INVENTORY CONTROL BY MODEL PREDICTIVE CONTROL METHODS

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Abstract: This paper describes a model predictive control strategy to tackle the inventory control problem in Supply Chains. The problem is formulated as a receding-horizon optimization problem and the market demand is considered an external unknown disturbance. The behaviour of the Supply Chain is modelled by discrete time difference equations, both the control variables and the output variables are constrained to satisfy suitable bounds. Quantitative performance indices are introduced and the effects of the choice of the control parameters are discussed.

Keywords: predictive control, decision support systems, control applications, closed-loop controllers, constraints, inventory control.

1. INTRODUCTION

In the continuous attempt to reduce production costs, the attention has shifted, in the last ten years, from assembly-line automation and machine automation to the government of logistics costs and to the control of the coupled dynamics of supply chains. In that situation, the use of control and system theory methodologies in supply chain management is gaining interest and it appears to be capable of providing useful insights and results. Works in this direction are, e.g, those of Forrester (1961), Towill (1992), (1994) and Tzafestas and Kapsiotis (1994), that address, under various hypothesis, the inventory control problem in supply chains. Here, an approach in the same spirit is developed and a possible methodology for tackling inventory control problems in a supply chain is proposed, working under constraints on the production levels and assuming to have uncertain information on the market demand over a finite time horizon.

The basic idea employed consists, first, in solving an optimization problem over the finite horizon $[t_0, t_0 + T]$, using a prediction of the system behaviour. The functional considered in the optimization problem depends on the difference between the actual inventory level (the output of the system) and a desired set point and, possibly, on the relative control effort, namely on the incremental variation of the production level (the control variable). The market demand acts as a partially unknown disturbance, since we assume that a prediction of its value on $[t_0, t_0 + T]$ is available. After solving the optimization problem over $[t_0, t_0 + T]$, the optimal value of the control at time t_0, namely $u(t_0)$, is applied and the horizon is moved one step forward. Then, the optimization process is repeated on $[t_0 + 1, (t_0 + 1) + T]$. This allows us in particular to update the information on the market demand and to take into account the error made in predicting it. This scheme follows that of the classical Model Predictive Control (MPC) (Garcia, *et al.*, 1989; Morari, *et al.*, 1994), mainly developed in the area of industrial control to facilitate dealing with constraints and to contrast the effect of unknown disturbances in a closed-loop fashion.

Although in inventory control problems variables often take only integer values (as they measure lots of good), we let them vary in $\mathbb{R}$, in order to reduce the computational burden required to solve the optimization

problem. It should be noted, however, that conceptually the approach here developed can be applied to the case in which all variables are constrained to take only integer values. In that case, obviously, it would be necessary to employ a different algorithm to solve, on the integer, the optimization problem.
The aim of our study is to analyse the dependency of the convergence and the robustness properties of the described MPC technique on the parameters that appear in the chosen cost function. Since no theoretical results can be invoked in order to clarify this issue, extensive simulations in various conditions were performed. As a result, the control method is shown to be capable of assuring satisfactory performances. In particular, the actual inventory level, on one side, is forced to match a given set point and the amplification effect of the disturbance through the various levels of the supply chain, on the other side, is drastically reduced. In the end, the ability to regulate the inventory levels may be exploited for reducing the set point with economic benefits and without incurring in backlogs of demand.
The remainder of the paper is organized as follows. Section 2 introduces the notation used, it describes the model of the Supply Chain and states the Inventory Control Problem (ICP). Section 3 gives a brief introduction to the Model Predictive Control (MPC) and its application to the Inventory Control Problem. Section 4 shows numerical example of the application of the MPC to the ICP. Section 5 summarizes the ideas presented in the paper.

2. PROBLEM DESCRIPTION

The model considered in this paper describes a pull-system, driven by the market demand. The model architecture consists of L interconnected levels, which represent various business units (echelons) of a Supply Chain. All units are modelled in a similar way, as processes characterized by a constant lead-time, whose output represents an inventory level. The Supply Chain is vertically integrated, in the sense that each business unit is the only customer of the unit in the level beneath it and, at the same time, it is the only supplier of the unit in the level above it. In other words, this means that the control variable of each unit acts as a disturbance on the unit in the level beneath it. The customer for the first level is the market, that drives the whole system by means of its demand. The last, lower level does not have a supplier in the chain.
Formally, the system described above is represented by the following set of equations:

$$\begin{cases} y_1(t+1) = y_1(t) + u_1(t-\vartheta_1) - m(t) \\ y_i(t+1) = y_i(t) + u_i(t-\vartheta_i) - u_{i-1}(t), \quad i = 2,...,L \end{cases} \tag{1}$$

$$y_i^l \le y_i(t) \le y_i^h, \quad t \ge 0, \quad i = 1,...,L \tag{2}$$

$$u_i^l \le u_i(t) \le u_i^h, \quad t \ge 0, \quad i = 1,...,L \tag{3}$$

where:

- $L \in \mathbb{N}_+$ is the number of business units (echelons)
- $y_i(t) \in \mathbb{R}$ is the inventory level of the i-th echelon at time t, in the following $y(t)$ will denote the vector $y(t) = [y_1(t),..., y_L(t)]^T$
- $u_i(t) \in \mathbb{R}$ is the order to produce made by the i-th echelon at time t and received by the (i+1)-th echelon, in the following $u(t)$ will denote vector $u(t) = [u_1(t),...,u_L(t)]^T$
- $m(t) \in \mathbb{R}_+$ is the market demand at time t
- $\vartheta_i \in \mathbb{N}$ is the lead-time for the i-th echelon
- $y_i^l, y_i^h \in \mathbb{R}$ are, respectively, the lower bound and the upper bound on the inventory level of the i-th echelon, in the following y^l will denote the vector $y^l = [y_1^l,..., y_L^l]^T$ and y^h will denote the vector $y^h = [y_1^h,..., y_L^h]^T$
- $u_i^l, u_i^h \in \mathbb{R}$ are, respectively, the lower bound and the upper bound on the replenishment capacity of the i-th echelon, in the following u^l will denote the vector $u^l = [u_1^l,...,u_L^l]^T$ and u^h will denote the vector $u^h = [u_1^h,...,u_L^h]^T$

Remark that in (1) $u(t)$ is viewed as the control variable and $y(t)$ is viewed as the output variable. Moreover, it should be noted that $m(t)$ can assume negative values, because it is necessary to represent the returned goods.
With the notation introduced above, it is possible to state the following problem. Given:

- a system Σ described by equations of the form (1);
- the constraints on $y(t)$ and $u(t)$ of the form (2) and (3);
- T functions $\hat{m}_1(t+1|t),...,\hat{m}_T(t+T|t)$ that predict, respectively, the market demand at time $t+i$, for $i = 1,...,T$, on the basis of information available at time t;
- L set points y_i^*, $i = 1,...,L$, for the inventory levels of the different echelons, in the following y^* will denote the vector $y^* = [y_1^*,..., y_L^*]^T$

the **Inventory Control Problem (ICP)** consists in finding $u(t), t \ge 0$ which minimize the expression

$$\sum_{t=0}^{+\infty} \left\| y(t) - y^* \right\|^2 \tag{4}$$

Several study (Towill, *et al.*, 1992; Towill and Del Vecchio, 1994) show that an inventory control policy, based only on the actual inventory level and the desired inventory level, could generate the amplification of the market demand. The oscillations of the market demand $m(t)$, that disturb the first level of the Supply Chain, generate a control sequence $u_1(t)$ with oscillations of greater amplitude, that disturb the second level of the Supply Chain. This effect spreads from the first to the last level of the Supply Chain and it is very disturbing for the production system, because great oscillations on the production do not allow the plant to work at full capacity. So the control sequence $u(t), t \geq 0$ minimizing (4) has also to avoid or to reduce the amplification effect.

3. MODEL PREDICTIVE CONTROL APPROACH TO THE ICP

In this paragraph we briefly describe how the Model Predictive Control (MPC) method (Garcia, *et al.*, 1989; Morari, *et al.*, 1994) can be applied to the Inventory Control Problem.
In the MPC approach, the control action is computed by solving an optimization problem over a finite horizon, using a prediction of the system behaviour. The computed optimal control is applied at the next time instant and the optimization process is repeated, using a new, updated prediction of the system behaviour.
In our situation, namely in dealing with the ICP defined above, the functional J employed in the optimization process takes the following form

$$J = \sum_{j=1}^{P} \left\| \Gamma \left[\hat{y}(t+j \mid t) - y^* \right] \right\|^2 + \sum_{j=1}^{M} \left\| \Lambda \Delta u(t+j-1) \right\|^2 \tag{5}$$

where :

- $\hat{y}(t+j \mid t)$ denotes a prediction of the output of Σ, namely of the vector of inventory levels, at time $t+j$, made at time t by exploiting the predicting functions $\hat{m}_1(t+1 \mid t), \ldots, \hat{m}_T(t+T \mid t)$;
- $\Delta u(t) = u(t+1) - u(t)$ represents the incremental variation of the control variable $u(t)$, it is assumed that $\Delta u(t+j-1) = 0$ for $(j-1) \geq M$;
- $\Gamma = diag\{\gamma_1, \ldots, \gamma_L\}$ and $\Lambda = diag\{\lambda_1, \ldots, \lambda_L\}$ are matrices of weights;
- P and M are, respectively, the prediction horizon and the control horizon.

Clearly, the prediction horizon P cannot be greater than T, the number of predicting functions, while, P has to be greater than M (recall that $\Delta u(t+j-1) = 0$ for $(j-1) \geq M$).

The application of the MPC procedure can be described as follows:

1. at time t solve the optimization problem $\min_{\{u(t),\ldots,u(t+M-1)\}} J$, with constraints (2) and (3);
2. feed $u(t)$ to the system
3. iterate with $t = t+1$

The performances of the above procedure depend, of course, on the choice of the parameters Γ, Λ, P and M.
We choose the parameters Γ and Λ the following structure: $\Gamma = \gamma diag\{\gamma_1, \ldots, \gamma_L\}$ and $\Lambda = \lambda diag\{\lambda_1, \ldots, \lambda_L\}$ where $\gamma_1, \ldots, \gamma_L$ and $\lambda_1, \ldots, \lambda_L$ are fixed by economic consideration on the relative value of each level of the Supply Chain, while γ and λ, in particular the ratio γ/λ, are parameters that have to be tuned in order to achieve the performances and the robustness proprieties desired. Also the parameters P and M are connected, to avoid great error on the output prediction we choose the control horizon close to the prediction horizon ($P = M + \max_{i \in [1,\ldots,L]} \vartheta_i$)
In presence of an estimation error $e(i \mid t) = m(t+i) - \hat{m}_i(t+i \mid t)$ on the market demand general rules for tuning the controller parameters cannot be derived by theoretical results, that are still missing (Bemporad and Morari, 1999). Simulations are commonly employed for analysing the behaviour of the controlled system and for tuning the control parameters Γ, Λ, P and M (Bemporad and Morari, 1999; Gallestey, *et al.*, 2003). Here, we propose the use of two quantitative indices for evaluating the performances of the above procedure:
- the **mean square regulation error** E, defined by

$$E = \sum_{i=1}^{L} E_i = \frac{1}{t_f - t_0} \sum_{t=t_0}^{t_f} (y_i(t) - y^*)^2 \tag{6}$$

where $[t_0, t_f]$ is an interval of interest;
- the **amplification index** G, defined by

$$G = \sum_{i=1}^{L} G_i = \frac{\max_{[t_0,t_f]} u_i(t) - \min_{[t_0,t_f]} u_i(t)}{\max_{[t_0,t_f]} u_{i-1}(t) - \min_{[t_0,t_f]} u_{i-1}(t)} \tag{7}$$

where $[t_0, t_f]$ is an interval of interest.
The mean square regulation error measures the ability of the control methods to regulate each component of the output, that is each inventory level, to the chosen set point. The amplification index measures the propagation of the oscillations in the market demand through the various levels of the supply chain. It is well known that, in regulating the inventory level, oscillations may dramatically increase and cause ultimately the practical failure of a control policy. Beside limiting E, therefore, one is usually interested in reducing G. The fact that

both indices are evaluated over an interval $\left[t_0, t_f\right]$ takes in particular into account the scarce interest for the transient behaviour in $t \le t_0$, which cannot be regulated due to the intrinsic delays of the system.

4. SIMULATION RESULTS

To test the MPC method on the inventory control of a supply chain, several simulations were performed. The simulation were performed varying the uncertainty on the estimation of the market demand, to evaluate the robustness of the method, and modifying the controller parameters, i.e. the length M of the control horizon and the ratio γ/λ, to tune the control action on the basis of the quantitative indices E and G. The simulations were performed for a supply chain consisting of three levels, $L = 3$, and characterized by constant lead-times ϑ_i, given respectively by $\vartheta_1 = 2$, $\vartheta_2 = 4$, $\vartheta_3 = 3$. The constraints (2) and (3) has been chosen as $0 \le y_i(t) \le 2000$ and $0 \le u_i(t) \le 300$ for $i = 1,...,L$. The set points were fixed as $y_i^* = 1000$ for $i = 1,...,L$. The market demand $m(t)$ is a random bounded signal, $-100 \le m(t) \le 200$ whose estimation is $\hat{m}(t)$.
The equations of the system are the following:

$$\begin{cases} y_1(t+1) = y_1(t) + u_1(t-2) - m(t) \\ y_2(t+1) = y_2(t) + u_2(t-4) - u_1(t) \\ y_3(t+1) = y_3(t) + u_3(t-3) - u_2(t) \end{cases} \tag{8}$$

$$0 \le y_i(t) \le 2000, \quad t \ge 0, \quad i = 1,2,3 \tag{9}$$

$$0 \le u_i(t) \le 300, \quad t \ge 0, \quad i = 1,2,3 \tag{10}$$

$$y_i^* = 1000, \quad i = 1,2,3 \tag{11}$$

Table 1 reports the best values for the parameters of the controller on the basis, respectively of E and G, and the values of the indices, at different level of error on the estimation of the market demand. The estimation error is quantify by

$$err = \left\{ \left[\frac{1}{P} \sum_{i=1}^{P} e(i \mid t) \right] \Big/ \left(\max_t m(t) - \min_t m(t) \right) \right\} 100 \tag{12}$$

Figure 1 and Figure 2 show the regulation results, respectively, in presence of low and high estimation error on the market demand. Each figure is composed by three graphs, the first one represents the output of the three levels of the system, the second graph shows the input feed to the system at each one of the three levels, the third graph shows both the real demand and the predicted demand used in the simulation. Both in Figure 1 and in Figure 2 after a transient due to the delays of the system the outputs stay close to the set point and the demand oscillation does not propagate in the inputs . The relatively large oscillations on the output are due to the market demand, that to be realistic shows great and quickly variations in time, and on the error in its prediction.

Table 1 controller parameter and performance indices

err	M_E	γ/λ_E	$E/(10\wedge 3)$	M_G	γ/λ_G	G
10%	20	70	2.05	40	70	426
20%	80	15	2.22	40	85	250
30%	45	65	3.22	100	70	247
40%	90	35	2.04	90	65	274
50%	80	20	4.17	90	75	266
60%	30	90	3.97	45	100	190
70%	25	50	5.45	80	85	165
80%	65	25	5.55	75	95	184
90%	15	80	8.74	100	100	215
100%	45	50	10.66	35	95	208

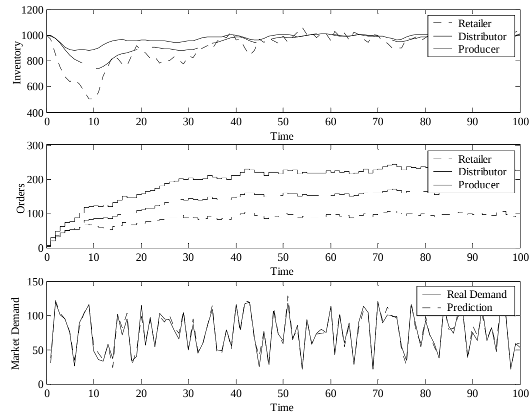

Fig. 1. Inventory and order levels for the three echelons of the Supply Chain in presence of 10% of error on the prediction of the market demand

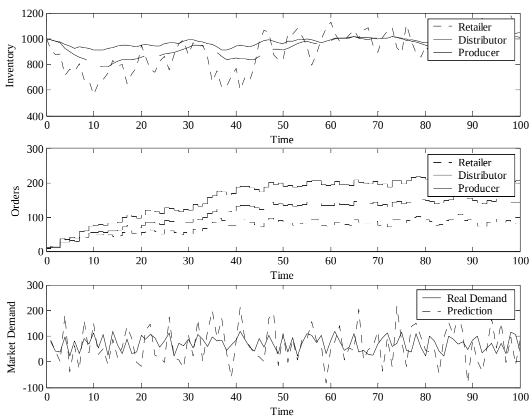

Fig. 2. Inventory and order levels for the three echelons of the Supply Chain in presence of 100% of error on the prediction of the market demand

Figure 3 and Figure 4 show the dependencies of E on the controller parameters M and γ/λ, in presence of low and high estimation error on the market demand.

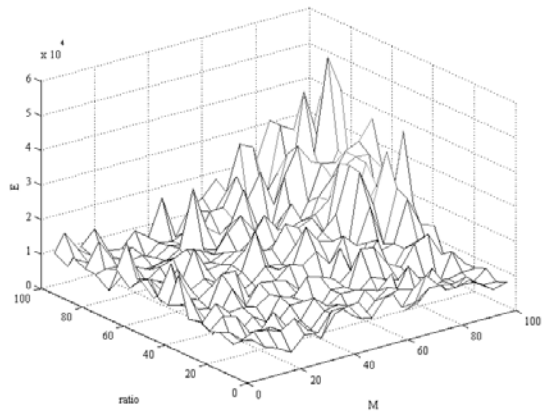

Fig. 3. Values of the quantitative index E, varying the controller parameters, in presence of 10% of error on the prediction of the market demand

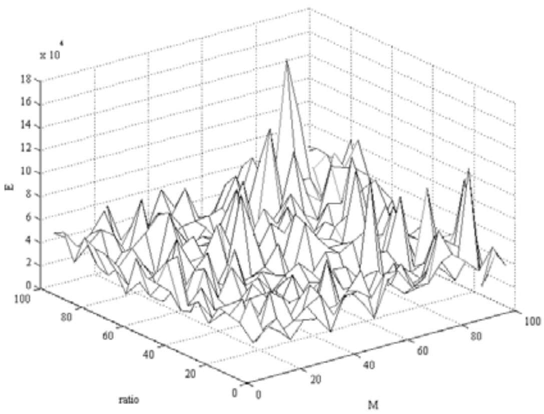

Fig. 4. Values of the quantitative index E, varying the controller parameters, in presence of 100% of error on the prediction of the market demand

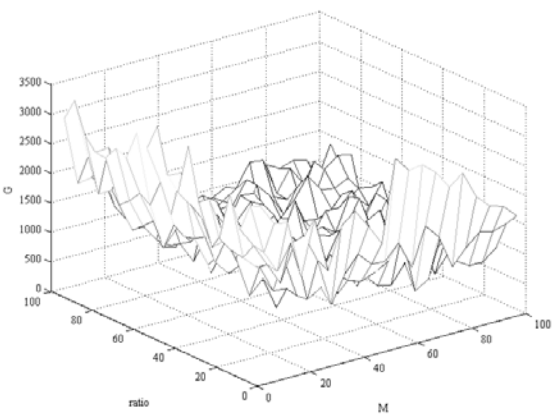

Fig. 5. Values of the quantitative index G, varying the controller parameters, in presence of 10% of error on the prediction of the market demand

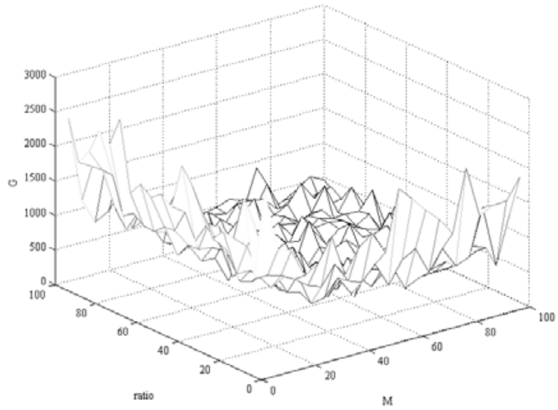

Fig. 6. Values of the quantitative index G, varying the controller parameters, in presence of 100% of error on the prediction of the market demand

While, Figure 5 and Figure 6 show the dependencies of G on the controller parameters M and γ/λ, in presence of low and high estimation error on the market demand.

Since there is a trade-off between regulation of the output and reduction of the amplification of demand oscillation, it is important to remark that both E and G increase (denoting worse performances) their values as prediction error on the market demand goes higher but this does not influence the qualitative ways in which they depend on the controller parameters. This is useful to choose the controller parameters, in order to achieve the desired performances.

5. CONCLUSION

In conclusion MPC methods prove to be suitable to dealing with the Inventory Control Problem in presence of un certainty on the market demand. The introduction of quantitative indices allows us to analyze the performances in relation to the chosen control parameters and, in principle, to tune the controller.

Further study will concern the case of more complex structure on the supply chain and uncertain lead-times.

REFERENCES

Bemporad, A. and M. Morari (1999). Robust Model Predictive Control: A Survey. In *Robustness in Identification and Control* (A. Garulli, A. Tesi and A. Vicino (Eds.)), pp. 207-226. Lecture Notes in Control and Information Sciences, vol. 245, Springer-Verlag.

Forrester, J.W (1961). *Industrial dynamics*. MIT press, Cambridge, MA.

Gallestey, E., A. Stothert, D. Castagnoli, G. Ferrari-Trecate and M. Morari (2003). Using Model Predictive Control and Hybrid Systems for Optimal

Scheduling of Industrial Processes. *Automatisierungtechnik,* **Vol. 51 No. 6**, pp. 285-293.

Garcia, C.E., D.M. Prett and M. Morari (1989). Model Predictive Control: Theory and Practice – a Survey. *Automatica,* **Vol. 25 No. 3**, pp. 335-348.

Morari, M., C.E. Garcia, J.H. Lee and D.M. Prett (1994). *Model Predictive Control*, Prentice Hall.

Towill, D.R., M.M. Naim and J. Wilkner (1992). Industrial Dynamics Simulation Models in the Design of Supply Chains. *International Journal of Physical Distribution & Logistics Management,* **Vol. 22.**

Towill, D.R. and A. Del Vecchio (1994). The application of filter theory to the study of supply chain dynamics. *Production Planning & Control,* **Vol. 5 No. 1**, pp. 82-96.

Tzafestas and S., G. Kapsiotis (1994). Co-ordinated control of manufacturing/supply chains using multi-level techniques. *Comp. Int. Manuf. Systems*, **Vol. 7 No. 3**, pp. 206-212.