM}(j\varphi)$ and Relevant Convergence Proposition

We implement a two-parameter-skew truncation to the operator components $\underline{A}$, $\underline{B}$ and $\underline{C}$ in $\underline{G}(j\varphi)$, which are truncated to $\underline{A}_N$, $\underline{B}_M$ and $\underline{C}_M$, respectively. Then, the skew-truncated version of $\underline{G}(j\varphi)$ is given by

$$\underline{G}^{(\text{sk})}_{NM}(j\varphi) := \underline{C}_M(\underline{E}(j\varphi) - \underline{A}_N)^{-1}\underline{B}_M$$

where the superscript (sk) stands for skew truncation. Here the truncation sizes on $\underline{A}$, $\underline{B}$ and $\underline{C}$ are differently taken for our purpose. The reason for deliberately doing so will become clear from the up-coming discussions.

By Remark 1, the inverse of $\underline{E}(j\varphi) - \underline{A}_N$ is well-defined for each $\varphi \in I\!I_0$ if N is sufficiently large. That is, $\underline{G}^{(\text{sk})}_{NM}(j\varphi)$ is well-defined as the frequency response operator of the FDLCP system $(A_N(t), B_M(t), C_M(t))$. Indeed, $\underline{G}^{(\text{sk})}_{NM}(j\varphi) \in C_{\mathcal{HS}}(l_2)$ for each $\varphi \in I\!I_0$ and the H_2 norm of the system $(A_N(t), B_M(t), C_M(t))$ can be given by

$$||\mathcal{G}^{(\text{sk})}_{NM}||_2^2 := \frac{1}{2\pi}\int_{-\omega_h/2}^{\omega_h/2} \text{tr}\Big(\underline{G}^{(\text{sk})}_{NM}(j\varphi)^*\underline{G}^{(\text{sk})}_{NM}(j\varphi)\Big)d\varphi$$

The following proposition can be proved by following arguments similar to those in Zhou and Hagiwara 2002a).

Proposition 3. Suppose in the FDLCP system (1) that $A(t) \in L_{\text{CPCD}}[0,h]$ is asymptotically stable, and that $B(t)$ and $C(t)$ belong to $L_{\text{CAC}}[0,h]$. Then, for any $\epsilon > 0$, there exists an integer $N_0(\epsilon) > 0$ such that for each fixed $N \ge N_0(\epsilon)$, one can have an integer $M_0(N,\epsilon) > 0$ satisfying

$$\Big|||\mathcal{G}||_2^2 - ||\mathcal{G}^{(\text{sk})}_{NM}||_2^2\Big| < \epsilon, \quad \forall M \ge M_0(N,\epsilon)$$

3.2 Truncating $\underline{G}^{(\text{sk})}_{NM}(j\varphi)$ to $\underline{G}^{(\text{sq})}_{NM}(j\varphi)$ and its Convergence Proposition

Proposition 3 means that $||\mathcal{G}^{(\text{sk})}_{NM}||_2$ can be an estimate of $||\mathcal{G}||_2$ if N and M are large enough. However, it is still hard to compute $||\mathcal{G}^{(\text{sk})}_{NM}||_2$ since $\underline{G}^{(\text{sk})}_{NM}(j\varphi)$ is infinite-dimensional. To surmount this problem, we introduce a square truncation to the operator component $(\underline{E}(j\varphi) - \underline{A}_N)^{-1}$ in $\underline{G}^{(\text{sk})}_{NM}(j\varphi)$ as the second step. This leads us to the square-truncated version of $\underline{G}(j\varphi)$ as follows.

$$\underline{G}^{(\text{sq})}_{NM}(j\varphi) := \underline{C}_M(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1}\underline{B}_M$$

where the superscript (sq) stands for square truncation. By Remark 1, $(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1}$ is a well-defined mapping from l_2 to l_2. Indeed, it is trivial to show that $\underline{G}^{(\mathrm{sq})}_{NM}(j\varphi)$ is also a Hilbert-Schmidt operator for each $\varphi \in \mathbb{I}_0$. Therefore, it is meaningful to define

$$||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2^2 := \frac{1}{2\pi} \int_{-\omega_h/2}^{\omega_h/2} \mathrm{tr}\Big(\underline{G}^{(\mathrm{sq})}_{NM}(j\varphi)^* \underline{G}^{(\mathrm{sq})}_{NM}(j\varphi)\Big) d\varphi$$

where $||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2$ denotes the H_2 norm of $\underline{G}^{(\mathrm{sq})}_{NM}(j\varphi)$. Naturally, $||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2$ can be an approximation of $||\mathcal{G}^{(\mathrm{sk})}_{NM}||_2$ as long as the truncation convergence between them is verified.

Proposition 4. Suppose in the FDCLP system (1) that $A(t) \in L_{\mathrm{CPCD}}[0,h]$ is asymptotically stable, and that $B(t), C(t) \in L_{\mathrm{CAC}}[0,h]$. Then for any fixed N that is sufficiently large and for each $\epsilon > 0$, there exists an integer $M_0(N,\epsilon) > 0$ such that

$$\Big| ||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2^2 - ||\mathcal{G}^{(\mathrm{sk})}_{NM}||_2^2 \Big| < \epsilon, \quad \forall M \geq M_0(N,\epsilon)$$

3.3 Finite-Dimensional Harmonic Models for the H_2 Norm

Now we establish an algorithm to compute $||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2$. Let us introduce some notations.

$$\mathcal{E}_M(j\varphi) - \mathcal{A}_{NM} := E_M(j\varphi) - A_{NM} \tag{8}$$

$$\mathcal{B}_{MM} := \underbrace{\begin{bmatrix} B_M & \cdots & B_{-M} & & 0 \\ & \ddots & & \ddots & \\ 0 & & B_M & \cdots & B_{-M} \end{bmatrix}}_{(4M+1)\text{ block-columns}} \tag{9}$$

$$\mathcal{C}_{MM} := \underbrace{\begin{bmatrix} C_{-M} & & 0 \\ \vdots & \ddots & \\ C_M & & C_{-M} \\ & \ddots & \vdots \\ 0 & & C_M \end{bmatrix}}_{(2M+1)\text{ block-columns}} \tag{10}$$

Based on these notations, we define the following harmonic continuous-time model.

$$\mathcal{G}_{NM}(j\varphi) := \mathcal{C}_{MM}(\mathcal{E}_M(j\varphi) - \mathcal{A}_{NM})^{-1}\mathcal{B}_{MM} \tag{11}$$

which can be an approximate frequency response derived by truncating $\underline{G}(j\varphi)$ in the H_2 norm sense. It should be stressed that the approximate model itself is linear time-invariant continuous-time in form and expressed with only finitely many Fourier coefficients of the system matrices $A(t)$, $B(t)$ and $C(t)$.

Note that $(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1}$ in $\underline{G}^{(\mathrm{sq})}_{NM}(j\varphi)$ is block-diagonal. Then, the following deductions follow from the fact that $\mathrm{tr}(\cdot)$ is the sum of all terms on the diagonal of $(\cdot)$.

$$\begin{aligned} ||\mathcal{G}^{(\mathrm{sq})}_{NM}||_2^2 &= \frac{1}{2\pi} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{tr}\Big(\underline{B}_M^*(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-*} \\ &\quad \underline{C}_M^*\underline{C}_M(\underline{E}_M(j\varphi) - \underline{A}_{NM})^{-1}\underline{B}_M\Big)d\varphi \\ &= \frac{1}{2\pi} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{tr}\Big(\mathcal{B}_{MM}^*\Big(\mathcal{E}_M(j\varphi) - \mathcal{A}_{NM}\Big)^{-*} \\ &\quad \mathcal{C}_{MM}^*\mathcal{C}_{MM}\Big(\mathcal{E}_M(j\varphi) - \mathcal{A}_{NM}\Big)^{-1}\mathcal{B}_{MM}\Big)d\varphi \\ &= \frac{1}{2\pi} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{tr}\Big(\mathcal{G}_{NM}(j\varphi)^*\mathcal{G}_{NM}(j\varphi)\Big)d\varphi \end{aligned}$$

Summarizing these arguments leads us to the main result.

Theorem 1. Suppose in the FDCLP system (1) that $A(t) \in L_{\mathrm{CPCD}}[0,h]$ is asymptotically stable, and that $B(t)$ and $C(t)$ belong to $L_{\mathrm{CAC}}[0,h]$. Then, the following results hold

(1) For sufficiently large truncation sizes N and M satisfying $M \geq N+1$, the frequency response function $\mathcal{G}_{NM}(j\varphi)$ of the finite-dimensional harmonic linear time-invariant continuous-time model $(\mathcal{E}_M(j0) - \mathcal{A}_{NM}, \mathcal{B}_{MM}, \mathcal{C}_{MM})$ is well-defined over $\varphi \in \mathbb{I}_0$ in the sense that the square-truncated harmonic state matrix $\mathcal{E}_M(j\varphi) - \mathcal{A}_{NM}$ is invertible for each $\varphi \in \mathbb{I}_0$;

(2) For any small number $\epsilon > 0$, there exists an integer $N_0(\epsilon) > 0$ such that for each $N \geq N_0(\epsilon)$, one can determine another integer $M_0(N,\epsilon) > 0$ such that for all $M \geq M_0(N,\epsilon)$

$$\Big|\frac{1}{2\pi} \int_{-\frac{\omega_h}{2}}^{\frac{\omega_h}{2}} \mathrm{tr}\Big(\mathcal{G}_{NM}(j\varphi)^*\mathcal{G}_{NM}(j\varphi)\Big)d\varphi - ||\mathcal{G}||_2^2\Big| < \epsilon$$

4. NUMERICAL EXAMPLES

Consider the asymptotic evaluation of the H_2 norm of the following π-periodic FDLCP system (Farkas, 1994) when the input weighting parameter β varies from 0 to 0.5.

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 - \sin^2(2t) & 2 - \frac{1}{2}\sin(4t) \\ -2 - \frac{1}{2}\sin(4t) & -1 - \cos^2(2t) \end{bmatrix} x \\ \qquad\qquad + \begin{bmatrix} 0 \\ 1 - 2\beta\rho(t) \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{cases}$$

Here, the function $\rho(\cdot)$ is given by

$$\rho(t) = \begin{cases} \sin(2t) & (0 \le t \le \pi/2) \\ 0 & (\pi/2 < t \le \pi) \end{cases}$$

which contains infinitely many sinusoid harmonic waves. The transition matrix has a Floquet factorization of the form

$$P(t,0) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}, Q = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Since the transition matrix is available and $C(t)$ is constant, the exact formula by Zhou, *et al* (2003) can be applied to get the H_2 norm of the above FDCLP system exactly. The exact H_2 norms for all cases about β are given in the last column of Table 1 (Zhou, *et al* (2003)).

Now we apply our main result suggested in Theorem 1. The computation results are listed in Table 1, in which the square truncation parameter M varies from 1 to 45, while the skew truncation parameter N changes only from 0 to 2. This is because the state matrix $A(t)$ does not contain harmonic components higher than the second harmonic wave so that it makes no difference when N is taken larger than 2.

Table 1. Finite-Dimensional Models

(N, M)	(1, 2)	(2, 5)	(2, 15)	(2, 45)	exact
$\beta = 0$	0.7205	0.7270	0.7304	0.7316	0.7323
0.1	0.6742	0.6793	0.6821	0.6831	0.6836
0.2	0.6335	0.6375	0.6396	0.6404	0.6408
0.3	0.5996	0.6027	0.6043	0.6049	0.6052
0.4	0.5735	0.5761	0.5774	0.5780	0.5783
0.5	0.5566	0.5590	0.5604	0.5608	0.5611

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