DIAGONALLY-INVARIANT EXPONENTIAL STABILITY

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Abstract: The diagonally-invariant exponential stability (DIES) is introduced as a special type of exponential stability (ES) which incorporates information about the sets invariant with respect to the state-space trajectories. DIES is able to unify, at the conceptual level, issues in stability analysis that have been separately addressed by previous researches Unlike ES, DIES is a norm-dependent property and its study requires appropriate instruments. These instruments are derived in terms of matrix measures from the characteristics of the system trajectories; their convenient exploitation in practice is ensured by methods based on matrix comparisons. The developed framework presents a noticeable generality and its applicability is illustrated for several classes of linear and non-linear systems. This framework can be simply adapted to discrete-time systems.

Keywords: stability analysis, invariant sets, continuous-time systems, matrix measures, comparison methods.

1. INTRODUCTION

The concept of "diagonal stability" is pertaining to linear algebra, where its usage refers to the following definition (Kaszkurewicz and Bhaya, 2000). The matrix $A \in \mathbf{R}^{n\times n}$ is said to be Hurwitz diagonally stable if there exists a positive diagonal matrix $P \in \mathbf{R}^{n\times n}$ that makes $A^T P + PA$ negative definite. Hence, the diagonal stability of *A*, regarded from the point of view of the *linear and time-invariant* (LTI) system $\dot{x}(t) = Ax(t)$, means not only the global *exponential stability* (ES), but also the *invariance of the sets* $\{x \in \mathbf{R}^n \mid \| P^{1/2} x \|_2 \leq c\}$, $c > 0$, with respect to the state-space trajectories.

A deeper insight into the aforementioned invariance problem proves the existence of values $r < 0$ for which the *time-dependent* sets described by $\{x(t) \in \mathbf{R}^n \mid \| P^{1/2} x(t) \|_2 \leq ce^{rt}\}$, $c > 0$, are also invariant with respect to the system trajectories. This fact suggested us to further explore the link between the ES and the time-dependent *invariant sets* (ISs) of the form $\{x(t) \in \mathbf{R}^n \mid \| \Delta x(t) \|_p \leq ce^{rt}\}$, $c > 0$, $r < 0$, constructed with *diagonal positive* matrices $\Delta \in \mathbf{R}^{n\times n}$, for *arbitrary* Hölder norms $\| \, \|_p$, in the general case of *nonlinear*, *time-variant*, dynamical systems. The existence of such ISs defined by diagonal positive matrices presents a great interest for practice due to their symmetry with respect to the axes corresponding to the state variables $x_1, \ldots, x_n$. For the usual Hölder norms $p = 1, 2, \infty$, these ISs have well-known shapes: hyper-diamonds, hyper-ellipses and hyper-rectangles, respectively.

Our research has yielded a framework that refines the qualitative analysis by defining a special type of ES, called by us *diagonally-invariant exponential stability* (DIES), able to incorporate the ISs into the classical concept of ES. Unlike ES, which is a norm-independent property, DIES depends on the considered norm $\| \, \|_p$, and, therefore, its study requires appropriate instruments. The definition of the local and global DIES in terms of the standard $\varepsilon - \delta$ formalism, as well as general characterizations of DIES are given in Section 2. Section 3 formulates DIES conditions in terms of matrix measures; these conditions are exploited in Section 4 via methods based on matrix comparisons. Section 5 applies the results from the previous sections to several classes of linear and nonlinear dynamical systems. Some final remarks on the relevance of our work are presented in Section 6.

Although not directly related to the diagonal stability, the papers cited below supported our investigation on DIES by valuable ideas about ES and/or ISs. Thus, (Fang and Kincaid, 1996) proposed a generalization of Coppel's inequality (Coppel, 1965) for studying the ES of nonlinear and time-variant systems, but the information available about the ISs is not taken into discussion. The set invariance problem for arbitrary norms is addressed within the context of ES in (Kiendl *et al*, 1992), but only for LTI systems. Moreover, that paper considers only constant ISs (i.e. time-independent), regarded, in the traditional manner, as attraction sets for the trajectories. The exponential time-dependence of the ISs (called "contractive" sets) is considered for LTI systems by the survey paper (Blanchini, 1999) and some papers cited therein, but without a special interest for the link between the ES and the ISs. The way toward a profound interpretation of this link as a special type of ES, has been opened by several works on componentwise stability, such as (Pavel, 1984), (Voicu 1984, 1987), (Hmamed and Benzaouia, 1997), (Matcovschi and Pastravanu, 2004), (Pastravanu and Voicu, 2004), that carefully revealed the properties of the time-dependent *rectangular* ISs. Some results of these works will be commented by the current paper from the DIES point of view.

To get a reasonable size for this text, the framework we have developed refers only to continuous-time systems, but it applies *mutatis mutandis* to the discrete-time case. Also for brevity, the proofs of our results are limited to the key elements.

2. GENERAL CHARACTERIZATION

Consider the dynamical system

$$\dot{x}(t) = f(x(t),t), \quad x \in \mathbf{R}^n, x(t_0) = x_0, \qquad (1)$$

where $f: \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}^n$ is continuously differentiable in $x \in \mathbf{R}^n$, continuous in $t \in \mathbf{R}_+$, and $\forall t \in \mathbf{R}_+$, $f(0,t) = 0$, i.e. $\{0\}$ is an *equilibrium point* (EP) of system (1). For referring to the state-space trajectory initialized in $x(t_0) = x_0$, we write $x(t;t_0,x_0)$.

Let $\| \|_p : \mathbf{R}^n \to \mathbf{R}_+$ denote the Hölder norm p in $\mathbf{R}^n$. If D is a positive diagonal matrix $D = \mathrm{diag}\{d_1,..., d_n\}$, $d_i > 0, i = 1,...,n$, denote by $\| \|_p^D$ the vector norm given by $\|x\|_p^D = \|D^{-1}x\|_p$.

Definition 1. The EP $\{0\}$ of system (1) is locally *diagonally-invariant exponentially stable* in the Hölder norm p (abbreviated as locally DIES_p) if there exist a positive diagonal matrix D, a constant $r < 0$ and a constant $\eta > 0$ such that

$$\forall \varepsilon \in (0,\eta), \ \forall t,t_0 \in \mathbf{R}_+, t \ge t_0, \forall x_0 \in \mathbf{R}^n,$$
$$\|x_0\|_p^D \le \varepsilon \Rightarrow \|x(t;t_0,x_0)\|_p^D \le \varepsilon e^{r(t-t_0)}. \qquad \blacksquare(2)$$

The link between the local DIES_p and the local ES defined by the $\varepsilon-\delta$ formalism for an arbitrary norm $\| \|$ (e.g. Michel and Wang, 1995, pp.107)

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 : \forall t,t_0 \in \mathbf{R}_+, t \ge t_0, \forall x_0 \in \mathbf{R}^n,$$
$$\|x_0\| \le \delta(\varepsilon) \Rightarrow \|x(t;t_0,x_0)\| \le \varepsilon e^{r(t-t_0)}, \qquad (3)$$

is given by the following proposition (easy to prove).

Proposition 1. The EP $\{0\}$ of system (1) is locally DIES_p *iff* there exist a positive diagonal matrix D, a constant $r < 0$ and a constant $\eta > 0$ such that $\{0\}$ is locally ES in the sense of condition (3) applied with $\delta(\varepsilon) = \varepsilon \in (0,\eta)$, for the vector norm $\| \|_p^D$. ■

Besides the local ES, the local DIES_p also ensures (easy to prove) the invariance property formulated below:

Proposition 2. The EP $\{0\}$ of system (1) is locally DIES_p *iff* there exist a positive diagonal matrix D, a constant $r < 0$ and a constant $\eta > 0$ such that $\forall \varepsilon \in (0,\eta)$, $\forall t_0 \in \mathbf{R}_+$, the *time-dependent* sets

$$X_\varepsilon(t;t_0) = \left\{ x(t) \in \mathbf{R}^n \,\middle|\, \|x(t)\|_p^D \le \varepsilon e^{r(t-t_0)} \right\}, t \ge t_0 \ge 0, \quad (4)$$

are *invariant* (*positively invariant*) with respect to the trajectories of system (1). ■

These ISs correspond to time-dependent level sets associated with strong Lyapunov functions.

Proposition 3. The EP $\{0\}$ of system (1) is locally DIES_p *iff* there exists $\eta > 0$ such that $V: \Omega = \{x \in \mathbf{R}^n \mid \|x\|_p^D < \eta\} \to \mathbf{R}_+$, $V(x) = \|x\|_p^D$ is a strong Lyapunov function with the decreasing rate r, i.e

$$\forall t \ge 0, \forall x \in \Omega, x = x(t), D^+ V(x(t)) \le r V(x(t)), \quad (5)$$

where D^+ denotes the Dini right derivative.

Proof. Inequality (5) is equivalent to

$$\forall t_0, t \ge 0, t \ge t_0, \forall x_0 = x(t_0) \in \Omega,$$
$$V(x(t)) \le e^{r(t-t_0)} V(x(t_0)). \qquad (6)$$

Necessity: Inequality (6) results from the invariance of the time-dependent ISs $X_\varepsilon(t;t_0)$ with $\varepsilon = \|x(t_0)\|_p^D$.

Sufficiency: It can be proved by contradiction. ` ■

Definition 2. If $\eta = \infty$ in Definition 1, then the EP $\{0\}$ of system (1) is *globally* DIES_p (or DIES_p in *the large*). ■

If the EP $\{0\}$ of system (1) is *globally* DIES_p, then inequality (6) is true for $\Omega = \mathbf{R}^n$ and it points out the link with the definition of *global* ES formulated for an arbitrary norm $\| \|$ (e.g. Michel and Wang, 1995 pp.108)

$$\exists K \ge 1 : \forall t,t_0 \in \mathbf{R}_+, t \ge t_0, \forall x_0 \in \mathbf{R}^n,$$
$$\|x(t;t_o,x_0)\| \le K e^{r(t-t_0)} \|x_0\|. \qquad (7)$$

Proposition 4. The EP $\{0\}$ of system (1) is globally DIES_p *iff* it is globally ES in the sense of condition (7) applied with $K = 1$, for the vector norm $\| \|_p^D$. ■

Remark 1. Propositions 2 and 3 with $\eta = \infty$ characterize the *global* DIES$_p$ of the EP $\{0\}$ of system (1). ■

3. DIES$_P$ AND MATRIX MEASURES

Given a square matrix *Q*, consider its measure

$$\mu_{\|\ \|_p^D}(Q) = \lim_{\theta \downarrow 0} (\| I + \theta Q \|_p^D - 1)/\theta \qquad (8)$$

associated with the matrix norm $\| Q \|_p^D = \| D^{-1} Q D \|_p$. Using the material presented in (Fang and Kincaid, 1996), the following three Lemmas can be proved.

Lemma 1. System (1) is equivalent to the system:

$$\dot{x}(t) = A(x(t),t)x(t), \qquad (9)$$

where the $n \times n$ matrix $A(x,t)$ is defined by:

$$A(x,t) = \int_0^1 J(sx,t)ds, \qquad (10)$$

and $J(x,t) = [\partial f(x,t)/\partial x] \in \mathbf{R}^{n \times n}$ denotes the Jacobian matrix of the function $f: \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}^n$ w.r.t. $x \in \mathbf{R}^n$. ■

Lemma 2. For any norm $\|\ \|_p^D$, $x \in \mathbf{R}^n$ and $t \in \mathbf{R}_+$,

$$\mu_{\|\ \|_p^D}(A(x,t)) \le \int_0^1 \mu_{\|\ \|_p^D}(J(sx,t))ds. \quad ■ \ (11)$$

Lemma 3. For any $t \in \mathbf{R}_+$,

$$A(0,t) = J(0,t). \quad ■ \ (12)$$

Relying on this background, let us explore the DIES$_p$ of system (1) in terms of matrix measures.

Theorem 1. The EP $\{0\}$ of system (1) is locally DIES$_p$ if there exist a positive diagonal matrix *D*, a constant $r < 0$ and a vicinity $U\{0\}$ of $\{0\}$ such that at least one of the following two conditions is fulfilled

(a) $\forall x \in U\{0\}, \forall t \in \mathbf{R}_+, \mu_{\|\ \|_p^D}(A(x,t)) \le r$; (13)

(b) $\forall x \in U\{0\}, \forall t \in \mathbf{R}_+$, $\mu_{\|\ \|_p^D}(J(x,t)) \le r$. (14)

Proof. (a) There exists a constant $\eta > 0$, such that, $\forall x \in \Omega = \{x \in \mathbf{R}^n \mid \|x\|_p^D < \eta\}$, $\forall t \in \mathbf{R}_+$ we have $D^+ V(x(t)) \le rV(x(t))$ and apply Proposition 3. (b) If (14) is true, use Lemma 2 and the result in part (a). ■

Remark 2. If inequalities (13) or (14) hold for $\forall x \in \mathbf{R}^n$, Theorem 1 provides sufficient conditions for the *global* DIES$_p$ of the EP $\{0\}$ of system (1). ■

For the *local* DIES$_p$ we can formulate a necessary and sufficient condition, by using $A(0,t)) = J(0,t))$.

Theorem 2. The EP $\{0\}$ of system (1) is locally DIES$_p$ *iff* there exist a positive diagonal matrix *D* and a constant $r < 0$ such that

$$\forall t \in \mathbf{R}_+, \ \mu_{\|\ \|_p^D}(A(0,t)) = \mu_{\|\ \|_p^D}(J(0,t)) \le r. \ (15)$$

Proof. Sufficiency: Use the continuity of $\mu_{\|\ \|_p^D}(J(x,t))$ to show $\mu_{\|\ \|_p^D}(J(x,t)) \le \sigma < 0$ for a vicinity of $\{0\}$ and, then, Theorem 1. *Necessity:* For any trajectory belonging to a vicinity of $\{0\}$, we write $\forall t \in \mathbf{R}_+, \forall \theta > 0, x(t+\theta) =$ $x(t) + \theta J(0,t)x(t) + \theta F(x(t),t) + \theta O(\theta)$, $\lim_{\theta \downarrow 0} O(\theta) = 0$ and $\lim_{x \to 0} \| F(x,t) \| / \| x \| = 0$ for any norm $\|\ \|$. Hence,

$$\mu_{\|\ \|_p^D}(J(0,t)) = \lim_{\theta \downarrow 0} \left[\frac{1}{\theta} \sup_{\|x(t)\|_p^D = c} \frac{\|(I + \theta J(0,t))x(t)\|_p^D}{\|x(t)\|_p^D} - \frac{1}{\theta} \right]$$

$$= \lim_{\theta \downarrow 0} \left[\frac{1}{\theta} \frac{\|(I + \theta J(0,t))x^*(t)\|_p^D}{\|x^*(t)\|_p^D} - \frac{1}{\theta} \right],$$

where x^* is the generic notation for those *x* that allow reaching the supremum. We can continue to write $\mu_{\|\ \|_p^D}(J(0,t)) \le$

$$\lim_{\theta \downarrow 0} \left[\left(\frac{\|x^*(t+\theta)\|_p^D}{\theta \|x^*(t)\|_p^D} + \frac{\|F(x^*(t),t)\|_p^D}{\|x^*(t)\|_p^D} + \frac{O(\theta)}{\|x^*(t)\|_p^D} \right) - \frac{1}{\theta} \right] =$$

$$\lim_{\theta \downarrow 0} \left[\frac{\|x^*(t+\theta)\|_p^D}{\theta \|x^*(t)\|_p^D} - \frac{1}{\theta} \right] + \frac{\|F(x^*(t),t)\|_p^D}{\|x^*(t)\|_p^D},$$

where the first term is dominated by the decreasing rate appearing in inequality (6), and the second term is negligible, once the considered vicinity of $\{0\}$, $\|x\|_p^D \le c$, is small enough..■

4. DIES$_P$ AND MATRIX COMPARISONS

Given a square matrix $Q = (q_{ij})$, $i, j = 1, \ldots, n$, denote by $\overline{Q} = (\overline{q}_{ij})$, $i, j = 1, \ldots, n$, the *essentially nonnegative* matrix built from the matrix *Q* as follows:

$$\overline{q}_{ii} = q_{ii}, i = 1, \ldots, n; \ \overline{q}_{ij} = |q_{ij}|, i \ne j, i, j = 1, \ldots, n. \ (16)$$

First, we prove four technical results referring to essentially nonnegative matrices.

Lemma 4. If *P* is an essentially nonnegative matrix, then it has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(P)$, which dominates the whole spectrum of *P*, i.e. $\mathrm{Re}(\lambda_i(P)) \le \lambda_{\max}(P)$, $i = 1, \ldots, n$.

Proof: $\lambda_{\max}(sI + P)$ is the spectral radius of the nonnegative matrix $sI + P$, where $s \ge p_{ii}, i = 1, \cdots, n$. ■

Lemma 5. Consider an arbitrary square matrix *Q* and an essentially nonnegative matrix of the same size *P*, such that $\overline{Q} \le P$. Then **(a)** $\mathrm{Re}(\lambda_i(Q)) \le \lambda_{\max}(\overline{Q}) \le \lambda_{\max}(P)$, $i = 1, \ldots, n$; **(b)** $\mu_{\|\ \|_p^\Delta}(Q) \le \mu_{\|\ \|_p^\Delta}(\overline{Q}) \le \mu_{\|\ \|_p^\Delta}(P)$, for any positive diagonal matrix Δ and for any Hölder norm *p*.

Proof: (a) It results from Theorem 8.1.18 in (Horn and Johnson, 1985). (b) For any Δ positive diagonal, $\overline{Q} \le P$ implies $\Delta^{-1} Q \Delta \le \Delta^{-1} \overline{Q} \Delta \le \Delta^{-1} P \Delta$ that yields $\mu_{\|\ \|_p}(\Delta^{-1} Q \Delta) \le \mu_{\|\ \|_p}(\Delta^{-1} \overline{Q} \Delta) \le \mu_{\|\ \|_p}(\Delta^{-1} P \Delta)$, since any Hölder norm *p* is monotonic. ■

Lemma 6. Consider an arbitrary square matrix *Q*. For any positive diagonal matrix Δ and for the Hölder norms $p = 1$ and $p = \infty$, there exists the equality $\mu_{\|\ \|_p^\Delta}(Q) = \mu_{\|\ \|_p^\Delta}(\overline{Q})$.

Proof: It results from the expressions of the measures. ■

Lemma 7. Let P be an essentially nonnegative matrix. If $r>\lambda_{\max}(P)$, for any Hölder norm p there exists a positive diagonal matrix $\Delta=\mathrm{diag}\{\delta_1,\ldots,\delta_n\}$, such that $\lambda_{\max}(P)\le\mu_{\|\;\|_p}(\Delta^{-1}P\Delta)=\mu_{\|\;\|_p^{\Delta}}(P)<r$.

Proof: First, consider the situation (*i*) when P is nonnegative. If E is a square matrix with all its entries 1, then $\lambda_{\max}(P+\varepsilon E)$ as a function of $\varepsilon\ge 0$ is continuous and nondecreasing, according to Theorem 8.1.18 in (Horn and Johnson, 1985). If $r>\lambda_{\max}(P)$, we can find an $\varepsilon^*>0$ such that $\lambda_{\max}(P+\varepsilon^* E)\le r$. On the other hand, the matrix $P+\varepsilon^* E$ is positive and there exist its right and left Perron eigenvectors $v=[v_1\ldots v_n]^T>0$, $w=[w_1\cdots w_n]^T>0$. From (Stoer and Witzgall, 1962), if $1/p+1/q=1$ ($p=1$ meaning $1/q=0$, $p=\infty$ meaning $1/q=1$), for the positive diagonal matrix $\Delta=\mathrm{diag}\{v_1^{1/q}/w_1^{1/p},\cdots,v_n^{1/q}/w_n^{1/p}\}$ we can write $\|\Delta^{-1}(P+\varepsilon^* E)\Delta\|_p=\lambda_{\max}(P+\varepsilon^* E)$. We also have $\|\Delta^{-1}P\Delta\|_p<\|\Delta^{-1}(P+\varepsilon^* E)\Delta\|_p$, since any Hölder norm p is monotonic. Consequently, $\lambda_{\max}(P)\le\|\Delta^{-1}P\Delta\|_p<r$. Similarly we develop the proof for situation (*ii*) when P is essentially nonnegative, by considering $s>0$ such that $sI+P$ is nonnegative. Note that when P is irreducible, the positive diagonal matrix Δ can be built directly from the right and left Perron eigenvectors, yielding $\lambda_{\max}(P)=\mu_{\|\;\|_p}(\Delta^{-1}P\Delta)$. ■

The following results exploit the DIES$_p$ conditions formulated in terms of matrix measures via matrix majorization applied to $\overline{A(x,t)}$, $\overline{J(x,t)}$.

Theorem 3. If there exist a vicinity $U\{0\}$ of the EP $\{0\}$ and a matrix S Hurwitz stable, such that at least one of the following two matrix inequalities is fulfilled:

(a) $$\forall x\in U\{0\},\forall t\in\mathbf{R}_+,\overline{A(x,t)}\le S, \qquad (17)$$

(b) $$\forall x\in U\{0\},\forall t\in\mathbf{R}_+,\overline{J(x,t)}\le S, \qquad (18)$$

then $\{0\}$ is locally DIES$_p$ for any Hölder norm p.

Proof: (a) For any Hölder norm p, Lemma 7 applied to the matrix S ensures the existence of a positive diagonal matrix D and of a constant $r<0$ such that $\mu_{\|\;\|_p^D}(S)<r<0$. At the same time, from Lemma 5b) we have $\forall x\in U\{0\},\forall t\in\mathbf{R}_+$ $\mu_{\|\;\|_p^D}(A(x,t))\le\mu_{\|\;\|_p^D}(S)$, for any Hölder norm p. Thus, we can apply the first part of Theorem 1. (b) The proof is similar to part (a). ■

Remark 3. If inequalities (17) or (18) hold for $\forall x\in\mathbf{R}^n$, Theorem 3 provides sufficient conditions for the *global* DIES$_p$, of EP $\{0\}$ for any Hölder norm p.■

Remark 4. The usage of Lemma 7 for the essentially nonnegative matrix S in the proof of Theorem 3 provides a *concrete procedure* for finding both the positive diagonal matrix D and the constant $r<0$. ■

Remark 5. Theorem 3 can also offer sufficient conditions for the *robustness analysis* of the DIES$_p$ with respect to some uncertainties affecting the expression of the vector function $f:\mathbf{R}^n\times\mathbf{R}_+\to\mathbf{R}^n$ that defines the dynamics of system (1). ■

Although Theorem 2 provides a necessary and sufficient condition for *local* DIES$_p$ based on $A(0,t)=J(0,t)$, sometimes matrix majorization can be more convenient.

Theorem 4. If there exists a matrix S Hurwitz stable such that the following matrix inequality:

$$\forall t\in\mathbf{R}_+,\overline{A(0,t)}=\overline{J(0,t)}\le S \qquad (19)$$

is fulfilled, then the EP $\{0\}$ of system (1) is locally DIES$_p$ for any Hölder norm p.

Proof: Reasoning as in the proof of Theorem 3, we show there exists a positive diagonal matrix D such that $\forall t\in\mathbf{R}_+$, $\mu_{\|\;\|_p^D}(A(0,t))=\mu_{\|\;\|_p^D}(J(0,t))<0$. Then we apply the sufficiency part of Theorem 2. ■

Theorem 5. If there exists $t^*\in\mathbf{R}_+$ such that

$$\forall t\in\mathbf{R}_+,\overline{A(0,t)}=\overline{J(0,t)}\le\overline{A(0,t^*)}=\overline{J(0,t^*)}, \quad (20)$$

then the Hurwitz stability of the matrix $\overline{A(0,t^*)}=\overline{J(0,t^*)}$ is **(a)** *sufficient* for the local DIES$_p$ of the EP $\{0\}$, for any Hölder norm p; **(b)** *necessary and sufficient* for the local DIES$_p$ of the EP $\{0\}$, for the Hölder norms $p = 1$ and $p = \infty$.

Proof: (a) It is a direct consequence of Theorem 4. (b) According to Theorem 2, for each of the Hölder norms $p = 1$, $p = \infty$, there exists a positive diagonal matrix D, such that $\mu_{\|\;\|_p^D}(A(0,t^*))<0$. Lemma 6 allows writing $\lambda_{\max}(\overline{A(0,t^*)})\le\mu_{\|\;\|_p^D}(\overline{A(0,t^*)})<0$. ■

Theorem 6. If there exists $t^*\in\mathbf{R}_+$ such that

$$\forall t\in\mathbf{R}_+,\overline{A(0,t)}=\overline{J(0,t)}\le A(0,t^*)=J(0,t^*), \quad (21)$$

then the Hurwitz stability of the matrix $A(0,t^*)=J(0,t^*)$ is *necessary and sufficient* for the local DIES$_p$ of the EP $\{0\}$, for any Hölder norm p.

Proof: *Sufficiency:* It is a consequence of Theorem 5, since, in this case $\overline{A(0,t^*)}=A(0,t^*)$. *Necessity:* For any Hölder norm p, Theorem 2 ensures the existence of a positive diagonal matrix D, such that $\lambda_{\max}(A(0,t^*))\le\mu_{\|\;\|_p^D}(A(0,t^*))<0$. ■

Remark 6. All our procedures for DIES$_p$ analysis need testing if an essentially nonnegative matrix is Hurwitz stable, or, equivalently, if it is a -M matrix.■

Remark 7. Theorem 3 can be regarded in terms of system comparison theory in the sense that matrix S defines the dynamics of a linear system used as comparison system for (1). If $\forall x\in\mathbf{R}^n,\forall t\in\mathbf{R}_+$, $\overline{A(x,t)}\le S$ and $-S$ is an M matrix, then a classical result for differential systems (e.g. Michel and Wang, 1994, pp. 271) concludes that the EP $\{0\}$ of system (1) is globally ES. Under the same hypothesis, our Theorem 3a) guarantees a more refined property, namely the global DIES$_p$ of the EP $\{0\}$, for any Hölder norm p. ■

5. DIES$_P$ OF SOME CLASSES OF SYSTEMS

In this section we will briefly illustrate the applicability of our results in analyzing the DIES$_p$ for some classes of systems. We will also show that the framework developed by us for the DIES$_p$ is able to incorporate, as particular cases, results reported by different works dealing with ES and/or ISs.

5.1. DIES$_p$ of Linear Systems

We analyse the *global* DIES$_p$ as a system property.

5.1.1. Time-Variant Systems. Consider the system:

$$\dot{x}(t) = A(t)x(t), \ A(t) \text{ continuous.} \qquad (22)$$

Corollary 1. System (22) is DIES$_p$ *iff* there exist a positive diagonal matrix D and a constant $r<0$ such that:

$$\forall t \in \mathbf{R}_+, \ \mu_{\|\|_p^D}(A(t)) \le r. \qquad (23)$$

Proof: It is a consequence of Theorem 2. ■

Corollary 2. Denote by $\Phi(t,t_0)$ the transition matrix of system (22). System (22) is DIES$_p$ *iff* there exist D positive diagonal and a constant $r<0$ such that:

$$\forall t,t_0 \in \mathbf{R}_+, \ t \ge t_0, \ \|\Phi(t,t_0)\|_p^D \le e^{r(t-t_0)}. \quad (24)$$

Proof: *Sufficiency:* If (24) is true, then $\forall t,t_0 \in \mathbf{R}_+$, $t \ge t_0$, $\|x(t)\| \le e^{r(t-t_0)}\|x(t_0\|$ and use the sufficiency part of Proposition 3. *Necessity:* From the necessity part of Proposition 3, we have $\forall t,t_0 \in \mathbf{R}_+$, $t \ge t_0$,

$$\|\Phi(t,t_0)\|_p^D = \sup_{\|x(t_0)\|_p^D \ne 0} \frac{\|\Phi(t,t_0)x(t_0)\|_p^D}{\|x(t_0)\|_p^D} \le e^{r(t-t_0)}. \blacksquare$$

Remark 8. The ES of system (22) means $\forall t,t_0 \in \mathbf{R}_+$, $t \ge t_0$, $\|\Phi(t,t_0)\| \le ke^{r(t-t_0)}$, with $k \ge 1$, for any norm $\| \ \|$, whereas DIES$_p$ means $k \equiv 1$ for the norm $\| \ \|_p^D$. ■

The following Corollaries result from Theorems 4 - 6.

Corollary 3. System (22) is DIES$_p$ for any Hölder norm p if there exists a Hurwitz stable matrix S such that:

$$\forall t \in \mathbf{R}_+, \overline{A(t)} \le S. \qquad \blacksquare \ (25)$$

Corollary 4. If there exists $t^* \in \mathbf{R}_+$ such that

$$\forall t \in \mathbf{R}_+, \overline{A(t)} \le \overline{A(t^*)}, \qquad (26)$$

then $\overline{A(t^*)}$ Hurwitz stable is **(a)** *sufficient* for the DIES$_p$ of (22), for any Hölder norm p; **(b)** *necessary and sufficient* for the DIES$_p$ of (22), for $p = 1, \infty$. ■

Corollary 5. If there exists $t^* \in \mathbf{R}_+$ such that

$$\forall t \in \mathbf{R}_+, \overline{A(t)} \le A(t^*), \qquad (27)$$

then $A(t^*)$ Hurwitz stable is *necessary and sufficient* for the DIES$_p$ of (22), for any Hölder norm p. ■

5.1.2. Time-Invariant Systems. Consider the system:

$$\dot{x}(t) = Ax(t). \qquad (28)$$

The next results are derived from Corollaries 1 - 5.

Corollary 6. System (28) is DIES$_p$ *iff* there exist D positive diagonal and a constant $r<0$ such that:

$$\mu_{\|\|_p^D}(A) = \mu_{\|\|_p}(D^{-1}AD) \le r < 0. \quad \blacksquare \ (29)$$

Remark 9. For the Hölder norm p=2, (29) means a Lyapunov inequality defining the *diagonal stability* of the matrix A (Kaszkurewicz and Bhaya, 2000), $A^T(D^{-1})^2 + (D^{-1})^2 A - 2r(D^{-1})^2 \le 0$, $r<0$. This fact has motivated us to regard inequality (29) as *a generalized Lyapunov inequality* that allows introducing the concept of DIES$_p$ for any Hölder norm p. ■

Remark 10. The usage of the vector norms in defining ISs (with general forms) for system (28) is proposed in (Kiendl *et. al*, 1992). Since the ISs are regarded as independent of time, those results do not express a link between the measure of A and the decreasing rate of the ISs, r, as formulated in (29). ■

Corollary 7. System (28) is DIES$_p$ *iff* there exist D positive diagonal and a constant $r<0$ such that:

$$\forall t,t_0 \in \mathbf{R}_+, \ t \ge t_0 \ \|e^{A(t-t_0)}\|_p^D \le e^{r(t-t_0)}. \ \blacksquare \ (30)$$

Corollary 8. $\overline{A}$ Hurwitz stable (or $-\overline{A}$ an M matrix) is **(a)** *sufficient* for the DIES$_p$ of (28), for any Hölder norm p; **(b)** *necessary and sufficient* for the DIES$_p$ of (28) for the Hölder norms $p = 1, \infty$. ■

Remark 11. Corollary 8b) gives a remarkable algebraic interpretation to inequality (29) with $p=\infty$ ($p=1$): the n generalized Gershgorin's disks of A and $\overline{A}$, written for rows (columns), coincide, and they are located in the region $\mathrm{Re}\, s \le r < 0$ of the complex plane. Moreover Corollary 8b) accommodates as DIES$_\infty$ the symmetrical case of the *componentwise exponential asymptotic stability* of system (28) investigated in (Voicu, 1984), (Hmamed and Benzaouia, 1997). ■

Corollary 9. If the matrix A is essentially nonnegative, then the Hurwitz stability of A (or -A an M matrix) is a necessary and sufficient condition for the DIES$_p$ of system (28), for any Hölder norm p. ■

5.1.3. Interval Matrix Systems. Consider the system:

$$\dot{x}(t) = Ax(t), \ A \in A^I = \{Q \in \mathbf{R}^{n\times n} \mid A^- \le Q \le A^+\}. (31)$$

Corollary 10. Select the matrix $A^* \in A^I$ satisfying

$$\forall A \in A^I, \ \overline{A} \le \overline{A^*}. \qquad (32)$$

The Hurwitz stability of $\overline{A^*}$ is **(a)** *sufficient* for the DIES$_p$ of the interval matrix system (31) for any Hölder norm p; **(b)** *necessary and sufficient* for the DIES$_p$ of system (31) for the Hölder norms $p = 1, \infty$.

Proof: (a) It results from Theorem 3. (b) For each of the Hölder norms $p = 1, \infty$, there exist D positive diagonal and a constant $r<0$, such that $\mu_{\|\|_p^D}(A^*) \le r$.

Lemma 5b) yields $\lambda_{\max}(\overline{A^*}) \le \mu_{\|\|_p^D}(\overline{A^*}) \le r$. ■

Remark 12. The Hurwitz stability of $\overline{A^*}$ was proposed as an easy-to-apply sufficient condition for testing the standard ES of interval systems, e.g. (Chen, 1992), (Sezer and Šiljak, 1994). Corollary 10a) shows that the discussed hypothesis ensures more refined properties. ■

Remark 13. Corollary 10b) for DIES$_\infty$ provides the same necessary and sufficient condition as formulated in (Pastravanu and Voicu, 2004) for the symmetrical componentwise exponential asymptotic stability. ■

Corollary 11. If there exists $A^* \in A^I$, such that

$$\forall A \in A^I,\ \overline{A} \le A^*, \qquad (33)$$

then the Hurwitz stability of the matrix A^* is a *necessary and sufficient* condition for the DIES$_p$ of the interval matrix system (31) for any Hölder norm *p*.

Proof: Sufficiency: It results from Theorem 3. *Necessity:* For an arbitrary Hölder norm *p*, there exist D positive diagonal and a constant $r<0$, such that $\mu_{\|\ \|_p^D}(A^*) \le r$. We also have $\lambda_{\max}(A^*) \le \mu_{\|\ \|_p^D}(A^*)$. ■

5.2. DIES$_p$ *of Recurrent Neural Networks*

Consider the nonlinear system

$$\dot{x}(t) = Bx(t) + Wg(x(t)), \qquad (34)$$

where $B = \mathrm{diag}\{b_1,\cdots,b_n\}$, $b_i<0$, $i=1,\ldots,n$, $W \in \mathbf{R}^{n\times n}$, and $g: \mathbf{R}^n \to \mathbf{R}^n \in C^1(\mathbf{R}^n)$, $g_i(x(t)) = g_i(x_i(t))$, with $g_i(0)=0$ and $0 \le g_i'(x_i) \le L_i$, $\forall x_i \in \mathbf{R}^n$, $i=1,\ldots,n$. This type of systems characterizes the dynamics of the Hopfield neural networks without delay, described with respect to the EP $\{0\}$.

5.2.1. Global DIES$_p$ *of Hopfield Networks*

Corollary 12. Define the matrices $\widetilde{W} \in \mathbf{R}^{n\times n}$, whose entries are $\widetilde{w}_{ii} = \max\{0, w_{ii}\}$, $i=1,\ldots,n$, $\widetilde{w}_{ij} = |w_{ij}|$, $i \ne j$, $i,j=1,\ldots,n$, and $\widetilde{\Lambda} = \mathrm{diag}\{L_i,\cdots,L_n\}$. If the matrix $\widetilde{\Pi} = B + \widetilde{W}\widetilde{\Lambda}$ is Hurwitz stable, then the EP$\{0\}$ of system (34) is *globally* DIES$_p$ for any Hölder norm *p*.

Proof: $\forall x \in \mathbf{R}^n$, $\overline{J(x)} = \overline{B + W\mathrm{diag}\{g_1'(x_1),\cdots,g_n'(x_n)\}} \le B + \widetilde{W}\widetilde{\Lambda} = \widetilde{\Pi}$ and apply Theorem 3b) with $S = \widetilde{\Pi}$. ■

Remark 14. Theorem 3.8c) in (Fang and Kincaid, 1996) uses the same inequality as ours, i.e. $\overline{J(x)} \le \widetilde{\Pi}$, but only for the analysis of the *global* ES. ■

Remark 15. Our Corollary 12 for DIES$_\infty$ provides the same sufficient condition as formulated in (Matcovschi and Pastravanu, 2004) for the symmetrical *componentwise exponential asymptotic stability*. ■

5.2.2. Local DIES$_p$ *of Hopfield neural networks*

Corollary 13. Define the matrices $\hat{W} \in \mathbf{R}^{n\times n}$, whose entries are $\hat{w}_{ii} = w_{ii}$, $i=1,\ldots,n$, $\hat{w}_{ij} = |w_{ij}|$, $i \ne j$, $i,j=1,\ldots,n$, and $\Lambda = \mathrm{diag}\{g_1'(0),\cdots,g_n'(0)\}$. The Hurwitz stability of the matrix $\hat{\Pi} = B + \hat{W}\Lambda$ is **(a)** *sufficient* for the local DIES$_p$ of the EP $\{0\}$ of (34) for any Hölder norm *p*; **(b)** *necessary and sufficient* for the local DIES$_p$ of the EP $\{0\}$ of (34) when $p = 1, \infty$.

Proof: $\overline{J(0)} = \overline{B + W\mathrm{diag}\{g_1'(0),\cdots g_n'(0)\}} = B + \hat{W}\Lambda = \hat{\Pi}$ and apply Theorem 4 with $S = \hat{\Pi}$. ■

Corollary 14. The EP $\{0\}$ of (34) is DIES$_p$ *iff* there exist *D* positive diagonal such that $\mu_{\|\ \|_p^D}(B + W\Lambda) < 0$.

Proof: Apply Theorem 2 for $J(0) = B + W\Lambda$ ■

Remark 16. As expected, the sufficient condition in Corollary 13a) for *local* DIES$_p$ is weaker than the one in Corollary 12 for *global* DIES$_p$, i.e. when $\hat{\Pi}$ is Hurwitz stable, $\widetilde{\Pi}$ may be not. This is because $\hat{\Pi} \le \widetilde{\Pi}$ implies $\lambda_{\max}(\hat{\Pi}) \le \lambda_{\max}(\widetilde{\Pi})$ - see Lemma 5a). In its turn, the condition in Corollary 14 is weaker than the condition in Corollary 13a), for $p \ne 1, \infty$, since $\mu_{\|\ \|_p^D}(J(0)) \le \mu_{\|\ \|_p^D}(\overline{J(0)})$ for any positive diagonal matrix *D* - see Lemma 5b). ■

6. CONCLUSIONS

The concept of DIES brings a refinement in the qualitative analysis of the dynamical systems because it expresses the link between the classical ES and different types of ISs. Matrix measures represent the instruments able to capture information about the both properties, ES and the existence of ISs. To simplify the exploitation of these instruments, we also formulate some results based on matrix comparisons. The developed framework offers a deeper interpretation for the usage of the algebraic tools in exploring the system stability. The paper also presents specialized and relevant results derived for some classes of linear and nonlinear systems, which are frequently encountered in automatic control.

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