FIXED POLES FOR NON MINIMAL SYSTEMS: A GEOMETRIC APPROACH

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Abstract: When considering the set of all possible solutions, *e.g.* by state or measurement feedback, of a given control problem (*e.g.* disturbance decoupling), complete pole placement is usually not possible, due to the so-called fixed poles: these are present as dynamics of the compensated system, whatever be the way the solution has been designed (within the chosen class). These fixed poles have two different origins: some are present, whatever be the considered control problem, because of possible non minimality of the state description (in the Kalman sense, i.e. controllability and observability). The other are due to the very particular control problem for which the feedback law is a solution. We show here how non minimality impacts the corresponding geometric solvability conditions and how the global set of the fixed poles of such control problems can be characterised in the general case (i.e. without any controllability or observability assumption).

Keywords: Linear control systems, disturbance rejection, decoupling, pole assignment, geometric approach.

1. INTRODUCTION

When analysing a control problem for a linear system having both control and disturbance inputs, there is no reason for the dynamics of the disturbance signal to be controllable by the control input. Hence, assuming controllability of the state description would be restrictive. Faced to such non minimal situations, and in the objective of designing solutions insuring some nice pole placement properties (*e.g.* stability or, more generally, pole placement in a desired region of the complex plan), one is limited by the so-called fixed poles of the considered control problem. These fixed poles are present as dynamics of the compensated system, whatever be the way the solution has been designed (within the chosen class). Since the set of those fixed poles obviously contains one subset which is problem independent, namely the uncontrollable poles (and, in the case of measurement feedbacks, also the unobservable poles), one is naturally tempted to first minimise the state realisation (i.e. to get rid of the uncontrollable part, and similarly the unobservable one), and then to consider the given control problem for the reduced system. However, can we claim (as intuition tells us) that the fixed poles of the solutions are the union of the uncontrollable and the unobservable poles, with the fixed poles of the control problem computed for the minimal realisation? How are the solvability conditions related, are these two problems equivalent before and after minimisation? Namely, could we loose some potential solutions through the reduction process? We give answers to these questions for classical control problems like disturbance rejection and non interaction, through the use of the so-called geometric approach.

2. NOTATION AND BACKGROUND

Let us consider a linear time-invariant strictly proper system described by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dq(t) \\ z(t) = Ex(t) \\ y(t) = Cx(t) \end{cases} \qquad (1)$$

where x, u, q, z, and y are respectively the state, control input, disturbance input, output to be controlled and measurement. These signals belong to the vector spaces X, U, Q, Z, and Y, respectively.

The Disturbance Rejection problem by dynamic Measurement Feedback (DRMF) amounts to finding (if any) a dynamic compensator with input y and output u, such that, for the compensated system, the transfer function matrix from q to z be zero. The general form of such compensators is the following, where $w \in W$ is the state of the compensator:

$$\begin{cases} \dot{w}(t) = Nw(t) + My(t) \\ u(t) = Lw(t) + Ky(t) \end{cases} \qquad (2)$$

The system (1) controlled by (2) can be described by the following description, where $X_e = X \oplus W$, with $\oplus$ denoting the direct sum :

$$\begin{cases} \dot{x}_e(t) = A_e x_e(t) + D_e q(t) \\ z(t) = E_e x_e(t) \end{cases} \qquad (3)$$

where:

$$A_e = \begin{bmatrix} A+BKC & BL \\ MC & N \end{bmatrix}, D_e = \begin{bmatrix} D \\ 0 \end{bmatrix}, E_e = \begin{bmatrix} E & 0 \end{bmatrix} \qquad (4)$$

The DRMF control objective thus amounts to cancelling all the Markov parameters of the compensated system, i.e.:

$$E_e A_e^{\,i} D_e = 0 \text{, for all } i \geq 0 \qquad (5)$$

This problem has been receiving a lot of complementary contributions since the major solutions proposed by (Schumacher, 1980) and (Willems and Commault, 1981). These authors gave geometric necessary and sufficient conditions for the solvability of this problem, without any minimality assumption on the considered system. Conditions for the existence of internally stable solutions have also been provided (see for instance (Basile and Marro, 1992), with the use of the so-called self-bounded and self-hidden invariant subspaces). An alternative way to characterise the existence of stable solutions came through the characterisation of the fixed poles. As an interesting by product, this way also gives an answer, without any extra cost, to a more general question related to pole placement within a pre-specified region of the complex plan. Indeed, thanks to some minimality assumptions on the considered model, it is possible to place all the other poles except the fixed ones. And thus, the location of these fixed poles with respect to the desired pole placement region, directly gives the answer. Concerning the characterisation of the DRMF fixed poles (poles which are present in any solution of the DRMF problem) the most advanced contribution (Del-Muro-Cuellar and Malabre, 2001) has been using the assumptions (A,[BD]) controllable and ([C^T E^T]^T, A) observable (where ^{T} denotes the transpose). We shall here consider the general case with none of these assumptions.

Some geometric concepts are first quickly recalled.

For systems described by (1), let us denote by R and N, respectively, the reachability and unobservability subspaces with respect to all input and output signals, namely:

$$R = \mathrm{Im}[BD] + A\mathrm{Im}[BD] + \ldots + A^{\dim(X)-1}\mathrm{Im}[BD]$$

$$N = \mathrm{Ker}[C^T E^T]^T \cap A^{-1}\mathrm{Ker}[C^T E^T]^T \cap \ldots \\ \ldots \cap A^{-[\dim(X)-1]}\,\mathrm{Ker}[C^T E^T]^T$$

where Im stands for the Image, Ker for the Kernel, and A^{-1}(.) for the inverse image of the subspace inside the brackets.

We shall here denote by S^* the infimal (C,A) (or conditioned)-invariant subspace containing ImD, and by V^* the supremal (A,B) (or controlled)-invariant subspace contained in Ker$E$. These are the respective limits of the following and famous algorithms (see (Basile and Marro, 1992) and (Wonham, 1985)):

$$\begin{cases} S_0 = 0 \\ S_{i+1} = \mathrm{Im}\,D + A(KerC \cap S_i) \end{cases} \qquad (6)$$

$$\begin{cases} V_0 = X \\ V_{i+1} = KerE \cap A^{-1}(\mathrm{Im}\,B + V_i) \end{cases} \qquad (7)$$

It is well known (Schumacher 1980) that the DRMF problem is solvable if and only if:

$$S^* \subset V^* \qquad (8)$$

Let us now consider one of the possible minimisation processes which from the general model (1) extracts a minimal one (controllable and observable with respect to all input and output signals, *i.e.* (u, q) and (z, y)). This amounts to keeping the trajectories which are controllable and getting rid of those which are unobservable. It is well known that the following diagrams (see Fig.1) "commute" and that the reduced system with maps (A", B", D", C", E") is indeed minimal. The terms "commute" (which is commonly used in the geometric approach, see *e.g.* (Wonham, 1985)) simply means that the following equations are satisfied:

$$\begin{cases} \Phi A' = A\Phi \\ A''\Pi = \Pi A' \end{cases} \text{, for the central parts} \qquad (9)$$

$$\begin{cases} \Phi\begin{bmatrix} B' & D' \end{bmatrix} = \begin{bmatrix} B & D \end{bmatrix} \\ \begin{bmatrix} B'' & D'' \end{bmatrix} = \Pi\begin{bmatrix} B' & D' \end{bmatrix} \end{cases} \text{, for the left parts} \qquad (10)$$

and,

$$\begin{cases} \begin{bmatrix} C' \\ E' \end{bmatrix} = \begin{bmatrix} C \\ E \end{bmatrix} \Phi \\ \begin{bmatrix} C'' \\ E'' \end{bmatrix} \Pi = \begin{bmatrix} C' \\ E' \end{bmatrix} \end{cases} \text{, for the right parts} \tag{11}$$

where Φ denotes the insertion map of $R+N$ into X, Π is the canonical projection onto $(R+N)/N$, the quotient space modulo N (for details about these standard tools of the geometric approach, see *e.g.* (Wonham, 1985)).

These commutations hold true because ImB+ImD is included in Im Φ and KerΠ is included in both KerE' and KerC'. The maps (A', B', D', C', E') and (A'', B'', D'', C'', E'') are unique since Φ is monic and Π epic.

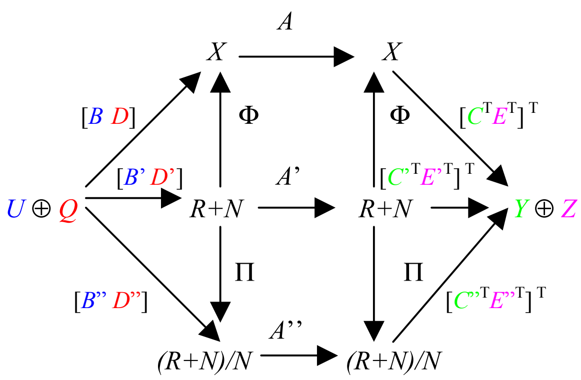

Fig. 1. Commutative diagrams

Let us denote by S'_i, respectively S''_i, and V'_i, respectively V''_i, the steps of the algorithms (6) and (7) with data A', B', … and respectively A'', B'', … in place A, B, … and with respective limits S'^*, S''^*, V'^*, and V''^*.

It can easily be shown that, for all i ≥ 0, and thus also for the limits:

$$\begin{aligned} S'_i &= \Phi^{-1} S_i \\ V'_i &= \Phi^{-1} V_i \\ S''_i &= \Pi S'_i \\ V''_i &= \Pi V'_i \end{aligned} \tag{12}$$

[*Sketch of the proof for the first relationship*:

$S'_0 = 0 = \Phi^{-1}(0) = \Phi^{-1} S_0$ (remember that Φ is monic)

Assume that $S'_i = \Phi^{-1} S_i$, for a given i ≥ 0.

Then:

$S'_{i+1} :=$ ImD' + A'(KerC' $\cap$ S'_i)
= Φ^{-1} ImD + A'(KerC' $\cap$ $\Phi^{-1}S_i$)
= Φ^{-1} ImD + A'(Φ^{-1}KerC $\cap$ $\Phi^{-1}S_i$)
= Φ^{-1} ImD + A'Φ^{-1}(KerC $\cap$ S_i)
= Φ^{-1} ImD + Φ^{-1}A(KerC $\cap$ S_i)
= Φ^{-1} [ImD + A(KerC $\cap$ S_i]

this distributes since ImD $\subset$ ImΦ = $R+N$, and thus:

$S'_{i+1} = \Phi^{-1} S_{i+1}$,

The other relationships can be proved in a similar way.■]

3. MAIN RESULTS

It is now possible to compare the DRMF solvability conditions for the initial system (1), for the reduced system (A', B', D', C', E'), and for the minimised realisation (A'', B'', D'', C'', E'').

Theorem 1: The DRMF problem is solvable for (A, B, D, C, E) if and only if it is solvable for the reduced realisation (A', B', D', C', E'), if and only if it is solvable for the minimal realisation (A'', B'', D'', C'', E''), both described in (9)-(11).

Proof: Thanks to (12):

$$S''^* = \Pi S'^* \text{ and } S'^* = \Phi^{-1} S^*$$

and

$$V''^* = \Pi V'^* \text{ and } V'^* = V^*.$$

Therefore,

$$\begin{aligned} & S''^* \subset V''^* \\ \Leftrightarrow\ & \Pi S'^* \subset \Pi V'^* && \text{(since (12))} \\ \Leftrightarrow\ & S'^* \subset V'^* + Ker\Pi \\ \Leftrightarrow\ & S'^* \subset V'^* && \text{(since Ker}\Pi \subset V'^*\text{)} \\ \Leftrightarrow\ & \Phi^{-1} S^* \subset \Phi^{-1} V^* && \text{(since (12))} \\ \Leftrightarrow\ & \text{Im}\Phi \cap S^* \subset \text{Im}\Phi \cap V^* && \text{(since } \Phi \text{ is monic)} \\ \Leftrightarrow\ & S^* \subset V^*, && [\text{since } S^* \subset \text{Im}\Phi] .■ \end{aligned}$$

Moreover, it is rather easy to show that the set of solutions is exactly the same for the initial description (A, B, D, C, E), for the reduced realisation (A', B', D', C', E'), and for the minimal realisation (A'', B'', D'', C'', E''):

Theorem 2: A dynamic controller (N, M, L, K), as described in (2), is a solution of the DRMF problem for (A, B, D, C, E), as described in (1), if and only if it is a solution of the DRMF problem for the reduced realisation (A', B', D', C', E'), if and only if it is a solution of the DRMF problem for the minimal realisation (A'', B'', D'', C'', E''), both described in (9)-(11).

Proof: Thanks to (9)-(11) it is quite direct to check that:

$$\begin{aligned} E_e D_e &= ED \\ &= E\Phi D' = E'D' \\ &= E''\Pi D' = E''D'' \end{aligned}$$

$$
\begin{aligned}
E_e A_e D_e &= E(A+BKC)D \\
&= E(A+BKC)\Phi D' \\
&= E(\Phi A'+\Phi B' KC')D' \\
&= E'(A'+B'KC')D' \\
&= E''\Pi(A'+B'KC')D' \\
&= E''(A''\Pi + B''KC''\Pi)D' \\
&= E''(A''+B''KC'')D''
\end{aligned}
$$

Similarly, it can be shown that:

$$
\begin{aligned}
E_e A_e^2 D_e &= E(A+BKC)^2 D + EBLMCD \\
&= E'(A'+B'KC')^2 D' + E'B'LMC'D' \\
&= E''(A''+B''KC'')^2 D'' + E''B''LMC''D''
\end{aligned}
$$

and so on …

Each Markov parameter of the compensated system (4) thus has exactly the same analytic expression, for the systems (*A*, *B*, *D*, *C*, *E*), (*A'*, *B'*, *D'*, *C'*, *E'*), and (*A''*, *B''*, *D''*, *C''*, *E''*), with exactly the same parameters *N*, *M*, *L*, and *K*, which ends the proof (see (5)).■

Using the results of (Del-Muro-Cuellar and Malabre, 2001), one can get as an obvious corollary the following general characterization of the DRMF Fixed Poles[1].

Theorem 3: Assume that the DRMF problem is solvable. The DRMF Fixed Poles for system (1) with parameters (*A*, *B*, *D*, *C*, *E*) are given as the union (with any common elements repeated) of:

- the uncontrollable poles of the pair (*A*, [*B D*])
- the unobservable poles of the pair $\left(A, \begin{bmatrix} E \\ C \end{bmatrix}\right)$
- the fixed poles of the DRMF problem for the minimal realisation (*A''*, *B''*, *D''*, *C''*, *E''*)[2].

Proof (sketch):
The uncontrollable poles of the pair (*A*, [*B D*]) are obviously not controllable by *u*. Similarly, the unobservable poles of the pair $\left(A^T, \begin{bmatrix} E^T & C^T \end{bmatrix}^T\right)$ are obviously not observable through *y*. This means that these are fixed dynamics for the closed-loop system controlled by *any* dynamic measurement feedback compensation like (2). The result then follows from the minimisation process, from Theorems 1 and 2, and from the results in (Del-Muro-Cuellar and Malabre, 2001).■

As another corollary, we can also easily characterise the Fixed Poles of the Disturbance Rejection problem by State Feedback (DRSF). Indeed, this just amounts to assuming full measurement, *i.e.* *C = Id (Identity)*. Remembering that for the particular case of state feedback, static and dynamic solutions are known to be equivalent (see Emre and Hautus, 1980), this gives an alternative description to the DRSF Fixed Poles characterised in (Malabre *et al*. 1997).

Theorem 4: Assume that the DRSF problem is solvable. The DRSF Fixed Poles for system (1) with parameters (*A*, *B*, *D*, *C=Id*, *E*) are given as the union (with any common elements repeated) of:

- the uncontrollable poles of the pair (*A*, [*B D*])
- the fixed poles of the DRSF problem for the minimal realisation (*A'*, *B'*, *D'*, *C'=Id*, *E'*).

Proof: This is exactly Theorem 3 with *C=Id*.■

From these Theorems directly follows that, when considering solvability conditions as well as pole assignment abilities, one is indeed fully allowed to first minimise the state space realisation and then work with the minimal version of the problem. It is important, however, to note that the minimisation process has to be performed with respect to all the input and output signals, *i.e.* with both control and disturbance inputs, as well as both controlled and measurement outputs.

4. EXTENSIONS

Similar results can be obtained for other control problems, such as (diagonal or block) decoupling or non interaction, model matching, and combined versions of those problems (*e.g.* simultaneous disturbance rejection and decoupling).

We shall just sketch some of these in the following.

Consider systems like (1) with *D*=0 (no disturbance) and with *C=Id* (state feedback case). The block-decoupling problem (by regular static state feedback) solved by (Wonham and Morse 1970) relies on the geometric condition:

$$\operatorname{Im} B = \sum_{i=1}^{k} \operatorname{Im} B \cap V_i^* \qquad (13)$$

where *k* is the number of blocks, and V_i^* denotes the supremal (*A,B*) or controlled invariant subspace contained in $\bigcap_{j \neq i} \operatorname{Ker} E_j$, where E_j denotes the j^{th} block of *E*.

It can easily be shown that relations similar to (12) hold between the controlled invariant subspaces of (*A,B,E*) and those of the reduced system (*A',B',E'*) obtained by just getting rid of the uncontrollable modes of (*A,B*). Indeed, $V'_i{}^* = T^{-1} V_i^*$, with *T* the insertion of Im*B* + *A*Im*B* +…+ $A^{\dim(X)-1}$Im*B* into *X*.

[1] As recalled earlier, assuming that the DRMF problem is solvable, the DRMF Fixed Poles are the dynamics which are present in any solution. Some come from uncontrollable and/or unobservable modes, the others are due to the specific problem to be solved (here DRMF).

[2] As characterised in (Del-Muro-Cuellar and Malabre, 2001) under the minimality assumptions: *R=X* and *N=0*.

Thanks to that, it can easily be shown that the Wonham and Morse condition (13) holds for the system (*A,B,E*) if and only if it holds for the reduced (controllable) system (*A*',*B*',*E*'). This "justifies" the usual assumption that "*for state feedback decoupling purposes, there is no loss of generality assuming that the pair (A,B) is controllable*".

From this, directly follows that the fixed poles of the block-decoupling problem, for any possibly non minimal state realisation (*A,B,E*), are the union of the uncontrollable poles of (*A,B*) with the set of the fixed poles of the block-decoupling problem computed on the minimal realisation (*A*',*B*',*E*'), as characterised for instance in (Koussiouris 1983).

A similar treatment can be done from the geometric conditions given by (Grizzle and Isidori 1989) for the same problem (but slightly differently formulated), namely, the block non interaction problem by static state feedback. This is solvable if and only if:

$$\operatorname{Im} B \cap L_i^* + \sum_{j \neq i, j=1}^{k} \operatorname{Im} B \cap L_j^* = \operatorname{Im} B \qquad (14)$$

where L_i^* denotes the supremal (*A,B*) or controlled invariant subspace contained in KerE_j (with E_j denoting the i^{th} block of *E*).

Similarly also, the geometric solvability conditions, as well as the characterisation of the Fixed Poles can be obtained for the simultaneous disturbance rejection and block decoupling (or non interaction) problem by static state feedback, following the lines of (Camart *et al*. 2001) (established under the assumption that $R = X$). This can be achieved by making use of such connections between the invariant subspaces of the initial system (1) and those of the minimal realisation described in (9)-(11) and Fig. 1.

5. CONCLUDING REMARKS

We have proposed here a simple frame, based on a geometric treatment, which fully justifies the very natural, but up to now "intuitive[3]" feeling, that, for several classical control problems by state or measurement feedback, we can indeed treat separately the non minimality of the description and the solvability conditions. This justifies the assertion that there is indeed no loss of generality assuming minimality. Note however, that in the presence of disturbances, minimality (and in particular controllability), has to be understood with respect to *all* the external signals (*i.e.* with both control and disturbance inputs).

This is not detailed here, but easy dualisations of the present results can also be obtained, leading to similar results for some dual problems such as state estimation in the presence of disturbances, failure detection and isolation (see *e.g.* (Massoumnia *et al*. 1989) …

[3] Indeed, up to the knowledge of the author, despite the fact that the results appear *a posteriori* as "obvious", such a rigorous proof was lacking.

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