CLOSED FORM ESTIMATION OF BOUNDARY VALUES IN ELECTRICAL DISTRIBUTION NETWORKS

Eduard Janeček [*,1] **Pavel Prautsch** [*,1]

[*] *Department of Cybernetics, University of West Bohemia, Univerzitní 8, 30614 Pilsen, Czech Republic*
email: janecek,prautsch@kky.zcu.cz

Abstract: The paper deals with the problem of estimating operating quantities in low-voltage distribution networks. The load is considered to be a stationary process and the proposed method gives the first two moments of current and voltage at an arbitrary node of the network. Boundary values that must not be exceeded with a given probability can thus be easily obtained. Moreover, the method calculates the estimates in a closed form, parameterized by coefficients of correlation between customers' loads. Hence, there is no need to recalculate the whole network if different coefficients are used.

Keywords: Distribution networks, Power distribution, Load flows, Load modelling, Stochastic modelling

1. INTRODUCTION

1.1 Motivation

Electric power utilities aim to provide customers with electric energy at optimal cost. Their service is characterized by the load supplied to customers, reliability and quality of voltage. Load data are needed for planning and analysis of electricity production, transmission and distribution (Handschin and Doernemann, 1988). In particular, consider the case of a designer, who is concerned with the task of determining the weakest point of the distribution network, that is a point, where the load maxima approach technical parameters of the network. Similarly, a sales manager needs to know whether a new customer can be connected at a certain point of the network without violating the technical capacities of the network, etc. Their available data usually consist of customers' individual loads in the form of functions of time obtained from energy meter readings. They may be provided for one year, sometimes on hourly basis (Janeček and Vacík, 2000; Seppälä, 1996).

Such tasks require some knowledge of the network load profile, i.e., knowledge of the active and reactive load (current and voltage) at any point of the network. Furthermore, static estimates (usually mean value) are clearly insufficient as they do not reveal much about the maximal values that may occur during the network operation. What is needed is a *boundary value*, which is a value that will not be exceeded with a given probability p. In other words, upper value of a $p\%$ confidence interval is searched for. Thus, instead of static estimates, the first two moments of a stationary process should be found.

[1] Supported by Ministry of Education of the Czech Republic – Project No. 1M6840770004

1.2 Approach to solution and result

The distribution network is considered as a radial network. It is perceived as being composed of orbits characterized by equal depth (number of branches) from the supply point. To each orbit, a set of state variables is defined. They allow an easy calculation of current and voltage profiles. During the analysis, one node of depth k, called the analyzed node k, is considered only and its state variables computed.

The load of one specific customer can not be exactly predicted - indeed, it is a randomly distributed variable, described by its mean and variance. In order to obtain the boundary values, all calculations are done at one fixed time a workday. Then, the load is considered as a stationary process defined on a discrete time set. In the following, current model will be used in order to linearize the problem. In case of a load model, the current model must be found iteratively, using a Newton-Raphson method (Krynský, 2003).

Correlation between customers' currents are considered equal for all customers (independent of a customer). The obtained result is in a *closed form parameterized by the correlation coefficients.* This enables to obtain the boundary values (of voltage and/or current) at any place of the network simply by changing the correlation coefficients, without recalculating the whole network load profile.

The proposed algorithm of recursive network analysis requires two computational passes. During computation, the radial network is seen as a tree graph. In the first pass of the algorithm, which runs from leave nodes to the root node (supply point), all currents are calculated according to Proposition 1. Second pass of the algorithm starts at the root node and proceeds towards the leaves. Voltage drops are computed according to Proposition 2. For the sake of obtaining boundary values of a network with many customers, it is reasonable to assume that the operating quantities have normal distributions. Moreover, a designer is more interested in the boundary values at places close to the supply point, where the Central limit theorem applies.

Notation. The following notation will be used through the paper. The real and imaginary part of a complex variable X are denoted by ${}^{\Re}X$ and ${}^{\Im}X$, respectively, and X^* denotes its complex conjugate. For a stochastic variable Y, the mean and standard deviation are denoted by $\bar{Y} = E\{Y\}$ and $\tilde{Y} = \sqrt{\text{var}(Y)} = \sqrt{E\{(Y-\bar{Y})(Y-\bar{Y})^*\}}$, respectively, where $E\{\cdot\}, \text{var}(\cdot)$ stand for the expected value and variance, respectively. Covariance between two stochastic variables is denoted by $\text{cov}(\cdot,\cdot)$.

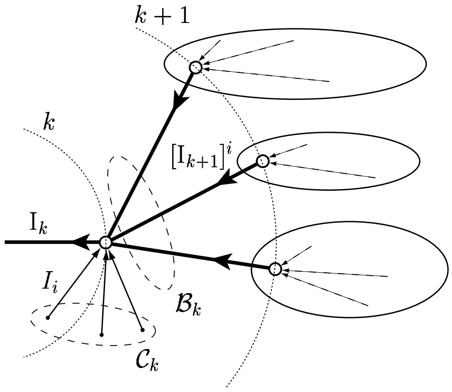

Fig. 1. Analysis of currents in radial distribution network

2. RECURSIVE NETWORK ANALYSIS

2.1 Model of the network

The aim of a network analysis is to obtain an accurate load profile. This profile should respect effect of customers' correlations. In particular, it is desirable to obtain results that would be parameterized by coefficients of correlation between customers' loads.

The radial distribution network, depicted in Figure 1, is considered as a tree graph composed of nodes and branches. The depth of each node is defined by a number of branches from the supply point of depth zero. The set of all branches from nodes at orbit $k+1$ connected to a particular node at orbit k is denoted $\mathcal{B}_k$. Similarly, $\mathcal{C}_k$ denotes the set of all customers connected directly to the node k. Clearly, $\mathcal{B}_k = \emptyset$ holds for a leaf node. Impedance of a branch l is denoted by Z_l. These physical properties are known before the analysis.

In the following, current of a node k means the current outputted by the node k. We distinguish the current I_k, which is the current of the analyzed node k, from I^j, which is a current outputted by a customer $j \in \mathcal{C}_k$. Note that unless necessary to avoid confusion, the subscript $_k$ will be omitted. Correlation coefficient between the real and imaginary part of a customer current is denoted by ρ. Then, ${}^{\Re}\rho, {}^{\Im}\rho, {}^{\Re\Im}\rho$ denote the correlation coefficients between the real parts, the imaginary parts and between the real and imaginary part of different customers' currents, respectively. Voltage drop between the supply point and a nodal point k is denoted by U_k.

The customer current is obtained from their respective estimated power load by Newton-Raphson iterative method that is started from the nominal voltage. Hence the variables ${}^{\Re}\bar{\mathrm{I}}^i, {}^{\Im}\bar{\mathrm{I}}^i, {}^{\Re}\tilde{\mathrm{I}}^i$ and ${}^{\Im}\tilde{\mathrm{I}}^i$ are known for all customers connected to the network.

2.2 Computational algorithm

Recall the radial network is considered as a set of orbits, defined by nodes of the same depth. The operational quantities of the network can then be described by a set of state variables of each such orbit. The state variables are defined inside their respective propositions. The computation starts at the leaves, with customers' currents as entry data.

During the *first pass of the algorithm*, state variables related to current are calculated from the leaves towards the supply point. Variance of the current of each node k is easily obtained as a function of the state variables, parameterized by the correlation coefficients.

Proposition 1. (Current calculation). Let $\bar{I}_k$, ${}^{\Re}V_k^I$, ${}^{\Im}V_k^I$, ${}^{\Re}D_k^I$ and ${}^{\Im}D_k^I$ be state variables associated to the analyzed node at the orbit k, defined by following recurrent formulas (state equations):

$$\begin{aligned}
\bar{I}_k &= \sum_{i\in\mathcal{B}_k} [\bar{I}_{k+1}]^i + \sum_{i\in\mathcal{C}_k} \bar{\mathrm{I}}^i \\
{}^{\Re}V_k^I &= \sum_{i\in\mathcal{B}_k} [{}^{\Re}V_{k+1}^I]^i + \sum_{i\in\mathcal{C}_k} ({}^{\Re}\tilde{\mathrm{I}}^i)^2 \\
{}^{\Im}V_k^I &= \sum_{i\in\mathcal{B}_k} [{}^{\Im}V_{k+1}^I]^i + \sum_{i\in\mathcal{C}_k} ({}^{\Im}\tilde{\mathrm{I}}^i)^2 \\
{}^{\Re}D_k^I &= \sum_{i\in\mathcal{B}_k} [{}^{\Re}D_{k+1}^I]^i + \sum_{i\in\mathcal{C}_k} {}^{\Re}\tilde{\mathrm{I}}^i \\
{}^{\Im}D_k^I &= \sum_{i\in\mathcal{B}_k} [{}^{\Im}D_{k+1}^I]^i + \sum_{i\in\mathcal{C}_k} {}^{\Im}\tilde{\mathrm{I}}^i,
\end{aligned}$$

where $[{}^{\Re}\bar{I}_{k+1}]^i$ stands for the mean of the real part of the current entering the node k via the branch $i \in \mathcal{B}_k$, and the same notation applies for the remaining state variables. Then the first two moments of the stochastic variable I_k are the mean $\bar{I}_k$ and variance

$$\begin{aligned}
\tilde{I}_k^2 &= (1-{}^{\Re}\rho){}^{\Re}V_k^I + {}^{\Re}\rho({}^{\Re}D_k^I)^2 \\
&\quad +(1-{}^{\Im}\rho){}^{\Im}V_k^I + {}^{\Im}\rho({}^{\Im}D_k^I)^2 \\
&=: ({}^{\Re}\tilde{I}_k)^2 + ({}^{\Im}\tilde{I}_k)^2.
\end{aligned} \tag{1}$$

Corollary 1. Assume the probability distribution of the current I_k at the analyzed node k can be described by the normal distribution $\mathcal{N}(\bar{I}_k, \tilde{I}_k)$. Then its boundary value $(I_k)_{bv_p}$, which will not be exceeded with probability p, is given by

$$(I_k)_{bv_p} = \bar{I}_k + \chi_p \tilde{I}_k,$$

where χ_p is the p^{th} quantile of the unit normal distribution $\mathcal{N}(0,1)$.

The *second pass of the algorithm* proceeds from the supply point towards the leaves. State variables related to current are computed for each node. Variance of the voltage drop between a node k and the supply point is a function of the state variables, impedance of the network branches and is parameterized by the correlation coefficients.

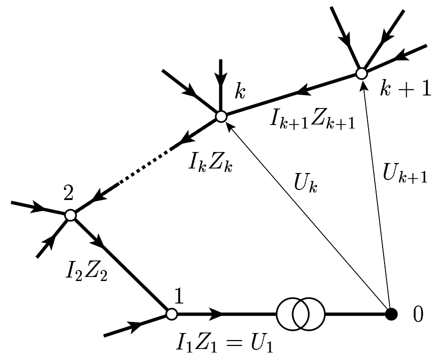

Fig. 2. Voltage analysis in radial distribution network

Proposition 2. (Voltage calculation). Let $\bar{U}_k$, ${}^{\Re}V_k^U$, ${}^{\Im}V_k^U$, ${}^{\Re}D_k^U$, ${}^{\Im}D_k^U$, ${}^{\Re\Im_z}D_k^U$ and ${}^{\Im\Re_z}D_k^U$ be state variables associated to the analyzed node at the orbit k, defined by the following recurrent formulas (state equations):

$$\begin{aligned}
\bar{U}_k &= \bar{U}_{k-1} + Z_k\bar{I}_k \\
{}^{\Re}V_k^U &= {}^{\Re}V_{k-1}^U + Z_0^k\,{}^{\Re}V_k^I \\
{}^{\Im}V_k^U &= {}^{\Im}V_{k-1}^U + Z_0^k\,{}^{\Im}V_k^I \\
{}^{\Re}D_k^U &= {}^{\Re}D_{k-1}^U + {}^{\Re}Z_k\,{}^{\Re}D_k^I \\
{}^{\Im}D_k^U &= {}^{\Im}D_{k-1}^U + {}^{\Im}Z_k\,{}^{\Im}D_k^I \\
{}^{\Re\Im_z}D_k^U &= {}^{\Re\Im_z}D_{k-1}^U + {}^{\Im}Z_k\,{}^{\Re}D_k^I \\
{}^{\Im\Re_z}D_k^U &= {}^{\Im\Re_z}D_{k-1}^U + {}^{\Re}Z_k\,{}^{\Im}D_k^I,
\end{aligned}$$

where Z_k denotes impedance of a branch from the analyzed node k, $k = 1, 2, \ldots$, towards the supply point, Z_0^k is given by

$$Z_0^k = {}^{\Re}Z_k^2 + {}^{\Im}Z_k^2 + 2\big({}^{\Re}Z_k \sum_{i=1}^{k-1} {}^{\Re}Z_i + {}^{\Im}Z_k \sum_{i=1}^{k-1} {}^{\Im}Z_i\big),$$

and $\bar{U}_0 = {}^{\Re}V_0^U = {}^{\Im}V_0^U = {}^{\Re}D_0^U = {}^{\Im}D_0^U = {}^{\Re\Im_z}D_0^U = {}^{\Im\Re_z}D_0^U = 0$.

The mean and variance of the voltage drop U_k between the supply point and the analyzed node k are given by $\bar{U}_k$ and

$$\begin{aligned}
\tilde{U}_k^2 &= (1-{}^{\Re}\rho){}^{\Re}V_k^U + (1-{}^{\Im}\rho){}^{\Im}V_k^U \\
&\quad + {}^{\Re}\rho\big(({}^{\Re}D_k^U)^2 + ({}^{\Re\Im_z}D_k^U)^2\big) \\
&\quad + {}^{\Im}\rho\big(({}^{\Im}D_k^U)^2 + ({}^{\Im\Re_z}D_k^U)^2\big) \\
&\quad - 2{}^{\Re\Im}\rho\big({}^{\Re}D_k^U\,{}^{\Im}D_k^U - {}^{\Re\Im_z}D_k^U\,{}^{\Im\Re_z}D_k^U\big),
\end{aligned} \tag{2}$$

respectively.

Corollary 2. Assume the probability distribution of the voltage drop U_k between the supply and the analyzed node k can be described by the normal distribution $\mathcal{N}(\bar{U}_k, \tilde{U}_k)$. Then its boundary value $(U_k)_{bv_p}$ is given by

$$(U_k)_{bv_p} = \bar{U}_k + \chi_p \tilde{U}_k,$$

where χ_p is the p^{th} quantile of the unit normal distribution $\mathcal{N}(0,1)$.

3. SUMMARY

A method of computing operational quantities in a radial electrical network was presented. It enables to obtain the second moment of current and voltage at any place of the network. Moreover, the variance formulae presented in Proposition 1 and Proposition 2 are in a closed form. They are functions of the state variables, impedance and correlation coefficient. As the physical properties of the network are known and the state variables can be precomputed, any change in assumed correlation among customers' currents can be quickly reflected in the estimated boundary values estimation.

The algorithm was implemented in a robust application INVYS, which provides a convenient user interface for various theoretical methods. It was successfully tested by West Bohemia Power Distribution Company on 10 low-voltage networks with hundreds of customers in each network (Krynský, 2003).

There remain several issues to be worked on. Extending the algorithm to the case where customers are split into several classes according to their load profiles (Janeček *et al.*, 1997) is one of them. Then the correlation coefficients must reflect different correlations among the customers groups. Also, fit of other probability distributions than normal should be tested. This is especially important when interested in the lower boundary value of the operational quantities.

Finally, note that the proposed algorithm can be used in any network where the analogy with electrical networks and Kirchhoff's current law applies.

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Appendix A. PROOFS

More on notation. All proofs presented bellow are based on induction. An explicit expression of the number of branches or customers directly connected to a node is then needed. Let $\mathcal{B}_k$ be a set. Its cardinality is denoted by $|\mathcal{B}_k|$. Furthermore, the argument (n_k), e.g. $\tilde{I}_k(n_k)$, implies that $\tilde{I}_k$ was calculated for n_k branches entering the node k together with customers from the set $\mathcal{C}_k$. With an abuse of notation, the argument may sometimes denote the number of customers connected to the node, that is, be equal to $|\mathcal{C}_k|$. As this notation is only used for leaf nodes, it should not lead to confusion.

A.1 Proof of Proposition 1

The formula for the mean $\bar{I}_k$ follows from Kirchhoff's current law, Figure 1 and from linearity of $E\{\cdot\}$. To prove the variance formula, recall that for any complex stochastic variable X holds that $\tilde{X}^2 = {}^{\Re}\tilde{X}^2 + {}^{\Im}\tilde{X}^2$. Hence we only need to prove (1) for ${}^{\Re}\tilde{I}_k^2$, the procedure for ${}^{\Im}\tilde{I}_k^2$ being analogous.

Claim 1. Let l be the depth of a leaf node, that is, $\mathcal{B}_l = \emptyset$. Then, variance of the real part of current I_l is given by

$$\begin{aligned}{}^{\Re}\tilde{I}_l^2 &= (1 - {}^{\Re}\rho)\sum_{i\in\mathcal{C}_l}({}^{\Re}\tilde{\mathrm{I}}^i)^2 + {}^{\Re}\rho(\sum_{i\in\mathcal{C}_l}{}^{\Re}\tilde{\mathrm{I}}^i)^2 \\ &=: (1 - {}^{\Re}\rho){}^{\Re}V_l^I + {}^{\Re}\rho({}^{\Re}D_l^I)^2.\end{aligned}$$

Proof. Let the cardinality of $\mathcal{C}_l$ be one. Then, the variance of the real part of the current can be rewritten as

$${}^{\Re}\tilde{I}_l^2(1) = (1 - {}^{\Re}\rho)({}^{\Re}\tilde{\mathrm{I}}^1)^2 + {}^{\Re}\rho({}^{\Re}\tilde{\mathrm{I}}^1)^2.$$

Assume now that

$${}^{\Re}\tilde{I}_l^2(n) = (1 - {}^{\Re}\rho)\sum_{i=1}^{n}({}^{\Re}\tilde{\mathrm{I}}^i)^2 + {}^{\Re}\rho(\sum_{i=1}^{n}{}^{\Re}\tilde{\mathrm{I}}^i)^2$$

holds for $|\mathcal{C}_l| = n$. Let $|\mathcal{C}_l| = n+1$. Then,

$$\begin{aligned}{}^{\Re}\tilde{I}_l^2(n+1) &= \mathrm{var}(\textstyle\sum_{i=1}^{n}{}^{\Re}\mathrm{I}^i) + \mathrm{var}({}^{\Re}\mathrm{I}^{n+1}) \\ &\quad +2\,\mathrm{cov}(\textstyle\sum_{i=1}^{n}{}^{\Re}\mathrm{I}^i, {}^{\Re}\mathrm{I}^{n+1}) \\ &= {}^{\Re}\tilde{I}_l^2(n) + ({}^{\Re}\tilde{\mathrm{I}}^{n+1})^2 \\ &\quad +2\textstyle\sum_{i=1}^{n}\mathrm{cov}({}^{\Re}\mathrm{I}^i, {}^{\Re}\mathrm{I}^{n+1}).\end{aligned}$$

Using the fact that $\mathrm{cov}({}^{\Re}\mathrm{I}^i, {}^{\Re}\mathrm{I}^{n+1}) = {}^{\Re}\rho^{\Re}\tilde{\mathrm{I}}^{i\Re}\tilde{\mathrm{I}}^{n+1}$, substituting for ${}^{\Re}\tilde{I}_l^2(n)$ from above and rearranging terms, we obtain

$$\begin{aligned}{}^{\Re}\tilde{I}_l^2(n+1) &= (1-{}^{\Re}\rho)\textstyle\sum_{i=1}^n({}^{\Re}\tilde{\mathrm{I}}^i)^2 + {}^{\Re}\rho(\sum_{i=1}^n {}^{\Re}\tilde{\mathrm{I}}^i)^2 \\ &\quad +(1-{}^{\Re}\rho)({}^{\Re}\tilde{\mathrm{I}}^{n+1})^2 + {}^{\Re}\rho({}^{\Re}\tilde{\mathrm{I}}^{n+1})^2 \\ &\quad +2\textstyle\sum_{i=1}^n {}^{\Re}\rho^{\Re}\tilde{\mathrm{I}}^{i\Re}\tilde{\mathrm{I}}^{n+1} \\ &= (1-{}^{\Re}\rho)\underbrace{\textstyle\sum_{i=1}^{n+1}({}^{\Re}\tilde{\mathrm{I}}^i)^2}_{{}^{\Re}V_l^I(n+1)} + {}^{\Re}\rho(\underbrace{\textstyle\sum_{i=1}^{n+1}{}^{\Re}\tilde{\mathrm{I}}^i}_{{}^{\Re}D_l^I(n+1)})^2.\end{aligned}$$

Induction argument gives the claim. □

The computation proceeds from the leaves towards the root (supply node). Assume that

$${}^{\Re}\tilde{I}_{k+1}^2 = (1-{}^{\Re}\rho){}^{\Re}V_{k+1}^I + {}^{\Re}\rho({}^{\Re}D_{k+1}^I)^2$$

holds for a node with depth $k+1$. Now, if there is only one branch entering the node k, i.e., if $|\mathcal{B}_k| = 1$ and $|\mathcal{C}_k| = \emptyset$, then clearly ${}^{\Re}\tilde{I}_k^2 = {}^{\Re}\tilde{I}_{k+1}^2$, ${}^{\Re}V_k^I = {}^{\Re}V_{k+1}^I$ and ${}^{\Re}D_k^I = {}^{\Re}D_{k+1}^I$.

Suppose (1) holds for $|\mathcal{B}_k| = n_k$. Then, if $|\mathcal{B}_k| = n_k+1$, it follows for the current of the node k that

$$\begin{aligned}{}^{\Re}\tilde{I}_k^2(n_k+1) &= \mathrm{var}\left(\textstyle\sum_{i=1}^{n_k}[{}^{\Re}I_{k+1}]^i + \sum_{i\in\mathcal{C}_k}{}^{\Re}\mathrm{I}^i\right. \\ &\qquad \left.+[{}^{\Re}I_{k+1}]^{n_k+1}\right) \\ &= (1-{}^{\Re}\rho){}^{\Re}V_k^I(n_k) + {}^{\Re}\rho({}^{\Re}D_k^I(n_k))^2 \\ &\quad +(1-{}^{\Re}\rho)[{}^{\Re}V_{k+1}^I]^{n_k+1} + {}^{\Re}\rho([{}^{\Re}D_{k+1}^I]^{n_k+1})^2 \\ &\quad +2\,\mathrm{cov}({}^{\Re}I_k(n_k), [{}^{\Re}I_{k+1}]^{n_k+1}).\end{aligned}$$

Let $\mathcal{C}_k^i$ denote a set of all customers connected to the node k via branch i. Expanding the arguments of the covariance term, applying linearity property and regrouping with the use of the definition of ${}^{\Re}D_k^I$,

$$\begin{aligned}&\mathrm{cov}\left({}^{\Re}I_k(n), [{}^{\Re}I_{k+1}]^{n+1}\right) = \\ &= \mathrm{cov}\Big(\sum_{i=1}^n\sum_{j\in\mathcal{C}_k^j}{}^{\Re}\mathrm{I}^j + \sum_{i\in\mathcal{C}_k}{}^{\Re}\mathrm{I}^i, \sum_{l\in\mathcal{C}_k^{n+1}}{}^{\Re}\mathrm{I}^l\Big) \\ &= \sum_{l\in\mathcal{C}_k^{n+1}}\mathrm{cov}\Big(\sum_{i=1}^n\sum_{j\in\mathcal{C}_k^j}{}^{\Re}\mathrm{I}^j + \sum_{i\in\mathcal{C}_k}{}^{\Re}\mathrm{I}^i, {}^{\Re}\mathrm{I}^l\Big) \\ &= {}^{\Re}\rho\Big(\sum_{i=1}^n\sum_{j\in\mathcal{C}_k^j}{}^{\Re}\tilde{\mathrm{I}}^j + \sum_{i\in\mathcal{C}_k}{}^{\Re}\tilde{\mathrm{I}}^i\Big)\sum_{l\in\mathcal{C}_k^{n+1}}{}^{\Re}\tilde{\mathrm{I}}^l \\ &= {}^{\Re}\rho^{\Re}D_k^I(n)[{}^{\Re}D_{k+1}^I]^{n+1}.\end{aligned}$$

Substituting into above yields

$$\begin{aligned}{}^{\Re}\tilde{I}_k^2(n_k+1) &= (1-{}^{\Re}\rho){}^{\Re}V_k^I(n_k+1) \\ &\quad +{}^{\Re}\rho({}^{\Re}D_k^I(n_k+1))^2,\end{aligned}$$

which, together with Claim 1 and the induction argument finishes the proof. □

A.2 *Proof of Proposition 2*

Before proceeding to the proof, it is useful to give the following lemma.

Lemma 1. Let I_k and I_l be currents supplying disjunct set of customers. Their covariance is given by

$$\begin{aligned}\mathrm{cov}(I_k, I_l) &= {}^{\Re}\rho^{\Re}D_k^{I\Re}D_l^I + {}^{\Im}\rho^{\Im}D_k^{I\Im}D_l^I \\ &\quad +j^{\Re\Im}\rho({}^{\Im}D_k^{I\Re}D_l^I - {}^{\Re}D_k^{I\Im}D_l^I), \quad (\mathrm{A.1})\end{aligned}$$

where the variables ${}^{\Re}D_i^I, {}^{\Im}D_i^I$, $i = k, l$, are as defined in Proposition 1.

Proof. As Lemma 1 is principally similar to Proposition 1, the proof will be stated briefly.

Claim 2. Consider two different leaf nodes of depth k and l, that is, $\mathcal{B}_k = \mathcal{B}_l = \emptyset$. Covariance of their respective currents I_k and I_l is given by

$$\begin{aligned}\mathrm{cov}(I_k, I_l) &= {}^{\Re}\rho\sum_{i\in\mathcal{C}_k}{}^{\Re}\tilde{\mathrm{I}}^i\sum_{j\in\mathcal{C}_l}{}^{\Re}\tilde{\mathrm{I}}^j + {}^{\Im}\rho\sum_{i\in\mathcal{C}_k}{}^{\Im}\tilde{\mathrm{I}}^i\sum_{j\in\mathcal{C}_l}{}^{\Im}\tilde{\mathrm{I}}^j \\ &\quad + j\ {}^{\Re\Im}\rho\Big(\sum_{i\in\mathcal{C}_k}{}^{\Im}\tilde{\mathrm{I}}^i\sum_{j\in\mathcal{C}_l}{}^{\Re}\tilde{\mathrm{I}}^j - \sum_{i\in\mathcal{C}_k}{}^{\Re}\tilde{\mathrm{I}}^i\sum_{j\in\mathcal{C}_l}{}^{\Im}\tilde{\mathrm{I}}^j\Big).\end{aligned}$$

Proof. Let $|\mathcal{C}_k| = |\mathcal{C}_l| = 1$. By definition

$$\begin{aligned}\mathrm{cov}(I_k, I_l) &= E\{(\mathrm{I}_k^1 - \bar{\mathrm{I}}_k^1)(\mathrm{I}_l^1 - \bar{\mathrm{I}}_l^1)^*\} \\ &= {}^{\Re}\rho^{\Re}\tilde{\mathrm{I}}_k^{1\Re}\tilde{\mathrm{I}}_l^1 + {}^{\Im}\rho^{\Im}\tilde{\mathrm{I}}_k^{1\Im}\tilde{\mathrm{I}}_l^1 \\ &\quad +j^{\Re\Im}\rho({}^{\Im}\tilde{\mathrm{I}}_k^{1\Re}\tilde{\mathrm{I}}_l^1 - {}^{\Re}\tilde{\mathrm{I}}_k^{1\Im}\tilde{\mathrm{I}}_l^1).\end{aligned}$$

Assume now that the claim holds for $|\mathcal{C}_k| = n_k$ and $|\mathcal{C}_l| = n_l$. Then, it follows for $|\mathcal{C}_k| = n_k+1$ that

$$\begin{aligned}&\mathrm{cov}\left(I_k(n_k+1), I_l(n_l)\right) = \\ &\quad = \mathrm{cov}\left(\textstyle\sum_{i=1}^{n_k}\mathrm{I}_k^i + \mathrm{I}_k^{n_k+1}, \sum_{j=1}^{n_l}\mathrm{I}_l^j\right) \\ &\quad = \mathrm{cov}\left(I_k(n_k), I_l(n_l)\right) + \textstyle\sum_{j=1}^{n_l}\mathrm{cov}(\mathrm{I}_k^{n_k+1}, \mathrm{I}_l^j).\end{aligned}$$

By hypothesis and rearranging terms

$$\begin{aligned}&\mathrm{cov}\left(I_k(n_k+1), I_l(n_l)\right) = \\ &= {}^{\Re}\rho\sum_{i=1}^{n_k}{}^{\Re}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Re}\tilde{\mathrm{I}}_l^j + {}^{\Im}\rho\sum_{i=1}^{n_k}{}^{\Im}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Im}\tilde{\mathrm{I}}_l^j \\ &\quad + j\ {}^{\Re\Im}\rho\Big(\sum_{i=1}^{n_k}{}^{\Im}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Re}\tilde{\mathrm{I}}_l^j - \sum_{i=1}^{n_k}{}^{\Re}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Im}\tilde{\mathrm{I}}_l^j\Big) \\ &\quad + \sum_{j=1}^{n_l}\Big({}^{\Re}\rho^{\Re}\tilde{\mathrm{I}}_k^{n_k+1\Re}\tilde{\mathrm{I}}_l^j + {}^{\Im}\rho^{\Im}\tilde{\mathrm{I}}_k^{n_k+1\Im}\tilde{\mathrm{I}}_l^j \\ &\quad +j\ {}^{\Re\Im}\rho({}^{\Im}\tilde{\mathrm{I}}_k^{n_k+1\Re}\tilde{\mathrm{I}}_l^j - {}^{\Re}\tilde{\mathrm{I}}_k^{n_k+1\Im}\tilde{\mathrm{I}}_l^j)\Big) \\ &= {}^{\Re}\rho\sum_{i=1}^{n_k+1}{}^{\Re}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Re}\tilde{\mathrm{I}}_l^j + {}^{\Im}\rho\sum_{i=1}^{n_k+1}{}^{\Im}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Im}\tilde{\mathrm{I}}_l^j \\ &\quad + j\ {}^{\Re\Im}\rho\Big(\sum_{i=1}^{n_k+1}{}^{\Im}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Re}\tilde{\mathrm{I}}_l^j - \sum_{i=1}^{n_k+1}{}^{\Re}\tilde{\mathrm{I}}_k^i\sum_{j=1}^{n_l}{}^{\Im}\tilde{\mathrm{I}}_l^j\Big).\end{aligned}$$

Skew symmetry of covariance and induction argument conclude the proof. □

Assume that (A.1) holds for $|\mathcal{B}_k| = n_k$ and $|\mathcal{B}_l| = n_l$. For $|\mathcal{B}_k| = n_k + 1$ then follows that

$$\begin{aligned}\operatorname{cov}\big(I_k(n_k+1), I_l(n_l)\big) &= \operatorname{cov}\big(I_k(n_k), I_l(n_l)\big)\\ &\quad + \operatorname{cov}\big([I_{k+1}]^{n_k+1}, I_l(n_l)\big)\end{aligned}$$

Expanding the branch currents up to the individual customers' currents and regrouping again as in the proof of Proposition 1, we obtain that

$$\begin{aligned}\operatorname{cov}\big([I_{k+1}]^{n_k+1}, I_l(n_l)\big) &=\\ &= \textstyle\sum_{i\in\mathcal{C}_k^{n_k+1}} \operatorname{cov}\big(\mathrm{I}^i, I_l(n_l)\big)\\ &= {}^{\Re}\rho[{}^{\Re}D^I_{k+1}]^{n_k+1}\,{}^{\Re}D^I_l(n_l)\\ &\quad + {}^{\Im}\rho[{}^{\Im}D^I_{k+1}]^{n_k+1}\,{}^{\Im}D^I_l(n_l)\\ &\quad + j\ {}^{\Re\Im}\rho\big([{}^{\Im}D^I_{k+1}]^{n_k+1}\,{}^{\Re}D^I_l(n_l)\\ &\qquad - [{}^{\Re}D^I_{k+1}]^{n_k+1}\,{}^{\Im}D^I_l(n_l)\big).\end{aligned}$$

Substitute the result into the equation above, use the hypothesis and the fact that

$$\begin{aligned}{}^{\Re}D^I_i(n_i+1) &= {}^{\Re}D^I_i(n_i) + [{}^{\Re}D^I_{i+1}]^{n_i+1}\\ {}^{\Im}D^I_i(n_i+1) &= {}^{\Im}D^I_i(n_i) + [{}^{\Im}D^I_{i+1}]^{n_i+1}\end{aligned}$$

holds for any i, to obtain the required formula. Skew-symmetry of covariance finishes the proof. □

Proof of Proposition 2 Denoting the voltage drop between two nodes, k and l, by $\Delta U_{k,l}$, the voltage drop between the supply point and a node k is given by

$$U_k = U_{k-1} + \Delta U_{k,k-1} := U_{k-1} + Z_k I_k.$$

Application of the expected value operator yields the formula for $\bar{U}_k$.

By definition, substituting for $\tilde{I}_1^2$ from Proposition 1, rearranging and using the trick of adding zero we obtain that for a node of depth 1 holds

$$\begin{aligned}\tilde{U}_1^2 &= \operatorname{var}(U_1) = Z_1 \operatorname{var}(I_1) Z_1^* = ({}^{\Re}Z_1^2 + {}^{\Im}Z^2)\tilde{I}_1^2\\ &= ({}^{\Re}Z_1^2 + {}^{\Im}Z^2)\big((1-{}^{\Re}\rho){}^{\Re}V_1^I\\ &\quad + {}^{\Re}\rho({}^{\Re}D_1^I)^2 + (1-{}^{\Im}\rho){}^{\Im}V_1^I + {}^{\Im}\rho({}^{\Im}D_1^I)^2\big)\\ &= (1-{}^{\Re}\rho){}^{\Re}V_1^U + (1-{}^{\Im}\rho){}^{\Im}V_1^U\\ &\quad + {}^{\Re}\rho\big(({}^{\Re}D_1^U)^2 + ({}^{\Re\Im z}D_1^U)^2\big)\\ &\quad + {}^{\Im}\rho\big(({}^{\Im}D_1^U)^2 + ({}^{\Im\Re z}D_1^U)^2\big)\\ &\quad - 2\,{}^{\Re\Im}\rho\big({}^{\Re}D_1^U\,{}^{\Im}D_1^U - {}^{\Re\Im z}D_1^U\,{}^{\Im\Re z}D_1^U\big).\end{aligned}$$

Assume (2) holds for a node at an orbit k. Moving one step towards the leaves, variance of the voltage drop U_{k+1} is given by

$$\begin{aligned}\tilde{U}_{k+1}^2 &= \operatorname{var}(U_k + \Delta U_{k+1,k})\\ &= \tilde{U}_k^2 + \operatorname{cov}(U_k, \Delta U_{k+1,k})\\ &\quad + \operatorname{cov}(\Delta U_{k+1,k}, U_k) + \Delta\tilde{U}_{k+1}^2 \qquad \text{(A.2)}\\ &= \tilde{U}_k^2 + \Delta\tilde{U}_{k+1}^2 + 2Re\{\operatorname{cov}(U_k, \Delta U_{k+1,k})\},\end{aligned}$$

where $Re(X)$ gives the real part of a complex variable X. Let $\Delta I_{k,l} = I_k - I_l$ be an auxiliary variable denoting the current of customers supplied via the node k, but not via the node l. Then, $I_i = \Delta I_{i,k+1} + I_{k+1}$, $i = 1, \dots k$, and

$$\begin{aligned}\operatorname{cov}(U_k, \Delta U_{k+1,k}) &= \textstyle\sum_{i=1}^k Z_i \operatorname{cov}(I_i, I_{k+1}) Z_{k+1}^*\\ &= \textstyle\sum_{i=1}^k Z_i \operatorname{cov}(\Delta I_{i,k+1}, I_{k+1}) Z_{k+1}^*\\ &\quad + \textstyle\sum_{i=1}^k Z_i \tilde{I}_{k+1}^2 Z_{k+1}^*.\end{aligned}$$

Since $\Delta I_{i,k+1}$ does not contain the currents of customers supplied by I_{k+1}, Lemma 1 can be applied to evaluate $\operatorname{cov}(\Delta I_{i,k+1}, I_{k+1})$. Hence

$$\begin{aligned}&Re\{\operatorname{cov}(U_k, \Delta U_{k+1,k})\} =\\ &\quad = \textstyle\sum_{i=1}^k ({}^{\Re}Z_{k+1}{}^{\Re}Z_i + {}^{\Im}Z_{k+1}{}^{\Im}Z_i)\\ &\qquad \cdot\big(\tilde{I}_{k+1}^2 + {}^{\Re}\rho\,{}^{\Re}\Delta D^I_{i,k+1}\,{}^{\Re}D^I_{k+1}\\ &\qquad + {}^{\Im}\rho\,{}^{\Im}\Delta D^I_{i,k+1}\,{}^{\Im}D^I_{k+1}\big)\\ &\quad - {}^{\Re}\rho \textstyle\sum_{i=1}^k ({}^{\Re}Z_{k+1}{}^{\Im}Z_i - {}^{\Im}Z_{k+1}{}^{\Re}Z_i)\\ &\qquad \cdot({}^{\Im}\Delta D^I_{i,k+1}\,{}^{\Re}D^I_{k+1} - {}^{\Re}\Delta D^I_{i,k+1}\,{}^{\Im}D^I_{k+1}).\end{aligned}$$

Finally, use the facts that $\Delta D^I_{i,k+1} = D^I_i - D^I_{k+1}$, and that $\Delta\tilde{U}_{k+1}^2 = ({}^{\Re}Z_{k+1}^2 + {}^{\Im}Z_{k+1}^2)\tilde{I}_{k+1}^2$, where $\tilde{I}_{k+1}^2$ was given in Proposition 1 and note that

$$\begin{aligned}{}^{\Re}D_k^U &= \textstyle\sum_{i=1}^k {}^{\Re}Z_i{}^{\Re}D_i^I & {}^{\Re\Im z}D_k^U &= \textstyle\sum_{i=1}^k {}^{\Im}Z_i{}^{\Re}D_i^I\\ {}^{\Im}D_k^U &= \textstyle\sum_{i=1}^k {}^{\Im}Z_i{}^{\Im}D_i^I & {}^{\Im\Re z}D_k^U &= \textstyle\sum_{i=1}^k {}^{\Re}Z_i{}^{\Im}D_i^I.\end{aligned}$$

Substituting into (A.2) and regrouping judiciously, we obtain

$$\begin{aligned}\tilde{U}_{k+1}^2 &= (1-{}^{\Re}\rho)({}^{\Re}V_k^U + Z_0^{k+1}{}^{\Re}V_{k+1}^I)\\ &\quad + (1-{}^{\Im}\rho)({}^{\Im}V_k^U + Z_0^{k+1}{}^{\Im}V_{k+1}^I)\\ &\quad + {}^{\Re}\rho({}^{\Re}D_k^U + {}^{\Re}Z_{k+1}{}^{\Re}D^I_{k+1})^2\\ &\quad + {}^{\Re}\rho({}^{\Re\Im z}D_k^U + {}^{\Im}Z_{k+1}{}^{\Re}D^I_{k+1})^2\\ &\quad + {}^{\Im}\rho({}^{\Im}D_k^U + {}^{\Im}Z_{k+1}{}^{\Im}D^I_{k+1})^2\\ &\quad + {}^{\Im}\rho({}^{\Im\Re z}D_k^U + {}^{\Re}Z_{k+1}{}^{\Im}D^I_{k+1})^2\\ &\quad - 2\,{}^{\Re\Im}\rho\big(({}^{\Re}D_k^U + {}^{\Re}Z_{k+1}{}^{\Re}D^I_{k+1})\\ &\qquad \cdot({}^{\Im}D_k^U + {}^{\Im}Z_{k+1}{}^{\Im}D^I_{k+1})\\ &\qquad - ({}^{\Re\Im z}D_k^U + {}^{\Im}Z_{k+1}{}^{\Re}D^I_{k+1})\\ &\qquad \cdot({}^{\Im\Re z}D_k^U + {}^{\Re}Z_{k+1}{}^{\Im}D^I_{k+1})\big),\end{aligned}$$

which is the desired form concluding the proof. □