OVERLAPPING GUARANTEED COST CONTROL FOR UNCERTAIN CONTINUOUS-TIME DELAYED SYSTEMS

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Abstract: Overlapping guaranteed cost control design problem is solved for a class of linear continuous-time uncertain systems with state as well as control delays. Unknown arbitrarily time-varying uncertainties with known bounds are considered. A point delay is supposed. Conditions preserving closed-loop systems expansion-contraction relations including the identical bounds of performance indices are proved. A linear matrix inequality (LMI) delay independent procedure is used for control design in the expanded space. The results are specialized on the overlapping decentralized control design. A numerical illustrative example is supplied. *Copyright© 2005 IFAC*

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1. INTRODUCTION

Information structure constraints may be classified according to the structure of the gain matrix. In fact, three different practically important forms of the gain matrix are usually considered, i.e. a block diagonal gain matrix, a block tridiagonal gain matrix, and a double block bordered gain matrix. A systematic way of the controller design with a block tridiagonal gain matrix lead to the concept of overlapping decompositions. A general mathematical framework for this approach has been called the Inclusion Principle (Ikeda and Šiljak 1980), (Šiljak 1991). The Inclusion Principle has been applied to different classes of systems and problems as illustrated for instance in (Bakule *et al.* 2000*a*), (Bakule *et al.* 2000*b*).

The paper deals with the guaranteed cost control problem within the framework of overlapping decompositions for a class of uncertain state and control-delayed continuous-time systems with quadratic performance index. The overlapping delay independent controller design is performed using a linear matrix inequality. The LMI solution of centralized quadratic guaranteed cost controller for this class of systems is presented in (Mukaidani 2003). It is used as a control design tool. The main results concern the conditions on the expansion-contraction relations between closed-loop systems including the requirement on the equality of bounds on costs. The specialization of these results on the decentralized overlapping control design is given including an illustrative example.

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To the authors knowledge, the expansion-contraction relations have not been extended up to now on the concept of guaranteed cost control for the considered class of problems.

2. PROBLEM FORMULATION

Consider a linear continuous-time uncertain system with state and control delay described by the state equation:

$$\begin{aligned}\mathbf{S}: \dot{x}(t) &= [A+\Delta A(t)]x(t)+[B+\Delta B(t)]u(t)\\ &\quad +[C+\Delta C(t)]x(t-d)+[D+\Delta D(t)]u(t-d),\\ x(t) &= \varphi(t), \quad -d \leqslant t \leqslant 0,\end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control, $d>0$ is a point delay and $\varphi(t)$ is a given continuous vector valued initial function. The set $\{x(t),u(s)\}$, $s\in[t-d,t]$, defines the *complete state* of the system (1) as is usual in the case of delays. A, B, C, D are known constant matrices of appropriate dimensions. $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, $\Delta D(t)$ are real-valued matrices of uncertain parameters. Uncertainties are assumed to be norm-bounded as follows:

$$[\Delta A(t)\ \Delta B(t)\ \Delta C(t)\ \Delta D(t)] = E\ F(t)\ [E_1\ E_2\ E_3\ E_4], \tag{2}$$

where E, E_1, E_2, E_3, E_4 are known constant real matrices of appropriate dimensions and $F(t)\in\mathbb{R}^{p\times q}$ is an unknown matrix function with Lebesgue measurable elements and such that

$$F^T(t)F(t)\leqslant I_q. \tag{3}$$

The cost function associated with (1) has the form

$$J(x,u)=\int_0^\infty \left[x^T(t)Q^*x(t)+u^T(t)R^*u(t)\right]dt, \tag{4}$$

where Q^* and R^* are positive-definite symmetric matrices. In order to simplify the notation, denote:

$$\begin{aligned}A+\Delta A(t) &= \bar{A}(t), & B+\Delta B(t) &= \bar{B}(t),\\ C+\Delta C(t) &= \bar{C}(t), & D+\Delta D(t) &= \bar{D}(t).\end{aligned} \tag{5}$$

Then, the system (1) can be rewritten as:

$$\begin{aligned}\mathbf{S}:\ \dot{x}(t) &= \bar{A}(t)x(t)+\bar{B}(t)u(t)+\\ &\quad +\bar{C}(t)x(t-d)+\bar{D}(t)u(t-d).\end{aligned} \tag{6}$$

Having an initial complete state $\{x(0),u(s)\}$, it is well-known that the unique solution of (6) is given by

$$\begin{aligned}x(t) &= \Phi(t,0)x(0)+\int_0^t \Phi(t,s)\bar{C}(s)x(s-d)\,ds\\ &\quad +\int_0^t \Phi(t,s)\left[\bar{B}(s)u(s)+\bar{D}(s)u(s-d)\right]ds,\end{aligned} \tag{7}$$

where Φ is the *transition matrix* of $\bar{A}(t)$. Similarly, consider a new system in the form

$$\begin{aligned}\tilde{\mathbf{S}}: \dot{\tilde{x}}(t) &= \left[\tilde{A}+\Delta\tilde{A}(t)\right]\tilde{x}(t)+\left[\tilde{B}+\Delta\tilde{B}(t)\right]u(t)\\ &\quad +\left[\tilde{C}+\Delta\tilde{C}(t)\right]\tilde{x}(t-d)+\left[\tilde{D}+\Delta\tilde{D}(t)\right]u(t-d),\\ \tilde{x}(t) &= \tilde{\varphi}(t), \quad -d\leqslant t\leqslant 0,\end{aligned} \tag{8}$$

with an associated cost function given by

$$\tilde{J}(\tilde{x},u)=\int_0^\infty \left[\tilde{x}^T(t)\tilde{Q}^*\tilde{x}(t)+u^T(t)\tilde{R}^*u(t)\right]ds, \tag{9}$$

where $\tilde{x}(t)\in\mathbb{R}^{\tilde{n}}$ is the state, $\tilde{\varphi}(t)$ is a continuous vector valued initial function and $\tilde{Q}^*$, $\tilde{R}^*$ are positive-definite symmetric matrices. Suppose that $n\leqslant\tilde{n}$. The set $\{\tilde{x}(t),u(s)\}$, $s\in[t-d,t]$, defines a complete state for (8). The conditions (2) and (3) for the system $\mathbf{S}$ are analogous for $\tilde{\mathbf{S}}$, but considering all matrices with tilde ($\tilde{}$). Under these assumptions, the unique solution of (8) has the form

$$\begin{aligned}\tilde{x}(t) &= \tilde{\Phi}(t,0)\tilde{x}(0)+\int_0^t \tilde{\Phi}(t,s)\tilde{\bar{C}}(s)\tilde{x}(s-d)\,ds\\ &\quad +\int_0^t \tilde{\Phi}(t,s)\left[\tilde{\bar{B}}(s)u(s)+\tilde{\bar{D}}(s)u(s-d)\right]ds,\end{aligned} \tag{10}$$

where $\{\tilde{x}(0),u(s)\}$ is a given initial complete state.

2.1 *The Inclusion Principle*

Denote $x(t)$=$x(t;\varphi(t),u(t))$ and $\tilde{x}(t)$=$\tilde{x}(t;\tilde{\varphi}(t),u(t))$ the formal solutions of (1) and (8) for given inputs $u(t)$ and initial complete states $\{x(0),u(s)\}$, $\{\tilde{x}(0),u(s)\}$, $s\in[-d,0]$, respectively. Consider the standard relations between the states within the Inclusion Principle. It means that the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the following linear transformations:

$$\tilde{x}(t)=Vx(t), \quad x(t)=U\tilde{x}(t), \tag{11}$$

where V and U are constant full-rank matrices of appropriate dimensions (Šiljak 1991).

Definition 1. A system $\tilde{\mathbf{S}}$ *includes* the system $\mathbf{S}$, $\tilde{\mathbf{S}}\supset\mathbf{S}$, if there exists a pair of constant matrices (U,V) such that UV=I_n and for any initial function $\varphi(t)$ and any fixed input $u(t)$ of $\mathbf{S}$, $x(t;\varphi(t),u(t))$=$U\tilde{x}(t;V\varphi(t),u(t))$ for all t.

Definition 2. A pair $(\tilde{\mathbf{S}},\tilde{J})$ is an *expansion* of $(\mathbf{S},J)$, $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$, if $\tilde{\mathbf{S}}\supset\mathbf{S}$ and $J(x,u)$= $\tilde{J}(Vx,u)$.

Definition 3. A control law $u(t)$=$\tilde{K}\tilde{x}(t)$ for $\tilde{\mathbf{S}}$ is *contractible* to $u(t)$=$Kx(t)$ for $\mathbf{S}$ if the choice $\tilde{\varphi}(t)$=$V\varphi(t)$ implies $Kx(t;\varphi(t),u(t))$=$\tilde{K}\tilde{x}(t;V\varphi(t),u(t))$ for all t, any initial function $\varphi(t)$ and any fixed input $u(t)$ of $\mathbf{S}$.

Suppose a given pair of matrices (U,V). Then, the matrices $\tilde{A}$, $\Delta\tilde{A}$, $\tilde{B}$, $\Delta\tilde{B}$, $\tilde{C}$, $\Delta\tilde{C}$, $\tilde{D}$, $\Delta\tilde{D}$, $\tilde{Q}^*$ and $\tilde{R}^*$ can be described as:

$$\begin{aligned}\tilde{A} &= VAU+M, & \Delta\tilde{A}(t) &= V\Delta A(t)U,\\ \tilde{B} &= VB+N, & \Delta\tilde{B}(t) &= V\Delta B(t),\\ \tilde{C} &= VCU+M_d, & \Delta\tilde{C}(t) &= V\Delta C(t)U,\\ \tilde{D} &= VD+N_d, & \Delta\tilde{D}(t) &= V\Delta D(t),\\ \tilde{Q}^* &= U^TQ^*U+M_{Q^*}, & \tilde{R}^* &= R^*+N_{R^*},\end{aligned} \tag{12}$$

where M, N, M_d, N_d, M_{Q^*} and N_{R^*} are so called *complementary matrices*. Usually, the transformations

(U,V) are selected a priori to define structural relations between the state variables in both systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$. Given these transformations, the choice of the complementary matrices gives degrees of freedom to obtain different expanded spaces with desirable properties (Bakule *et al.* 2000*a*), (Bakule *et al.* 2000*b*).

Some conditions on the complementary matrices (12) must be imposed on $(\tilde{\mathbf{S}},\tilde{J})$ to be an expansion of $(\mathbf{S},J)$ by Definition 2. This provides the following theorem.

Theorem 4. Consider the problems (1) - (4) and (8) - (9). A pair $(\tilde{\mathbf{S}},\tilde{J})$ includes the pair $(\mathbf{S},J)$ if and only if

$$\begin{array}{ll} U\tilde{\Phi}(t,0)V=\Phi(t,0), & U\tilde{\Phi}(t,s)M_dV=0, \\ U\tilde{\Phi}(t,s)N=0, & U\tilde{\Phi}(t,s)N_d=0, \\ V^TM_{Q^*}V=0, & N_{R^*}=0 \end{array} \tag{13}$$

hold for all t and s.

Proof. The proof of a similar theorem can be found in (Bakule *et al.* 2002). □

Remark. It is well-known that the general solution of time-varying systems is a big problem. Because of this, an attempt has been made to approximate the solutions using transition matrices. However, even to compute such approximation via Peano-Baker series can be a complicated task excluding trivial cases (Stanković and Šiljak 2003). For this reason, we present in Section 3 conditions under which $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$ and which eliminate the necessity to compute the transition matrices.

2.2 *Guaranteed Cost Control*

Consider the problem (1) - (4). The objective is to implement a linear time-invariant quadratic guaranteed cost control law $u{=}Kx(t)$ in the delay system (1) but as a contraction of a quadratic guaranteed cost control law $u{=}\tilde{K}\tilde{x}(t)$ designed for the problem (8) - (9). Moreover, the corresponding expanded closed-loop system is quadratically stable and guarantees an upper bound of the expanded quadratic cost function (9).

Definition 5. A control law $u(t){=}\tilde{K}\tilde{x}(t)$ is said to be a *quadratic guaranteed cost control* with an associated cost matrix $\tilde{P}{>}0$ for the delay system (8) and cost function (9) if the corresponding closed-loop system is quadratically stable, that is

$$\dot{\tilde{x}}^T(t)\tilde{P}\tilde{x}(t)+\tilde{x}^T(t)\left[\tilde{Q}^*+\tilde{K}^T\tilde{R}^*\tilde{K}\right]\tilde{x}(t)<0 \tag{14}$$

for all nonzero $\tilde{x}{\in}\mathbb{R}^{\tilde{n}}$ and the closed-loop value of the cost function (9) satisfies the bound $\tilde{J}{\leq}\tilde{J}_0$ for all admissible uncertainties, where $\tilde{J}_0$ is a given constant.

2.3 *LMI Approach*

There are available different approaches to compute quadratic guaranteed cost control laws. A delay independent linear matrix inequality (LMI) approach is selected to design a linear state memoryless feedback controller guaranteeing that the system is quadratically stable with a desired upper bound on the quadratic cost function. The following proposition gives sufficient conditions to get a guaranteed cost control law (Mukaidani 2003). To simplify, the result is presented only for the system (1) - (4), but it evidently holds also for the expanded system.

Theorem 6. Suppose there exist a constant parameters $\mu{>}0$, $\varepsilon{>}0$, a symmetric positive-definite matrices X, S, $Z{\in}\mathbb{R}^{n\times n}$ and a matrix $Y{\in}\mathbb{R}^{m\times n}$ such that the following LMI

$$\begin{bmatrix} \Psi & DY & (E_1X+E_2Y)^T & X & Y^T & CS & 0 & X \\ Y^TD^T & -Z & Y^TE_4^T & 0 & 0 & 0 & 0 & 0 \\ E_1X+E_2Y & E_4Y & -\mu I_q & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & -(Q^*)^{-1} & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & -(R^*)^{-1} & 0 & 0 & 0 \\ SC^T & 0 & 0 & 0 & 0 & -S & SE_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 & E_3S & -\varepsilon I_q & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & -S \end{bmatrix}<0 \tag{15}$$

holds, with $\Psi{:=}AX{+}BY{+}(AX{+}BY)^T{+}Z{+}(\mu{+}\varepsilon)EE^T$. Then, the feedback control law $u(t){=}Kx(t){=}YX^{-1}x(t)$ is a guaranteed cost controller and

$$J<\varphi^T(0)X^{-1}\varphi(0)+\int_{-d}^{0}\varphi^T(s)\left[S^{-1}+X^{-1}ZX^{-1}\right]\varphi(s)\,ds \tag{16}$$

is the guaranteed cost for the closed-loop uncertain delay system.

2.4 *The Problem*

Suppose given a linear continuous-time uncertain delayed system by $\mathbf{S}$ (1) - (3) with an associated cost function J by (4). Consider an expanded system $\tilde{\mathbf{S}}$ by (8) with an associated cost function $\tilde{J}$ by (9). Suppose that $\tilde{\mathbf{S}}\supset\mathbf{S}$ holds by Definition 1. Then, the specific goals are as follows:

- Derive conditions under which $(\tilde{\mathbf{S}}_{\mathbf{C}},\tilde{J})\supset(\mathbf{S}_{\mathbf{C}},J)$. Use the concept of quadratic guaranteed cost control.
- Specialize the global system results into decentralized control design setting.
- Supply these results with a numerical example.

Derive all the above results in terms of complementary matrices. Use the delay independent LMI approach to compute the required gain matrices.

3. MAIN CONTRIBUTION

The following theorem gives equivalent conditions to Theorem 4 expressed in terms of complementary matrices.

Theorem 7. Consider the problems (1) - (4), (8) - (9). A pair $(\tilde{\mathbf{S}},\tilde{J})$ includes the pair $(\mathbf{S},J)$ if and only if

$$UM^iV = 0, \quad UM^{i-1}M_dV = 0, \quad UM^{i-1}N = 0,$$
$$UM^{i-1}N_d = 0, \qquad V^TM_{Q^*}V = 0, \qquad N_{R^*} = 0 \tag{17}$$

hold for all i=$1,2,...,\tilde{n}$.

Proof. Consider the transition matrix $\tilde{\Phi}(t,0)$ of $\tilde{\tilde{A}}$, where $\tilde{\tilde{A}}$ represents the state matrix of the expanded space as a function of two variables defined by the Peano-Baker series (Rugh 1996)

$$\tilde{\Phi}(t,0) = I + \int_0^t \tilde{\tilde{A}}(\sigma_1)\,d\sigma_1 + \int_0^t \tilde{\tilde{A}}(\sigma_1)\int_0^{\sigma_1}\tilde{\tilde{A}}(\sigma_2)\,d\sigma_2 d\sigma_1 + \int_0^t \tilde{\tilde{A}}(\sigma_1)\int_0^{\sigma_1}\tilde{\tilde{A}}(\sigma_2)\int_0^{\sigma_2}\tilde{\tilde{A}}(\sigma_3)\,d\sigma_3 d\sigma_2 d\sigma_1 + \cdots \tag{18}$$

where according to (5) and (12), $\tilde{\tilde{A}}(\sigma_i)$=$\tilde{A}+\Delta\tilde{A}(\sigma_i)$= $VAU+M+V\Delta A(\sigma_i)U$ for all i=$1,2,...$ From Theorem 4 pre and post-multiplying both sides of $\tilde{\Phi}(t,0)$ by U and V, respectively, it is easy to prove that UM^iV=0, i=$1,2,...,\tilde{n}$, is equivalent to $U\tilde{\Phi}(t,0)V$=$\Phi(t,0)$. Following a similar reasoning, $U\tilde{\Phi}(t,s)M_dV$=0 is equivalent to $UM^{i-1}M_dV$=0. The condition $U\tilde{\Phi}(t,s)N$=0 is equivalent to $UM^{i-1}N$=0 and $U\tilde{\Phi}(t,s)N_d$=0 is equivalent to $UM^{i-1}N_d$=0, for all i=$1,2,...,\tilde{n}$. The remaining conditions $V^TM_{Q^*}V$=0, N_{R^*}=0 are the same as in Theorem 4. □

Remark. Theorem 7 allows to obtain expanded systems satisfying the Inclusion Principle with the same cost function without a precise knowledge of transition matrices. This observation can be surprising because in fact the systems (1), (8) are time-varying systems.

Proposition 8. Consider the problems (1) - (4) and (8) - (9). A pair $(\tilde{\mathbf{S}},\tilde{J})$ includes the pair $(\mathbf{S},J)$ if $V^TM_{Q^*}V$=0, N_{R^*}=0 and

$$a)\quad MV = 0, \quad M_dV = 0, \quad N = 0, \quad N_d = 0 \quad \text{or}$$
$$b)\quad UM = 0, \quad UM_d = 0, \quad UN = 0, \quad UN_d = 0. \tag{19}$$

Proof. The proof is straightforward from Theorem 7. □

Remark. If M_d=0, N_d=0 in (19), i.e. if $\mathbf{S}$ is not a delayed system, then *a)* and *b)* correspond to particular cases within the Inclusion Principle called *restrictions* and *aggregations*, respectively, (Šiljak 1991).

Definition 3 presents the conditions under which a control law designed in the expanded system $\tilde{\mathbf{S}}$ can be contracted and implemented into the initial system $\mathbf{S}$. However, these requirements do not guarantee that the closed-loop system $\tilde{\mathbf{S}}_\mathbf{C}$ includes the closed-loop system $\mathbf{S}_\mathbf{C}$ in the sense of the Inclusion Principle, i.e. $\tilde{\mathbf{S}}_\mathbf{C}\supset\mathbf{S}_\mathbf{C}$. Now, consider conditions which include also cost functions. They are presented in the following theorem by using complementary matrices.

Theorem 9. Consider the problems (1) - (4) and (8) - (9) satisfying $\tilde{\mathbf{S}}\supset\mathbf{S}$. Suppose that $u(t)$=$\tilde{K}\tilde{x}(t)$ is a contractible control law designed in $\tilde{\mathbf{S}}$. If MV=0, N=0, M_dV=0, N_d=0, $V^TM_{Q^*}V$=0 and N_{R^*}=0 then $(\tilde{\mathbf{S}}_\mathbf{C},\tilde{J})\supset(\mathbf{S}_\mathbf{C},J)$.

Proof. Suppose $\tilde{\mathbf{S}}\supset\mathbf{S}$ and consider $u(t)$=$\tilde{K}\tilde{x}(t)$ a contractible control law designed in $\tilde{\mathbf{S}}$. The corresponding state matrix of the closed-loop expanded system $\tilde{\mathbf{S}}_\mathbf{C}$ is given by

$$\begin{aligned}\tilde{\mathbf{S}}_\mathbf{C}: \dot{\tilde{x}}(t) &= \left[\tilde{A}+\Delta\tilde{A}(t)+[\tilde{B}+\Delta\tilde{B}(t)]\tilde{K}\right]\tilde{x}(t)\\ &\quad+\left[\tilde{C}+\Delta\tilde{C}(t)+[\tilde{D}+\Delta\tilde{D}(t)]\tilde{K}\right]\tilde{x}(t-d)\\ &= \tilde{A}_p(t)\tilde{x}(t)+\tilde{A}_q(t)\tilde{x}(t-d).\end{aligned} \tag{20}$$

Similar expression can be obtained for the closed-loop system $\mathbf{S}_\mathbf{C}$, where the contracted gain matrix K is given by K=$\tilde{K}V$ from Definition 3. Consider the relation between the state matrices of the closed-loop systems $\tilde{\mathbf{S}}_\mathbf{C}$ and $\mathbf{S}_\mathbf{C}$. The relation $\tilde{A}_p(t)$=$VA_p(t)U$+M_p implies M_p=M+$N\tilde{K}$+$VB\tilde{K}$−$VB\tilde{K}VU$+$V\Delta B(t)\tilde{K}(I-VU)$, where M_p is a complementary matrix to be determined. Since $\tilde{\mathbf{S}}_\mathbf{C}\supset\mathbf{S}_\mathbf{C}$ is desired, the condition UM_p^iV=0, i=$1,2,...,\tilde{n}$, must be satisfied. Imposing this requirement and using equations (12), we can prove that MV=0, N=0 is a sufficient condition so that the relation UM_p^iV=0 holds for all i=$1,2,...,\tilde{n}$. Analogously, $\tilde{A}_q(t)$=$VA_q(t)U$+M_q implies M_q=$M_d+N_d\tilde{K}+VB\tilde{K}-VB\tilde{K}VU+V\Delta B(t)\tilde{K}(I-VU)$, where M_q is a new complementary matrix. In this case M_dV=0, N_d=0 is a sufficient condition under which UM_q^iV=0, i=$1,2,...,\tilde{n}$, holds. The conditions $V^TM_{Q^*}V$=0 and N_{R^*}=0 are the same as in Theorem 7. □

It remains to show that the controller considered in the above theorem is a contracted quadratic guaranteed cost controller and the the costs of both systems are equal. This condition presents the following theorem.

Theorem 10. Consider the systems (1) and (8) with their corresponding cost functions (4) and (9), respectively. Suppose that MV=0, N=0, M_dV=0, N_d, $V^TM_{Q^*}V$=0 and N_{R^*}=0 hold. Suppose that $u(t)$=$\tilde{K}\tilde{x}(t)$ is a quadratic guaranteed cost controller designed in the system $\tilde{\mathbf{S}}$ with an associated cost matrix $\tilde{P}$>0. Then, $u(t)$=$Kx(t)$=$\tilde{K}Vx(t)$ is a contracted quadratic guaranteed cost controller with an associated cost matrix P=$V^T\tilde{P}V$>0 for $\mathbf{S}$ and moreover J_0=$\tilde{J}_0$.

Proof. Consider $u(t)$=$\tilde{K}\tilde{x}(t)$ a quadratic contractible control law for the system $\tilde{\mathbf{S}}$. Then, by Definition 5,

$$\dot{\tilde{x}}^T(t)\tilde{P}\tilde{x}(t)+\tilde{x}^T(t)\left[\tilde{Q}^*+\tilde{K}^T\tilde{R}^*\tilde{K}\right]\tilde{x}(t)<0 \tag{21}$$

is satisfied. By using (12) and supposing $\tilde{x}(t)$=$Vx(t)$, $V^TM_{Q^*}V$=0, N_{R^*}=0, P=$V^T\tilde{P}V$ and K=$\tilde{K}V$ hold, it is easy to prove that the inequality matrix (21) implies

$$\dot{x}^T(t)Px(t)+x^T(t)\left[Q^*+K^TR^*K\right]x(t)<0. \tag{22}$$

Thus, if $u(t)$=$\tilde{K}\tilde{x}(t)$ is a quadratic guaranteed cost controller for $\tilde{\mathbf{S}}$ then $u(t)$=$Kx(t)$=$\tilde{K}Vx(t)$ is a quadratic guaranteed cost controller for $\mathbf{S}$. Moreover, and taking account that $X^{-1}=P$, we have:

$$J_0 = \varphi^T(0)X^{-1}\varphi(0) + \int_{-d}^{0} \varphi^T(s)\left[S^{-1}+X^{-1}ZX^{-1}\right]\varphi(s)\,ds$$
$$= \varphi^T V^T(0)\tilde{P}V\varphi(0) + \int_{-d}^{0} \varphi^T(s)\left[V^T\tilde{S}^{-1}V + V^T\tilde{P}(VZV^T)\tilde{P}V\right]\varphi(s)ds$$
$$= \tilde{\varphi}^T(0)\tilde{X}^{-1}\tilde{\varphi}(0) + \int_{-d}^{0} \tilde{\varphi}^T(s)\left[\tilde{S}^{-1}+\tilde{X}^{-1}\tilde{Z}\tilde{X}^{-1}\right]\tilde{\varphi}(s)\,ds = \tilde{J}_0.$$

Then, the values of cost J and $\tilde{J}$ are the same. □

3.1 *Overlapping State Feedback*

There exist three main structures of information structure constraints on the state feedback gain matrices. These structures correspond with the sparsity forms of gain matrices well known in the theory of sparse matrices. These particular forms are a block diagonal form, a block tridiagonal form, and a double bordered block diagonal form corresponding with decentralized, overlapping, and DBBD gain matrices, respectively. Control design of state feedback controllers possessing all these structures may be effectively performed using a LMI approach for linear systems, nonlinear but nominally linear uncertain systems as well as certain classes of nonlinear systems with quadratic nonlinearities (Gahinet *et al.* 1995), (Šiljak and Zečević 2004). Generally, this approach includes both delayed and non-delayed systems.

Standard way of reasoning supposes two overlapping subsystems with the structure of matrices A, Δ_A, C, Δ_C and B, Δ_B, D, Δ_D, respectively, in the form:

$$\left[\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ \hline * & * & * \end{array}\right], \quad \left[\begin{array}{c|c} * & * \\ \hline * & * \\ \hline * & * \end{array}\right], \tag{23}$$

where A_{ii}, ΔA_{ii}, C_{ii}, ΔC_{ii} for i=1,2,3 are $n_i \times n_i$; B_{ij}, ΔB_{ij}, D_{ij}, ΔD_{ij} for i=1,2,3 and j=1,2 are $n_i \times m_j$ dimensional matrices. The dimensions of the components x_1, x_2, x_3 are n_1, n_2, n_3 and satisfy n_1+n_2+n_3=n, respectively. The partition of u^t=(u_1^t, u_2^t) has two components of dimensions m_1, m_2 such that m_1+m_2=m. A standard particular selection of the matrix V is

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix} \tag{24}$$

leads in a simple natural way to an expanded system where the state vector x_2 appears repeated in $\tilde{x}^t$=$(x_1^t, x_2^t, x_2^t, x_3^t)$ (Šiljak 1991). The expanded controller has a block diagonal form with two subblocks of dimensions $m_1 \times (n_1+n_2)$ and $m_2 \times (n_2+n_3)$ as follows:

$$\tilde{K}_D = \left[\begin{array}{cc|cc} \tilde{K}_{11} & \tilde{K}_{12} & 0 & 0 \\ \hline 0 & 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \tag{25}$$

The corresponding contracted gain matrix has a tridiagonal block diagonal form as follows:

$$K_{TD} = \left[\begin{array}{c|c|c} \tilde{K}_{11} & \tilde{K}_{12} & 0 \\ \hline 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \tag{26}$$

However, the design of overlapping controllers depends on the structure of matrices B, Δ_B, D, Δ_D. Type I corresponds with all nonzero element of all input matrices in (23), while Type II corresponds with all elements $(*)_{21}=0$ and $(*)_{22}=0$. The LMI control design for Type I can be performed directly on the original system. Type II requires to perform the LMI control design in the expanded space because the direct design usually leads to infeasibility (Šiljak and Zečević 2004). To simplify the control design for Type II case, let us introduce the following concept. Suppose the problem (1) - (4) with K=K_{TD} in (14).

Definition 11. Consider the problem (1) - (4). A state feedback control law $u_{TD}(k)$=$K_{TD}x(k)$, where K_{TD} is a tridiagonal block matrix, is said to be a *td-quadratic guaranteed cost controller* with a symmetric definite-positive cost matrix P_{TD} if

$$\dot{x}^T(t)P_{TD}x(t) + x^T(t)\left[Q^* + K_{TD}^T R^* K_{TD}\right]x(t) < 0 \tag{27}$$

holds for all $x(k) \neq 0$ and all admissible uncertainties satisfying (2) - (3).

Theorem 12. Consider the systems (1) and (8) with their corresponding cost functions (4) and (9), respectively. Consider the subsystem structure (23) and the transformation matrix (24). Suppose that MV=0, N=0, $V^T M_{Q^*} V$=0 and N_{R^*}=0. If $u_D(k)$=$\tilde{K}_D\tilde{x}(k)$ is a contractible quadratic guaranteed cost controller with a cost matrix $\tilde{P}_D$>0 for the system $\tilde{\mathbf{S}}$, then $u_{TD}(k)$=$K_{TD}x(k)$=$\tilde{K}_D Vx(k)$ is a td-quadratic guaranteed cost controller with a cost matrix P_{TD}=$V^t\tilde{P}_D V$>0 for $\mathbf{S}$ and J_0=$\tilde{J}_0$.

Proof. It is straightforward because this theorem is a particular case of Theorem 10. □

4. EXAMPLE

4.1 *Problem Statement*

Consider the problem (1), (4) as follows:

$$A = \begin{bmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \; B=D=E=\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.1 & 0 & -0.1 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \end{bmatrix}, \; E_1=E_3=\begin{bmatrix} 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$
$$E_2 = E_4 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad \tilde{Q}^* = I_6, \quad \tilde{R}^* = I_2,$$
$$\varphi(t) = (1, 0, t, 0)^T, \quad d = 1. \tag{28}$$

The subsystem A_{22} is A_{22}=$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$. The other overlapped subsystems corresponding to the matrices $\Delta A(t)$, C and $\Delta C(t)$ are also 2×2 dimensional blocks. Choosing the matrices $\tilde{Q}^*$=I_6, $\tilde{R}^*$=I_2 in the expanded space, the corresponding matrices Q^*, R^* in the system $\mathbf{S}$ are Q^*=diag$\{1,2,2,1\}$, R^*=diag$\{1,1\}$. Find the overlapping quadratic guaranteed cost controller for the above system, where two overlapping subsystems are supposed with the dimensions $n_1 = n_3 = 1$, $n_2 = 2$ by (23). Compare these results with the centralized control design as a reference. Use the delay independent LMI approach for the control design.

4.2 *Results*

Decentralized controller. Consider the expansion of the system $\mathbf{S}$ with the transformation V given in (24), as follows:

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

Suppose that the complementary matrix M has the following structure:

$$M = \begin{bmatrix} 0 & 0 & -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -0.5 & 1 & 0.5 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0.5 & -1 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

which satisfies MV=0. The remaining complementary matrices are selected as N=0, M_d=0, N_d=0, M_{Q^*}=0, N_{R^*}=0. Such choice ensures that the results presented in Section 3 are satisfied. Denote $\tilde{X}_D$, $\tilde{T}_D$ and $\tilde{Y}_D$ block diagonal matrices and consider them as the matrices X, T and Y appearing in Theorem 6. To obtain the structure (25) for the gain matrix $\tilde{K}_D$, we must impose some conditions on the structure of the matrices $\tilde{X}_D$, and $\tilde{Y}_D$. They are as follows:

$$\tilde{X}_D = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{12} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{13} & x_{23} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & x_{45} & x_{55} & x_{56} \\ 0 & 0 & 0 & x_{46} & x_{56} & x_{66} \end{bmatrix}, \qquad \tilde{Y}_D = \begin{bmatrix} y_{11} & y_{12} & y_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{24} & y_{25} & y_{26} \end{bmatrix}. \tag{31}$$

By applying the LMI design by (15) on this expanded system $\tilde{\mathbf{S}}$, we get the gain matrix

$$\tilde{K}_D = \begin{bmatrix} -0.3132 & 0.0205 & 0.1872 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0172 & -0.0252 & -0.1670 \end{bmatrix}. \tag{32}$$

The corresponding contracted gain matrix has the following form

$$K_D = \begin{bmatrix} -0.3132 & 0.0205 & 0.1872 & 0 \\ 0 & -0.0172 & -0.0252 & -0.1670 \end{bmatrix}. \tag{33}$$

The associated bound on the cost is $J \leqslant J_0$=11.84.

Centralized controller. The direct computation on the original system and cost results in the controller:

$$K = \begin{bmatrix} -0.2189 & 0.1366 & 0.2644 & -0.1177 \\ -0.0352 & -0.1251 & -0.0529 & -0.1670 \end{bmatrix} \tag{34}$$

with the bound on the cost equal to $J \leqslant J_0$=2.76.

The centralized control design case serves only as a reference to compare the bounds on costs in both cases. The upper bound J_0 is greater than in the centralized case because of given information structural constraints as expected. All computations have been performed using Matlab LMI Control Toolbox (Gahinet *et al.* 1995).

5. CONCLUSION

The paper contributes to the solution of the overlapping guaranteed cost control design problem for a class of linear continuous-time uncertain systems with state and control delays. Arbitrarily time-varying unknown uncertainties with known bounds and a given point delay are considered. Conditions preserving closed-loop systems expansion-contraction relations including the bounds equality of costs have been proved. A LMI delay independent procedure has been used for control design in the expanded space. The results have been specialized on the overlapping state feedback design. A numerical illustrative example has been supplied.

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