# STABILIZATION OF SWITCHED SYSTEMS VIA OPTIMAL CONTROL

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Abstract: In this paper we consider switched systems composed of LTI non Hurwitz dynamics and we deal with the problem of computing an appropriate switching law such that the controlled system is globally asymptotically stable. We first present a method to design a feedback control law that minimizes a linear quadratic performance index when an infinite number of switches is allowed and at least one dynamics is Hurwitz. Then, we show that this approach can be applied to stabilize switched systems whose modes are all unstable, by simply applying the proposed procedure to a "dummy" system, augmented with a stable dynamics. If the system with unstable modes is globally exponentially stabilizable, then our method is guaranteed to provide the feedback control law that minimizes the chosen quadratic performance index, and that guarantees the closed loop system to be globally asymptotically stable. 

Keywords: Hybrid systems, switched systems, stabilization, optimal control.

## 1. INTRODUCTION

In this paper we show how it is possible to design stabilizing laws for switched systems $\{A_i\}_{i\in\mathcal{S}}$, whose evolution is given by

$$\dot{x}(t) = A_{i(t)}x(t), \qquad i(t) \in \mathcal{S} = \{1, \ldots, s\}, \quad (1)$$

where, $\forall\ i \in \mathcal{S}$ dynamics $A_i$ are non Hurwitz, by extending an optimal control technique we have developed for stable switched systems.

### *1.1 Motivation*

In a series of papers (Giua *et al.*, 2001; Bemporad *et al.*, 2002) we have considered the problem of designing optimal feedback laws for systems of the type $\{A_i\}_{i\in\mathcal{S}}$ with a positive definite quadratic cost. The technique we use requires that the number of allowed switches $N$ is finite; thus the optimal control problem is solvable (with a finite cost) if and only if at least one of the system dynamics $A_i$ is Hurwitz. The approach requires the computation of a set of tables, one for each switch. Whenever $k$ switches are still allowed and the current dynamics is $i(t) = i$, we use table $\mathcal{C}_k^i$ to determine if a switch should occur. Each table is partitioned in (up to) $s$ regions $\mathcal{R}_j$'s: as soon as the state reaches a point in the region $\mathcal{R}_j$ for a certain $j \in \mathcal{S} \setminus \{i\}$ we will switch to mode $j$; on the contrary, no switch will occur while the system's state belongs to $\mathcal{R}_i$.

The first result we present in this paper consists in allowing the number of switches to go to infinity. We show that in this case if the optimal control is solvable with a finite cost then $\forall\ i \in \mathcal{S}$ the tables converge to a single table $\mathcal{C}_\infty$. The optimal control problem with $N = \infty$ can be solved by using only the table $\mathcal{C}_\infty$.

Secondly, we observe that in the table $\mathcal{C}_\infty$ it may happen that the region $\mathcal{R}_j$ associated to a given dynamics $j$ never appears. In this case, the optimal evolution for the system $\{A_i\}_{i\in\mathcal{S}}$ and for the system $\{A_i\}_{i\in\mathcal{S}\setminus\{j\}}$ are identical. This in particular, may allow us to compute an optimal control law for an unstable system introducing a dummy stable dynamics $\bar{A}$, provided that the corresponding regions do not appear in $\mathcal{C}_\infty$.

Finally, we use this approach exploiting an intuitive relation between stability and optimal control. In (Giua *et al.*, 2001) we have proved that under very loose conditions if a switched system can be optimally controlled with a finite cost, then the closed loop system is *asymptotically stable*. We also prove that if the switched system is *exponentially stabilizable*, then our approach can always find an optimal control law with a finite cost that makes the closed loop system asymptotically stable.

We are aware of the small gap in our result: a switched system may be asymptotically (but not exponentially) stabilizable but if no finite-cost optimal control law exists, we cannot compute a stabilizing law. Furthermore, our approach requires a discretization of the state space, and for large dimensional systems this may be computationally burdensome. Nevertheless, the approach we present is extremely general, and we believe that the results we have obtained are significant. In fact, although there is a rich literature on stability analysis of hybrid systems, there are very few results on the design of stabilizing laws and they usually apply to restricted classes of systems or give only sufficient conditions.

### 1.2 Relevant literature

Many papers on the stability analysis of switched systems are based on the use of multiple Lyapunov functions (MLF's) (Branicky, 1998; Liberzon and Morse, 1999; Ye *et al.*, 1998; Michel and Hu, 1999) but in all these cases the proposed approaches only give sufficient conditions for the asymptotic stabilizability. Necessary and sufficient conditions are given in (Feron, 1996; Wicks *et al.*, 1998) in the case of two switched systems when the performance index under consideration is the quadratic stability of the switched systems. Iterative algorithms for constructing such common Lyapunov function can be found in (Liberzon and Tempo, 2003).

Antsaklis *et al.* in (Hu *et al.*, 1999; Xu and Antsaklis, 1999) using a geometric approach, were able to obtain necessary and sufficient conditions for asymptotic stabilizability of switched systems with an arbitrarily large number of *second-order* LTI unstable systems. When the switched system is asymptotically stabilizable, they also provide an approach to compute a stabilizing law.

The problem of stabilizing a switched system of the form (1) with unstable dynamics $A_i$'s was translated into the problem of solving a set of quadratic inequalities. This is appealing, but it turns to be a non convex problem (thus it only provides sufficient conditions) when the number of subsystems is greater than 2. Moreover many proposed solutions lean on LMI or BMI methods, which become computationally hard as the number of modes grows (DeCarlo *et al.*, 2000; Pettersson, 1999).

Recently, Ishii *et al.* in (2003) approached the problem of solving the quadratic inequalities by an iterative algorithm (whose convergence is guaranteed with a given probability), that exploits a gradient descent method on energy and multi modal Lyapunov functions.

## 2. PROBLEM FORMULATION

We briefly recall some basic definitions (Khalil, 2002). Consider the *nonautonomous system*

$$\dot{x}(t) = f(t, x) \tag{2}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$ on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

*Definition 1.* The *origin* is an *equilibrium point* for (2) if $f(t, 0) = 0$, $\forall t \geq 0$. ■

*Definition 2.* The equilibrium point $x = 0$ of (2) is
— *stable* if, $\forall\ \varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$, $\forall t \geq t_0 \geq 0$;
— *unstable* if it is not stable;
— *asymptotically stable* (AS) if it is stable and there is a positive constant $\delta = \delta(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, $\forall\ ||x(t_0)|| < \delta$;
— *exponentially stable* (ES) if there exist positive constants $\delta$, $K$, and $\lambda$ such that $||x(t)|| \leq K||x(t_0)||\ e^{-\lambda(t-t_0)}$, $\forall\ ||x(t_0)|| < \delta$.
If asymptotic (or exponential) stability holds for any initial state, the equilibrium point is said *globally* asymptotically (or exp.) stable. ■

Note that exponential stability implies asymptotic stability, which in turn implies stability.

In this paper we consider the following class of nonautonomous (hybrid) systems, commonly denoted as *switched systems*,

$$\dot{x}(t) = f(x, t) \triangleq A_{i(t)} x(t), \quad i \in \mathcal{S} \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $i(t) \in \mathcal{S}$ is the current mode and represents a control variable, and $\mathcal{S} \triangleq \{1, \cdots, s\}$ is a finite set of integers, each one associated with a matrix $A_i \in \mathbb{R}^{n \times n}$. We assume a continuous evolution of the state, i.e., when a switch occurs at time $\tau$, $x(\tau^-) = x(\tau^+)$.

*Definition 3.* The switched system $\{A_i\}_{i \in \mathcal{S}}$ is said *globally stabilizable* if there exists a switching control law $i(t)$ such that the controlled system is globally stable. Analogous definitions hold for global asymptotic (or exponential) stabilizability. ■

Note that if at least one dynamics $A_i$ is Hurwitz, then the system $\{A_i\}_{i \in \mathcal{S}}$ is obviously globally exponentially stabilizable.

The main goal of this paper is that of computing an appropriate switching law $i(t)$, when it does exist, such that the controlled system $\{A_i\}_{i \in \mathcal{S}}$ is globally asymptotically stable. In particular, we provide a procedure that guarantees to determine a globally asymptotically stable switching law whenever the system is globally exponentially stabilizable.

## 3. THE OPTIMAL CONTROL PROBLEM WITH A FINITE NUMBER OF SWITCHES

The proposed stabilizing procedure is based on the solution of an optimal control problem of the following form:

$$\begin{array}{l} V_N^*(x_0, i_0) \triangleq \min\limits_{I, \mathcal{T}} F(I, \mathcal{T}) \triangleq \displaystyle\int_0^\infty x'(t) Q_{i(t)} x(t) dt \\ \text{s.t. } \dot{x}(t) = A_{i(t)} x(t), \ \ x(0) = x_0, \quad i(0) = i_0 \\ \qquad i(t) = i_k \text{ for } \tau_k \leq t < \tau_{k+1}, k = 0, \ldots, N \\ \qquad \tau_0 = 0, \ \ \tau_{N+1} = +\infty \\ \qquad i_k \in \mathcal{S}, \qquad\qquad\qquad\quad k = 1, \ldots, N \\ \qquad x(\tau_k^+) = x(\tau_k^-), \qquad\qquad k = 1, \ldots, N \end{array} \tag{4}$$

where $N$, denoting the maximum number of allowed switches, is finite and fixed a priori.

Let us observe that in the general optimization problem (4) although the number of allowed switches is $N$, it may possible to consider also solutions where only $m < N$ switches effectively occur: this can be done choosing $i_m = i_{m+1} = \cdots = i_N$.

The initial state $x_0$ and location $i_0$ are given.

The control variables are $\mathcal{T} \triangleq \{\tau_1, \ldots, \tau_N\}$ and $I \triangleq \{i_1, \ldots, i_N\}$, where $\mathcal{T}$ is the set of switching times and $I$ is the sequence of indices associated with discrete locations.

In order to make the problem solvable with finite cost $V_N^*$, we assume the following:

*Assumption 1.* There exists at least one index $i \in \mathcal{S}$ such that $A_i$ is strictly Hurwitz.

In (Bemporad *et al.*, 2002) we showed that the optimal control law for the optimization problem (4) takes the form of a state-feedback, i.e., it is only necessary to look at the current system state $x$ in order to determine if a switch from linear dynamics $A_{i_{k-1}}$ to $A_{i_k}$, should occur.
More precisely, for a given mode $i \in \mathcal{S}$ when $k$ switches are still available, it is possible to construct a table $\mathcal{C}_k^i$ that partitions the state space $\mathbb{R}^n$ into $s$ regions $\mathcal{R}_j$'s, $j = 1, \cdots, s = |\mathcal{S}|$. Whenever $i_{N-k} = i$ we use table $\mathcal{C}_k^i$ to determine if a switch should occur: as soon as the state reaches a point in the region $\mathcal{R}_j$ for a certain $j \in \mathcal{S} \setminus \{i\}$ we will switch to mode $i_{N-k+1} = j$; no switch will occur if the system's state belongs to $\mathcal{R}_i$.
To prove this result, in (Bemporad *et al.*, 2002) we showed constructively how the tables $\mathcal{C}_k^i$ can be computed off-line using a dynamic programming argument. We first shown how the tables $\mathcal{C}_1^i$ ($i \in \mathcal{S}$) for the last switch can be determined. Then, we shown by induction how the tables $\mathcal{C}_k^i$ can be computed once the tables $\mathcal{C}_{k-1}^i$ are known.
Note that regions $\mathcal{R}_j$'s are homogeneous regions, namely if $x \in \mathcal{R}_j$ then $\lambda x \in \mathcal{R}_j$ for all $\lambda \in \mathbb{R}$. This implies that they can be computed by simply looking at the unitary semisphere.
To avoid repeating results already reported in previous papers we do not describe here how tables can be computed: the complete derivation can be found in (Giua *et al.*, 2001; Bemporad *et al.*, 2002). We only remind here that the computational cost of the proposed approach is of the order $\mathcal{O}(r^{n-1}Ns^2)$ where $n$ is the dimension of the state space and $r$ is the number of samples in each direction of the unitary semisphere. Therefore, the complexity is a quadratic function of the number of possible dynamics.
Let us finally observe that the idea of computing a control law based on the off-line computation of state space partitions has also been used by other authors. In particular, in the continuous time framework, Shaikh and Caines (Shaikh and Caines, 2003) considered a finite horizon optimal control problem for switched systems. They exploit the maximum principle integrated with dynamic programming arguments to construct an appropriate state space partition called *optimality zones.*

## 4. THE OPTIMAL CONTROL PROBLEM WITH AN INFINITE NUMBER OF SWITCHES

In this section we discuss how, under appropriate assumptions, the above procedure can be extended to the case of $N = \infty$. In particular, we consider an optimal control problem of the form (4) where
(i) for at least one $i \in \mathcal{S}$, $A_i$ is stable;
(ii) for all $i \in \mathcal{S}$, $Q_i > 0$.
Let us preliminary state a monotonicity result.

*Property 1.* Let $N, N' \in \mathbb{N}$. If $N < N'$ and the switched system evolves along an optimal trajectory, then for any continuous initial state $x_0$, and for all $i, j \in \mathcal{S}$, $+\infty > V_N^*(x_0, i) \geq V_{N'}^*(x_0, j)$.

**Proof:** We first observe that by assumption (i) $V_N^*(x_0, i)$ is finite for any $N \geq 1$. In fact, regardless of the value of the initial dynamics $i$, we can always switch to the stable dynamics whose cost to infinity is finite. Now, we prove the second inequality by contradiction. Assume that $\exists\ j \in \mathcal{S}$ such that $V_{N'}^*(x_0, j) > V_N^*(x_0, i)$. Then the same evolution that generates $V_N^*(x_0, i)$ is also admissible for (4) when a larger value $N'$ of switches is allowed. This leads to a contradiction. □

*Proposition 1.* For any continuous initial state $x_0$, and $\forall\ \varepsilon' > 0$, $\exists\ \bar{N} = \bar{N}(x_0)$ such that for all $N > \bar{N}$, $V_N^*(x_0, i) - V_{\bar{N}}{}^*(x_0, j) < \varepsilon'$, $\forall i, j \in \mathcal{S}$.
**Proof:** By definition $V_N^*(x_0, i) \geq 0$ for all $i \in \mathcal{S}$, hence $V_N^*$ is a lower bounded non-increasing (by Property 1) sequence. By the Axiom of Completeness it converges in $\mathbb{R}$, hence it is a Cauchy sequence. □

*Proposition 2.* For any continuous initial state $x_0$, $x_0 \neq 0$, and $\forall\ \varepsilon > 0$, $\exists\ \bar{N}$ such that for all $N > \bar{N}$,

$$\frac{V_N^*(x_0, i) - V_{\bar{N}}{}^*(x_0, j)}{V_N^*(x_0, i)} < \varepsilon, \quad \forall i, j \in \mathcal{S}.$$

**Proof:** We first observe that by assumption (ii) $V_N^*(x_0, i)$ is lower bounded by a strictly positive number. Moreover, the optimal costs are quadratic functions of $x_0$, i.e., if $x_0 = \lambda y_0$, then $V_N^*(\lambda y_0, i) = \lambda^2 V_N^*(y_0, i)$ . Finally, by Proposition 1 $\forall\ y_0$ and $\forall\ \varepsilon' > 0$, $\exists\ \bar{N}(y_0)$ such that $\forall\ N > \bar{N}(y_0)$, $V_N^*(y_0, i) - V_{\bar{N}}{}^*(y_0, j) < \varepsilon'$. Hence if we define

$$\bar{N} = \max_{y_0\ :\ ||y_0||=1} \bar{N}(y_0) \quad \Rightarrow$$

$$\frac{V_N^*(x_0, i) - V_{\bar{N}}{}^*(x_0, j)}{V_N^*(x_0, i)} = \frac{\lambda^2[V_N^*(y_0, i) - V_{\bar{N}}{}^*(y_0, j)]}{\lambda^2 V_N^*(y_0, i)}$$

$$\leq \frac{\varepsilon'}{\min\limits_{y_0\ :||y_0||=1} V_N^*(y_0, i)} = \varepsilon.$$

□

According to the above result, one may use a given fixed relative tolerance $\varepsilon$ to approximate two cost values, i.e.,

$$\frac{V_N^*(x, i) - V_{N'}{}^*(x, j)}{V_N^*(x, i)} < \varepsilon \implies V_N^*(x, i) \cong V_{N'}{}^*(x, j).$$

We can now prove the main result of this section.

*Theorem 1.* Given a fixed relative tolerance $\varepsilon$, if $\bar{N}$ is chosen as in Proposition 2 then for all $N > \bar{N}+1$ it holds that $\mathcal{C}_N^i = \mathcal{C}_{\bar{N}+1}^i$.
**Proof:** By definition (see (Bemporad *et al.*, 2002))

$$V_N^*(x_0, i) = \min_{j \in \mathcal{S}} \min_{\varrho \geq 0} \left\{ x_0' \bar{Q}_i(\varrho) x_0 + V_{N-1}^*(x(\varrho), j) \right\}$$

where $x(\varrho) = e^{A_i \varrho} x_0$ and $\bar{Q}_i(\varrho) = \int_0^\varrho e^{A_i' t} Q_i e^{A_i t} dt$.
Now, being by assumption $N - 1 > \bar{N}$, by virtue of Proposition 2 we may approximate

$$V_{N-1}^*(x(\varrho), j) \cong V_{\bar{N}}^*(x(\varrho), j) \quad \Rightarrow$$

$$V_N^*(x_0, i) \cong \min_{j \in \mathcal{S}} \min_{\varrho \geq 0} \left\{ x_0' \bar{Q}_i(\varrho) x_0 + V_{\bar{N}}^*(x(\varrho), j) \right\}$$

$$= V_{\bar{N}+1}^*(x_0, i).$$

Therefore, the optimal arguments $(\varrho^*, j^*)$ used to compute $\mathcal{C}_{\bar{N}}^i$ and $\mathcal{C}_{\bar{N}+1}^i$ are the same. $\square$

The above result allows one to compute with a finite procedure the optimal tables for a switching law when $N$ goes to infinity. In such a case, in fact, it holds that $\forall i \in \mathcal{S}$, $\mathcal{C}_\infty^i = \lim_{N\to\infty} \mathcal{C}_N^i = \mathcal{C}_{\bar{N}+1}^i$.

*Theorem 2.* Given a fixed relative tolerance $\varepsilon$, if $\bar{N}$ is chosen as in Proposition 2 then for all $i, j \in \mathcal{S}$ it holds that $\mathcal{C}_{\bar{N}+1}^i = \mathcal{C}_{\bar{N}+1}^j$.

**Proof:** It trivially follows from the fact that, by Proposition 2, $V_{\bar{N}+1}^*(x_0, i) = V_{\bar{N}+1}^*(x_0, j)$ for all $i, j \in \mathcal{S}$, and from the uniqueness of the optimal tables as discussed in Section 3. $\square$

This result also allows one to conclude that for all $i \in \mathcal{S}$ $\mathcal{C}_\infty = \lim_{N\to\infty} \mathcal{C}_N^i$, i.e., *all tables converge to the same one.*

To construct the table $\mathcal{C}_\infty$ the value of $\bar{N}$ is needed. We do not provide so far any analytical way to compute $\bar{N}$: our approach consists in constructing tables until a convergence criterion is met.

Table $\mathcal{C}_\infty$ can be used to compute the optimal feedback control law that solves an optimal control problem of the form (4) with $N = \infty$. More precisely, when an infinite number of switches is available, we only need to keep track of the table $\mathcal{C}_\infty$. We associate a different color to each region $\mathcal{R}_i \in \mathcal{C}_\infty$. If the current continuous state is $x$ and the current mode is $A_i$, on the basis of the knowledge of the color of $\mathcal{C}_\infty$ in $x$, we decide if it is better to still evolve with the current dynamics $i$ or switch to a different dynamics, that is univocally determined by the color of the table in $x$.

*Remark 1.* Note that the table $\mathcal{C}_\infty$ is *Zeno-free*, i.e., it guarantees that no Zeno instability may occur when it is used to compute the optimal feedback control law. This property is guaranteed by the procedure used for their construction. $\blacksquare$

## 5. STABILIZABILITY OF UNSTABLE SWITCHED SYSTEMS

In this section we deal with the problem of stabilizing a switched system $\{A_i\}_{i\in\mathcal{S}}$ whose linear dynamics $A_i$ are not stable. In particular, we show that a solution to this problem — when it does exist — can be obtained by solving an optimal control problem of the form (4) with $N = \infty$. More precisely, we show how this problem can be solved by applying the switching table procedure to a "dummy" problem that satifies the assumption that at least one dynamics $A_i$ is asymptotically stable. When the original switched system is stabilizable, we select among all stabilizing laws a switching law that minimizes a given quadratic performance index.

We first present the following preliminary result.

*Proposition 3.* Let us consider an optimal control problem (OP) of the form (4) with $N = \infty$, and whose possible modes are $A_i$, $i \in \mathcal{S}$, and the corresponding weighting matrices are $Q_i$, $i \in \mathcal{S}$. If the table $\mathcal{C}_\infty$ only contain colors associated to a subset of indices $\mathcal{S}' \subset \mathcal{S}$, then $\forall i_0 \in \mathcal{S}$ and $\forall x_0 \in \mathbb{R}^n$, the optimal control law that results by solving (OP) is also optimal for the optimal control problem (OP') of the same form (4) with $N = \infty$, and whose possible modes are $A_i$, $i \in \mathcal{S}'$ and the corresponding weighting matrices are $Q_i$, $i \in \mathcal{S}'$.

**Proof:** The validity of the statement follows from the definition of the table $\mathcal{C}_\infty$ and the possibility of using it to derive an optimal feedback control law for (OP). Thus, if a color corresponding to a certain mode $A_j$ does not appear in $\mathcal{C}_\infty$, this means that it is never convenient to switch to mode $A_j$, or to evolve with $A_j$ if it is the initial mode, regardless of the current continuous state. $\square$

The above result enables us to use the switching table procedure to compute a stabilizing switching law, if it does exist, for switched systems whose dynamics are unstable. In particular, the proposed approach is based on the construction of an *augmented optimal control problem*, defined as follows.

*Definition 4.* Let us consider an optimal control problem of the form (4) with $N = \infty$. Assume that all possible modes $A_i$, $i \in \mathcal{S}$, are not stable and the corresponding weighting matrices $Q_i$, $i \in \mathcal{S}$, are strictly positive definite.

Let $\bar{A} \in \mathbb{R}^{n\times n}$ be any matrix that is strictly Hurwitz and $\bar{Q} \in \mathbb{R}^{n\times n}$ be any strictly positive definite matrix. Let $A_{s+1} = \bar{A}$ and $Q_{s+1} = q \cdot \bar{Q}$ where $q \in \mathbb{R}^+$.

We define *augmented optimal control problem* an optimal control problem of the form (4) with $N = \infty$, and whose possible modes are $A_i$, $i \in \bar{\mathcal{S}}$, with $\bar{\mathcal{S}} = \mathcal{S} \cup \{s+1\}$, and the corresponding weighting matrices are $Q_i$, $i \in \bar{\mathcal{S}}$. $\blacksquare$

*Proposition 4.* Let us consider an optimal control problem (OP) of the form (4) with $N = \infty$. Assume that all possible modes $A_i$, $i \in \mathcal{S}$, are not stable and the corresponding weighting matrices $Q_i$, $i \in \mathcal{S}$, are strictly positive definite. Let $V_\infty^*(x_0, i_0)$ be the optimal value of the cost of (OP) when the initial state is $(x_0, i_0)$.

Let us consider an augmented optimal control problem $(\overline{\mathrm{OP}})$ with $A_{s+1} = \bar{A}$ and $Q_{s+1} = q \cdot \bar{Q}$ where $q \in \mathbb{R}^+$. Let $\overline{V}_\infty^*(x_0, i_0, q)$ be the optimal value of the cost of $(\overline{\mathrm{OP}})$ when the initial state is $(x_0, i_0)$.

The optimal cost $\overline{V}_\infty^*(x_0, i_0, q)$ is a strictly increasing function of $q$ for all values of $q$ such that the stable dynamics $A_{s+1}$ appears in the optimal evolution of the augmented optimal control problem.

**Proof:** We prove this by contradiction. Let us consider two different augmented optimal control problems ($\overline{\mathrm{OP}}$') and ($\overline{\mathrm{OP}}$") that differ for their value of $q$. In particular, let $q'$ and $q''$ be the values of the coefficient $q$ associated to ($\overline{\mathrm{OP}}$') and ($\overline{\mathrm{OP}}$" ) respectively, and let $q' > q''$. Assume that $\overline{V}_\infty^*(x_0, i_0, q') = \overline{V}_\infty^*(x_0, i_0, q'')$. If we consider the evolution that is optimal for ($\overline{\mathrm{OP}}$') and evaluate the cost using the the weights of ($\overline{\mathrm{OP}}$"), we find out that the resulting value of the cost is less than $\overline{V}_\infty^*(x_0, i_0, q'')$, that leads to a contradiction because by definition $\overline{V}_\infty^*(x_0, i_0, q'')$ is the optimal value of the cost when $q = q''$. $\square$

Now, we prove the main result of this paper that enables us to conclude that, if a switched system $\{A_i\}_{i\in\mathcal{S}}$ with unstable dynamics is stabilizable, if we associate an optimal control problem to the switched system, and then we define an augmented optimal control problem, a stabilizing switching law can always be computed using the switching table procedure. The main feature of the computed switching law is that it minimizes the chosen quadratic performance index.

*Theorem 3.* Given a switched system $\{A_i\}_{i\in\mathcal{S}}$, let us consider an optimal control problem of the form (4) with $N = \infty$ and weighting matrices $Q_i > 0$, $i \in \mathcal{S}$. Then, let us define an augmented optimal control problem with Hurwitz dynamics $A_{s+1} = \bar{A}$ and corresponding weighting matrix $Q_{s+1} = \bar{q} \cdot \bar{Q}$, where $\bar{Q} > 0$ and $\bar{q} \in \mathbb{R}^+$. Let $\bar{\mathcal{S}} = \mathcal{S} \cup \{s+1\}$.
*(i)* The switched system $\{A_i\}_{i\in\mathcal{S}}$ is globally exponentially stabilizable $\Longrightarrow \exists\ \bar{q} \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$, computed by solving the augmented optimal control problem, does not contain the color associated to $\bar{A}$.
*(ii)* The switched system $\{A_i\}_{i\in\mathcal{S}}$ is asymptotically stabilizable $\Longleftarrow \exists\ \bar{q} \in \mathbb{R}^+$ such that the table $\mathcal{C}_\infty$, computed by solving the augmented optimal control problem, does not contain the color associated to $\bar{A}$.
**Proof:** We denote $V^*_\infty(x_0, i_0)$ the optimal cost of the optimal control problem for the system $\{A_i\}_{i\in\mathcal{S}}$ when the initial state is $(x_0, i_0)$, and $\overline{V}^*_\infty(x_0, i_0, q)$ the corresponding optimal cost of the optimal control problem for the system $\{A_i\}_{i\in\bar{\mathcal{S}}}$.
The cost $\overline{V}^*_\infty(x_0, i_0, q)$ is obviously finite for all finite values of $q$ because of the assumption that $\bar{A}$ is stable. Moreover, it is upper limited by the value of $V^*_\infty(x_0, i_0)$, i.e., $\forall q \in \mathbb{R}^+$, $\overline{V}^*_\infty(x_0, i_0, q) \le V^*_\infty(x_0, i_0)$. Finally, $\overline{V}^*_\infty(x_0, i_0, q)$ is a quadratic function of $x_0$, i.e., if $x_0 = \lambda y_0$ then $\overline{V}^*_\infty(\lambda y_0, i_0, q) = \lambda^2 \overline{V}^*_\infty(y_0, i_0, q)$.
(i) Assume that the switched system $\{A_i\}_{\in\mathcal{S}}$, is globally exponentially stabilizable.
This implies that $V^*_\infty(x_0, i_0) < +\infty$, for all $x_0 \in \mathbb{R}^n$ and for all $i_0 \in \mathcal{S}$. In fact, any control law that is exponentially stable implies that along any trajectory it holds

$$\int_0^\infty x'(t) Q_{i(t)} x(t) dt = \int_0^\infty y'(t) Q_{i(t)} y(t)\ ||x(t)||^2 dt$$

$$\le K \int_0^\infty ||x(t)||^2 dt \le K c^2 ||x_0||^2 \int_0^\infty e^{-2\lambda t} dt < +\infty,$$

where we have written $x(t) = y(t)\ ||x(t)||$ with $||y(t)|| = 1$, $K = \max_{i\in\mathcal{S}, ||y||=1} y' Q_i y$, $c, \lambda \in \mathbb{R}^+$.

By Proposition 4 we know that $\overline{V}^*_\infty(x_0, i_0, q)$ is an increasing function of $q$ for all values of $q$ such that $A_{s+1}$ appears in the optimal evolution. Therefore, we may conclude that if $\{A_i\}_{i\in\mathcal{S}}$ is globally exponentially stabilizable then $\exists\ q'(x_0, i_0) \in \mathbb{R}^+$ such that $\overline{V}^*_\infty(x_0, i_0, q'(x_0, i_0)) = V^*_\infty(x_0, i_0)$. Moreover, if the equality holds for a certain value of $q = q'(x_0, i_0)$, then it also holds for all $q > q'(x_0, i_0)$. In fact, the above equality implies that the optimal control law of the augmented optimal control problem requires no evolution with the stable mode $A_{s+1}$. If this is the case when its weighting matrix is $Q_{s+1} = q' \cdot \bar{Q}$, all the more reason this is the case when its weighting matrix is $Q_{s+1} = q \cdot \bar{Q}$ with $q > q'(x_0, i_0)$.
Now, the result holds if we let

$$\bar{q} = \max_{i_0\in\mathcal{S},\ x_0\in\mathbb{R}^n} q'(x_0, i_0) = \max_{i_0\in\mathcal{S},\ ||y_0||=1} q'(y_0, i_0),$$

where the second equality follows from the fact that $\overline{V}^*_\infty(x_0, i_0, q)$ is a quadratic function of $x_0$.
If we define the augmented optimal control problem with $Q_{s+1} = \bar{q} \cdot \bar{Q}$, then for all values of $x_0 \in \mathbb{R}^n$ and all $i_0 \in \mathcal{S}$, it holds that $\overline{V}^*_\infty(x_0, i_0, \bar{q}) = V^*_\infty(x_0, i_0)$, i.e., the controlled system never switches to dynamics $A_{s+1}$, neither evolves with $A_{s+1}$ if it is the initial mode. This obviously implies that the table $\mathcal{C}_\infty$, computed applying the switching table procedure to the augmented optimal control problem with $Q_{s+1} = \bar{q} \cdot \bar{Q}$, does not contain the color associated to the stable mode $A_{s+1} = \bar{A}$.
(ii) Assume that $\exists\ \bar{q}$ such that the switching table $\mathcal{C}_\infty$, computed applying the switching table procedure to the augmented optimal control problem with $Q_{s+1} = \bar{q} \cdot \bar{Q}$, does not contain the color associated to the stable mode $A_{s+1} = \bar{A}$.
By Proposition 3 this implies that the control law that results using table $\mathcal{C}_\infty$ is also optimal for the optimal control problem with unstable modes $A_i$'s and weighting matrices $Q_i$'s, with $i \in \mathcal{S}$. Therefore, being $\overline{V}^*_\infty(x_0, i_0, \bar{q}) < +\infty$, and $V^*_\infty(x_0, i_0) = \overline{V}^*_\infty(x_0, i_0, \bar{q})$ for all $x_0 \in \mathbb{R}^n$ and all $i_0 \in \mathcal{S}$, it follows that $V^*_\infty(x_0, i_0) < +\infty$.
It is not difficult to show, with the same argument we used in (Giua *et al.*, 2001), that the finite value of the optimal cost for all initial states and dynamics implies that the switched system $\{A_i\}_{i\in\mathcal{S}}$ is globally asymptotically stabilizable. □

The above theorem provides a systematic way to deal with the problem of determining an asymptotic stabilizing switching law for a switched system $\{A_i\}_{i\in\mathcal{S}}$ with linear unstable modes, that can be summarized in the following steps.
— We associate to the switched system that we want to stabilize an optimal control problem of the form (4) with $N = \infty$.
— We define an augmented optimal control problem with a Hurwitz matrix $A_{s+1} = \bar{A}$ and weighting matrix $Q_{s+1} = q \cdot \bar{Q}$, where $\bar{Q}$ is any definite positive matrix and $q$ is a very large positive real number.
— We construct the switching table $\mathcal{C}_\infty$ solving the augmented optimal control problem.
— If this table does not contain the color associated to the stable mode $A_{s+1}$, by Theorem 3, item (ii), we may conclude that the switched system $\{A_i\}_{i\in\mathcal{S}}$ is globally asymptotically stabilizable. In such a case, we compute the stabilizing feedback control law that minimizes the chosen quadratic performance index using table $\mathcal{C}_\infty$.
Note, finally, that the procedure may also find control laws that locally stabilize a system, as shown in Example 2 in the next session.
We do not provide an a priori rule to establish if the switched system is stabilizable and in such a case, an analytical way to compute an appropriate value of $q$. Nevertheless in all numerical examples taken from the literature, we found out that if the system is stabilizable if was sufficient to use a large value of $q$ ($10^{10} \div 10^{20}$) to compute stabilizing laws.

## 6. A NUMERICAL EXAMPLE

As an example of the described approach we choose a variant of a very well-known switched system (Branicky, 1998) $\{A_i\}_{i\in\mathcal{S}}$, with $s = 3$ and $A_1 = [1\ {-10}; 100\ 1]$, $A_2 = [39.97\ {-77.5}; 32.5\ 37.97]$,

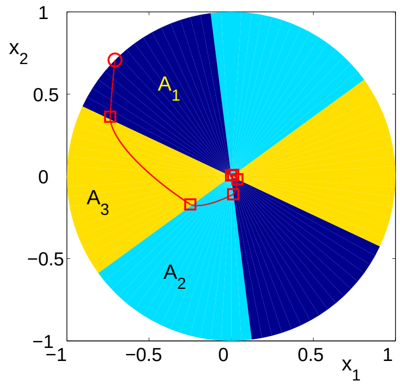


Fig. 1. The optimal trajectory of the switched system $\{A_i\}_{i\in\{1,2,3\}}$ of the Example in Section 6.

$A_3 = [-37.97 \;\; -77.50; 32.50 \;\; 39.97]$. Note that dynamics $A_2$ and $A_3$ are obtained from dynamics $A_1$ by an axis rotation of 120 and 240 degrees respectively. All dynamics $A_i$'s are unstable.

To determine a stabilizing switching law we first associate to the switched system $\{A_i\}_{i\in\mathcal{S}}$ an optimal control problem of the form (4) with $N = \infty$. In particular, we take $Q_i = I_2$, $i = 1, 2, 3$, where $I_2$ denotes the second order identity matrix.

We define an augmented optimal control problem with the stable dynamics $A_4 = -A_1$ and weighting matrix $Q_4 = \bar{q} \cdot \bar{Q}$, where $\bar{q} = 10^5$ and $\bar{Q} = I_2$.

We construct the table $\mathcal{C}_\infty$. More precisely, we apply the procedure to construct the tables $\mathcal{C}_N^i$ for finite values of $N$ and we find out that, for a sufficiently large value of $N$, namely $N = 15$, the tables converge to the same one. The table $\mathcal{C}_\infty$ is reported in Figure 1.

We can immediately observe that the color associated to the stable dynamics $A_4$ never appears. This means that, regardless of the initial state, the optimal trajectory of the augmented optimal control problem is obtained by infinitely switching among unstable dynamics $A_i$, $i = 1, 2, 3$.

This allows one to conclude that the switched system $\{A_i\}_{i\in\{1,2,3\}}$ is globally asymptotically stabilizable. Moreover, the table $\mathcal{C}_\infty$ can be used to compute the stabilizing feedback control law that minimizes the chosen quadratic performance index.

An example of an optimal trajectory is reported in Figure 1 when the initial state is $x_0 = [-1 \quad 1]^T/\sqrt{2}$, $i_0 = 1$. The optimal switching times $\mathcal{T}^*$, the optimal switching sequence $\mathcal{I}^*$ and the optimal cost $V_\infty^*(x_0, i_0)$ are: $\mathcal{T}^* = 10^{-2} \cdot \{0.48, 3.84, 3.78, 3.42, 3.72, 3.48, 3.18, \ldots\}$, $\mathcal{I}^* = \{1, 3, 2, 1, 3, 2, 1, \ldots\}$, and $V_\infty^*(x_0, i_0) = 0.0208$. Note that the system, because of the homogeneous regions, presents a periodic behaviour.

## 7. CONCLUSIONS

Based on our previous results on the optimal control of switched systems with a finite number of admissible switches and at least one Hurwitz dynamics, we first showed that a feedback control law that minimizes a given quadratic cost can also be computed when the number of allowed switches goes to infinity. Then, we showed that this approach can also be efficiently applied when all LTI dynamics are not stable, by simply solving an appropriate optimal control law, called the augmented optimal control problem that contains a Hurwitz dynamics. In particular, we showed that if the switched system with unstable modes is globally exponentially stabilizable, then an optimal feedback control law can be computed, that guarantees the closed-loop system to be globally asymptotically stable.

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