ROBUSTNESS PROPERTIES OF A CLASS OF OPTIMAL RISK-SENSITIVE CONTROLLERS FOR QUANTUM SYSTEMS [1]

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Abstract: In this note we described the robustness properties of optimal risk-sensitive controllers for quantum systems. We consider a quantum generalization of risk-sensitive criteria using the framework of (James, 2004). The robustness properties are derived by evaluating certain Radon-Nikodym derivatives of the quantum models and of the cost criteria. In addition to induced perturbations in the quantum statistics, perturbations in the cost function are allowed—evidently a non-classical feature.

Keywords: Quantum control, robust control, risk-sensitive stochastic control.

1. INTRODUCTION

In recent years, a number of papers have appeared concerning the feedback control of quantum systems; e.g., see (Wiseman and Milburn, 1994; Doherty *et al.*, 2000; Doherty and Jacobs, 1999). As is well known in the control theory literature, robustness is a critical issue in the design of feedback control systems. In particular, a feedback controller is commonly designed on the basis of a nominal model which only approximates the true behaviour of the system being controlled. If the robustness issue is not taken into account when designing a feedback controller, this may result in a significant degradation of the performance of the feedback control system or even instability. The importance of robustness in feedback control system design applies equally in the control of quantum systems as in the control of classical systems. Furthermore, when considering the feedback control of quantum systems, stochastic models naturally arise and thus it is useful to consider problems of robust stochastic optimal control for quantum systems. These considerations motivated the results of (James, 2004) which are concerned with the risk sensitive optimal control of quantum systems.

The use of a risk sensitive cost criterion in designing optimal feedback controllers is known to lead to useful robustness properties for the resulting controller; e.g., see (Boel *et al.*, 2002; Dupuis *et al.*, 2000). Indeed, in the paper (Boel *et al.*, 2002), it is shown that the use of a risk sensitive optimal controller enables an upper bound to be obtained for a certain (risk neutral) cost function for a class of true system dynamics which differ from the nominal system model which is used to design the risk sensitive controller. The main

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result to be presented in this paper shows that for a class of risk sensitive optimal controllers for quantum systems (designed using the approach of (James, 2004)) a bound on a corresponding risk neutral cost function can also be obtained. This result represents a quantum version of the result of (Boel *et al.*, 2002; Dupuis *et al.*, 2000). This result provides motivation for the risk sensitive optimal control procedure as providing a suitable methodology for designing robust feedback controllers for quantum systems. The main result of the paper is illustrated with an example involving a two level quantum system. Complete details will be given in the full version of this paper.

2. THE CONTROLLED QUANTUM SYSTEM

As in (James, 2004), we consider a controlled quantum system whose dynamics are described in discrete-time by the recursion

$$\omega_{k+1} = \Lambda_\Gamma(u_k, y_{k+1})\omega_k, \tag{1}$$

where

$$\Lambda_\Gamma(u, y)\omega = \frac{\Gamma(u, y)\omega}{p_\Gamma(y|u, \omega)}. \tag{2}$$

Here, $\Gamma(u, y)$ is a quantum operation that is used to model (via (1)) the state transfer if a control value u is applied and a measurement value y is observed. The probability of a measurement outcome y (assumed discrete-valued) is given by

$$p_\Gamma(y|u, \omega) = \langle \Gamma(u, y)\omega, I\rangle, \tag{3}$$

where I is the identity operator. The operator $\Gamma(u, y)$ is assumed to be normalized.

Thus if the quantum system is in state ω_k at time k, and at this time the control value u_k is applied, a measurement outcome y_{k+1} will be recorded, and the system will transfer to a new state ω_{k+1}. The probability of y_{k+1} is $p_\Gamma(y_{k+1}|u_k, \omega_k)$. Equation (1) is a discrete time *stochastic master equation (SME)*; e.g., see (Nielsen and Chuang, 2000; James, 2004).

On a time interval $0 \leq k \leq M-1$ a *feedback controller* is specified by a control law $u = K(y)$, where $K = \{K_0, K_1, \ldots, K_{M-1}\}$. To simplify notation, we often write sequences $u_{k_1}, u_{k_1+1}, \ldots, u_{k_2}$ as u_{k_1,k_2}. Then we can write $u_k = K_k(y_{1,k})$. We denote by $\mathcal{K}$ the class of all such feedback controllers.

The controller K therefore determines controlled stochastic processes ω_k, u_k and y_k on the interval $0 \leq k \leq M$. The resulting probability distribution $\mathbf{P}^K_{\omega_0,0}$ of the measurement process $y_{1,M} = y_1, \ldots, y_M$ is given by

$$\begin{aligned}\mathbf{P}^K_{\omega_0,0}(y_1, \ldots, y_M) &= \prod_{k=0}^{M-1} p_\Gamma(y_{k+1}|u_k, \omega_k) \\ &= \langle \prod_{k=0}^{M-1,\leftarrow} \Gamma(u_k, y_{k+1})\omega_0, I\rangle \end{aligned}\tag{4}$$

where p_Γ is defined in (3),

$$u_k = K_k(y_1, \ldots, y_k) \tag{5}$$

and ω_k is defined by the recursion (1). In the second line of (4), the product denotes a time-ordered composition with the most recent operator applied first.

3. RISK-SENSITIVE OPTIMAL CONTROL

We consider a risk-sensitive criterion which generalizes the well-known classical LEQG criterion. For each control value u, let $L(u)$ be a non-negative observable. Let N be a non-negative observable. These *cost observables* are used to define the following performance criterion (James, 2004):

$$J^\mu_{\omega,0}(K) \triangleq \mathbf{E}^K_{\omega,0}[\prod_{k=0}^{M-1} \langle \omega_k, e^{\mu L(u_k)}\rangle \langle \omega_M, e^{\mu N}\rangle] \tag{6}$$

Here, $\mu > 0$ is a positive risk parameter. In this expression, the conditional states ω_k are given by the quantum system model (1) and the expectation is evaluated with respect to the probability distribution $\mathbf{P}^K_{\omega_0,0}$ determined by a feedback controller K. Each term in the expression corresponds to a quantum average of an exponential cost, and when multiplied together provide a generalization of the LEQG criterion.

We wish to find a controller that minimizes this criterion. This was done in (James, 2004) for a class of multiplicative criteria that includes (6). We now explain how the risk-sensitive criterion (6) can be cast in this general form, and then describe the solution.

Define the operator

$$R(u)\hat{\omega} \triangleq \frac{\langle \hat{\omega}, e^{\mu L(u)}\rangle}{\langle \hat{\omega}, 1\rangle}\hat{\omega}. \tag{7}$$

This operator maps possibly unnormalized states to possibly unnormalized states, and is in general nonlinear but satisfies the real multiplicative homogeneity property $R(u)r\hat{\omega} = rR(u)\hat{\omega}$ for any real number r and any $\hat{\omega}$, u.

Define $\Gamma_R(u, y) = \Gamma(u, y)R(u)$ and

$$\Lambda_{\Gamma,R}(u, y)\hat{\omega} = \frac{\Gamma_R(u, y)\hat{\omega}}{p_R(y|u, \hat{\omega})} \tag{8}$$

where

$$p_R(y|u,\hat{\omega}) = \frac{\langle \Gamma_R(u,y)\hat{\omega}, I\rangle}{\langle R(u)\hat{\omega}, I\rangle}. \tag{9}$$

Associated with the operator $\Lambda_{\Gamma,R}$ are the dynamics

$$\hat{\omega}_{k+1} = \Lambda_{\Gamma,R}(u_k, y_{k+1})\hat{\omega}_k, \tag{10}$$

where y_{k+1} is distributed according to the probability distribution $p_R(y_{k+1}|u_k,\hat{\omega}_k)$ given by (9). This is a controlled Markov chain, with *unnormalized* states $\hat{\omega}_k$. It is a modified stochastic master equation corresponding to the operator Γ_R. Under the action of a controller $K \in \mathcal{K}$ the stochastic process $\hat{\omega}_k$ is determined by (10) and $u_k = K_k(y_{1,k})$.

Let M be a positive integer indicating a finite time interval $k = 0, \ldots, M$. For each k, given a sequence of control values $u_{k,M-1} = u_k, \ldots, u_{m-1}$ and measurement values $y_{k+1,M} = y_{k+1}, \ldots, y_M$, define random cost observables G_k by the recursion

$$\begin{aligned} G_k &= R^\dagger(u_k)\Gamma^\dagger(u_k, y_{k+1})G_{k+1}, \\ G_M &= F \end{aligned} \tag{11}$$

where $0 \le k \le M-1$ and F is a non-negative linear observable. Here, $R^\dagger$ denotes that adjoint of R, etc.

We next define the general risk-sensitive cost functional

$$J^\mu_{\hat{\omega},0}(K) = \sum_{y_{1,M}\in \mathbf{Y}^M} \langle \hat{\omega}, G_0\rangle. \tag{12}$$

In (James, 2004, Example 7) it is shown that the risk-sensitive criterion (6) can be expressed in the form (12), and by (James, 2004, Lemma 1) we have

$$J^\mu_{\hat{\omega},0}(K) = \mathbf{E}^K_{\hat{\omega},0}[\langle \hat{\omega}_M, F\rangle] \tag{13}$$

where $\hat{\omega}_i$, $i = k, \ldots, M$ is the solution of the recursion (10) with initial state $\hat{\omega}_0 = \hat{\omega}$ under the action of the controller K.

The optimal control problem is solved using dynamic programming in terms of the cost to go:

$$J^\mu_{\hat{\omega},k}(K) \triangleq \sum_{y_{k+1,M}\in \mathbf{Y}^{M-k}} \langle \hat{\omega}, G_k\rangle \tag{14}$$

The dynamic programming equation is

$$\begin{aligned} W(\hat{\omega},k) &= \inf_{u\in\mathbf{U}} \left\{ \begin{array}{l} \sum_{y\in\mathbf{Y}} W(\Lambda_{\Gamma,R}(u,y)\hat{\omega}, k+1). \\ \qquad\qquad .p_R(y|u,\hat{\omega}) \end{array} \right\} \\ W(\hat{\omega},M) &= \langle \hat{\omega}, F\rangle \end{aligned} \tag{15}$$

where $0 \le k \le M-1$.

Theorem 3.1. (James, 2004, Theorem 1) Let $W(\hat{\omega},k)$, $0 \le k \le M$, be the solution of the dynamic programming equation (15).

(i) Then for any $K \in \mathcal{K}$ we have

$$W(\hat{\omega},k) \le J^\mu_{\hat{\omega},k}(K) \tag{16}$$

(ii) Assume in addition that the minimizer

$$\hat{\mathbf{u}}^*(\hat{\omega},k) \in \underset{u\in\mathbf{U}}{\operatorname{argmin}} \left\{ \begin{array}{l} \sum_{y\in\mathbf{Y}} W(\Lambda_{\Gamma,R}(u,y)\hat{\omega}, k+1) \\ \qquad\qquad .p_R(y|u,\hat{\omega}) \end{array} \right\} \tag{17}$$

exists for all $\hat{\omega}$, $0 \le k \le M-1$. Then the separation structure controller $K^{\hat{\mathbf{u}}^*}_{\hat{\omega}_0}$ defined by (17) is optimal for problem (12); i.e. $J^\mu_{\hat{\omega}_0,0}(K) \ge J^\mu_{\hat{\omega}_0,0}(K^{\hat{\mathbf{u}}^*}_{\hat{\omega}_0})$ for all $K \in \mathcal{K}$.

4. ROBUSTNESS PROPERTIES

We seek a bound on the (risk-neutral) performance of the control system

$$J_{\omega,0}(K) = \mathbf{E}^K_{\omega,0}[\sum_{k=0}^{M-1} \langle \omega_k, L(u_k)\rangle + \langle \omega_M, N\rangle] \tag{18}$$

where the state $\omega_k = \omega_k^{true}$ evolves according to the true model (see (22) below) and the control is determined by the nominal model

$$u_k = \hat{\mathbf{u}}^*_{nom}(\hat{\omega}_k, k) \tag{19}$$

via (17) with $\Gamma = \Gamma_{nom}$ and

$$\hat{\omega}_{k+1} = \Lambda_{\Gamma_{nom},R}(u_k, y_{k+1})\hat{\omega}_k. \tag{20}$$

This controller is denoted $K^*_{nom} = K^{\hat{\mathbf{u}}^*_{nom}}_{\hat{\omega}_0}$.

A bound for (18) is sought in terms of a measure of the "distance" between $\Gamma_{nom}(u)$ and $\Gamma_{true}(u)$. The way in which we measure the "distance" between $\Gamma_{nom}(u)$ and $\Gamma_{true}(u)$ is to consider the "distance" between the probability distributions $\mathbf{P}_{nom}$ and $\mathbf{P}_{true}$ defined on the space of observation paths and determined by $\Gamma_{nom}(u)$ and $\Gamma_{true}(u)$ respectively, under the controller K^*_{nom}. This distance is defined in terms of the *relative entropy* (e.g., see (Nielsen and Chuang, 2000, Chapter 11)):

$$\Re(\mathbf{P}_{true} \parallel \mathbf{P}_{nom}) \triangleq \mathbf{E}_{\mathbf{P}_{true}}[\log \frac{d\mathbf{P}_{true}}{d\mathbf{P}_{nom}}] \tag{21}$$

provided $\mathbf{P}_{true}$ is absolutely continuous with respect to $\mathbf{P}_{nom}$.

From (4), the distributions $\mathbf{P}_{true}$ and $\mathbf{P}_{nom}$ are given explicitly by

$$\mathbf{P}_{true}(y_1,\ldots,y_M)=\prod_{k=0}^{M-1} p_{\Gamma_{true}}(y_{k+1}|u_k,\omega_k^{true})$$

and

$$\mathbf{P}_{nom}(y_1,\ldots,y_M)=\prod_{k=0}^{M-1} p_{\Gamma_{nom}}(y_{k+1}|u_k,\omega_k^{nom})$$

where $p_\Gamma(\cdot)$ is defined by (3), u_k is determined by K^*_{nom}, and

$$\omega_{k+1}^{true}=\Lambda_{\Gamma_{true}}(u_k,y_{k+1})\omega_k^{true} \quad (22)$$

and

$$\omega_{k+1}^{nom}=\Lambda_{\Gamma_{nom}}(u_k,y_{k+1})\omega_k^{nom} \quad (23)$$

respectively under these distributions. The next lemma computes the Radon-Nikodym derivative of these distributions.

Lemma 4.1. Suppose $\mathbf{P}_{true}$ is absolutely continuous with respect to $\mathbf{P}_{nom}$. Then the Radon-Nikodym derivative $\frac{d\mathbf{P}_{true}}{d\mathbf{P}_{nom}}$ can be written in the form

$$\frac{d\mathbf{P}_{true}}{d\mathbf{P}_{nom}}(y_1,\ldots,y_M)=\prod_{k=0}^{M-1} f_{k+1}(y_{k+1}|y_1,\ldots,y_k)$$

where

$$f_{k+1}(y_{k+1}|y_1,\ldots,y_k)=\frac{p_{\Gamma_{true}}(y_{k+1}|u_k,\omega_k^{true})}{p_{\Gamma_{nom}}(y_{k+1}|u_k,\omega_k^{nom})}\geq 0$$

and

$$\sum_{y_{k+1}} f_{k+1}(y_{k+1}|y_1,\ldots,y_k)p_{\Gamma_{nom}}(y_{k+1}|u_k,\omega_k^{nom})=1.$$

□

We consider $\Gamma_{nom}(u,y)$ in Kraus (operator sum) form (Nielsen and Chuang, 2000, Chapter 8),

$$\Gamma_{nom}(u,y)\omega=\sum_{a\in\mathbf{A}}\gamma_{nom,a}(u,y)\omega\gamma^\dagger_{nom,a}(u,y) \quad (24)$$

for suitable operators $\gamma_{nom,a}(u,y), a\in\mathbf{A}$ satisfying

$$\sum_{a\in\mathbf{A},y\in\mathbf{Y}}\gamma^\dagger_{nom,a}(u,y)\gamma_{nom,a}(u,y)=I. \quad (25)$$

Here $\mathbf{A}$ is a finite index set. The true model is assumed to be given by

$$\Gamma_{true}(u,y)\omega =\sum_{a\in\mathbf{A}}\lambda_a(u,y)\gamma_{nom,a}(u,y)\omega\gamma^\dagger_{nom,a}(u,y) \quad (26)$$

where $\lambda_a(u,y)$ are real numbers satisfying $0\leq\lambda_a(u,y)\leq d(u,y)$ for all a. Since $\Gamma_{true}(u,y)$ is required to be a normalized quantum operation, we also require

$$\sum_{a\in\mathbf{A},y\in\mathbf{Y}}\lambda_a(u,y)\gamma^\dagger_{nom,a}(u,y)\gamma_{nom,a}(u,y)=I \quad (27)$$

Thus we can think of the true model as a perturbation of the nominal model in the sense that the operator $\gamma_{nom,a}(u,y)$ is multiplicatively perturbed to $r_a(u,y)\gamma_{nom,a}(u,y)$, where $r_a(u,y)$ are complex numbers such that $|r_a(u,y)|^2=\lambda_a(u,y)$. In the terminology of (Belavkin and Staszewski, 1986), (Raginsky, 2003), we say that $\Gamma_{true}(u,y)$ is *completely dominated* by $d(u,y)\Gamma_{nom}(u,y)$.

Lemma 4.2. Let $\Gamma_{true}(u,y)$ be completely dominated by $d(u,y)\Gamma_{nom}(u,y)$ as described above. Then $\mathbf{P}_{true}$ is absolutely continuous with respect to $\mathbf{P}_{nom}$. □

We will need the following general convex duality formula (e.g., see (Boel *et al.*, 2002)):

$$\log\mathbf{E}_{\mathbf{P}}[e^f]=\sup_{\mathbf{Q}}\{\mathbf{E}_{\mathbf{Q}}[f]-\mathfrak{R}(\mathbf{Q}\parallel\mathbf{P})\} \quad (28)$$

where $\mathbf{P}$ and $\mathbf{Q}$ are probability distributions.

Let X be a non-negative observable (e.g. $L(u)$ or N), and let $P_X(dx)$ denote the projection-valued measure corresponding to the observable X.

Lemma 4.3. We have

$$\log\langle\omega,e^{\mu X}\rangle=\sup_{g(\cdot|X,\omega)}\{\mu\langle\omega,\tilde{X}\rangle-\mathfrak{C}(\tilde{X}\parallel X)\} \quad (29)$$

where $g(\cdot|X,\omega)\geq 0$,

$$\int g(x|X,\omega)\langle\omega,P_X(dx)\rangle=1, \quad (30)$$

$$\tilde{X}=Xg(X|u,\omega),$$

and

$$\mathfrak{C}(X\parallel X)=\langle\omega,g(X|X,\omega)\log g(X|X,\omega)\rangle. \quad (31)$$

□

Theorem 4.4. Suppose $\Gamma_{true}(u,y)$ is completely dominated by $d(u,y)\Gamma_{nom}(u,y)$ as described above. Consider the controller K^*_{nom} determined by the nominal model for the risk-sensitive criteria (6). Then

$$\begin{aligned}&\mathbf{E}^{true}_{\omega,0}[\sum_{k=0}^{M-1}\langle\omega_k,\tilde{L}_k(u_k)\rangle+\langle\omega_M,\tilde{N}_M\rangle]\\&\leq\frac{1}{\mu}\log J^\mu_{\omega,0}(K^{\hat{\mathbf{u}}^*,\mu}_{\hat{\omega}_0})\\&\quad+\frac{1}{\mu}\mathfrak{R}(\mathbf{P}_{true}\parallel\mathbf{P}_{nom})+\frac{1}{\mu}\mathfrak{C}(\tilde{L}\parallel L) \quad (32)\end{aligned}$$

where $\tilde{L}_k(u_k) = L(u_k) g_k(L(u_k)|L(u_k), \omega_k)$, $\tilde{N}_M = N g_M(N|N, \omega_M)$, g_k is as in Lemma 4.3, and

$$\mathfrak{C}(\tilde{L} \parallel L) \tag{33}$$
$$= \mathbf{E}^{true}_{\omega,0}[\sum_{k=0}^{M-1} \mathfrak{C}_{\omega_k}(\tilde{L}(u_k) \parallel L(u_k)) + \mathfrak{C}_{\omega_M}(\tilde{N} \parallel N)].$$

□

Note that the result of this Theorem provides for uncertainty in the induced measurement probability distribution and in the cost that is specified.

5. EXAMPLE

We consider an example from (James, 2004) of a two-level system on $\mathbf{H} = \mathbf{C}^2$. For this system, we consider a risk sensitive controller using criteria (6). Let $|-1\rangle$ and $|1\rangle$ denote the orthonormal basis of eigenvectors of the spin observable

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{34}$$

corresponding to ideal measurement values $a = -1$ and $a = 1$ (e.g. spin down and spin up respectively). The actual measurement values $y \in \{-1, 1\}$ recorded are imperfect, being subject to an error probability $0 < \alpha < 1$. The control actions available are to either do nothing, or to flip the state. Given an initial mixed state

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12}^* & \omega_{22} \end{pmatrix}, \tag{35}$$

it is desired to put the system into the up state

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ or } |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

through a series of measurements and feedback control actions (say over a time horizon $M = 2$).

The controlled dynamics are determined by a controlled quantum operation, with nominal value

$$\Gamma_{nom}(u, y)\omega \tag{36}$$
$$= q(y|-1) P_{-1} T^u \omega T^{u\,\dagger} P_{-1} + q(y|1) P_1 T^u \omega T^{u\,\dagger} P_1,$$

where

$$T^u = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } u = 0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } u = 1, \end{cases}$$

$$P_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned} q(-1|-1) &= q(1|1) &&= 1 - \alpha \\ q(-1|1) &= q(1|-1) &&= \alpha. \end{aligned}$$

The cost function is specified by

$$L(u) = C^2 + c(u) I$$

and

$$N = C^2$$

where $C = \frac{1}{2}(A - I)$ and $c(0) = 0$, $c(1) = p$, with $p \geq 0$. Note that the expected value of C^2 is

$$\begin{aligned} \langle 1|C^2|1\rangle &= \mathrm{tr}[C^2|1\rangle\langle 1|] &&= 0 \\ \langle -1|C^2|-1\rangle &= \mathrm{tr}[C^2|-1\rangle\langle -1|] &&= 1 \end{aligned}$$

giving zero cost in the desired state and non-zero cost otherwise.

From (37), the Kraus operators are

$$\gamma_{nom,a}(u, y) = \sqrt{q(y|a)} P_a T^u$$

where $a \in \{-1, 1\}$. We now suppose that the error probability α is not known exactly. Let $0 < \tilde{\alpha} < 1$ be the true value of this parameter and set

$$\begin{aligned} \tilde{q}(-1|-1) &= \tilde{q}(1|1) &&= 1 - \tilde{\alpha} \\ \tilde{q}(-1|1) &= \tilde{q}(1|-1) &&= \tilde{\alpha}. \end{aligned}$$

Then the probability distribution $\tilde{q}(y|a)$ is absolutely continuous with respect to $q(y|a)$, with Radon-Nikodym derivative

$$\lambda_a(u, y) = \frac{\tilde{q}(y|a)}{q(y|a)}.$$

Hence robustness can be considered relative to the class of "true" models with error parameter $\tilde{\alpha}$ described by

$$\Gamma_{true}(u, y)\omega \tag{37}$$
$$= \tilde{q}(y|-1) P_{-1} T^u \omega T^{u\,\dagger} P_{-1} + \tilde{q}(y|1) P_1 T^u \omega T^{u\,\dagger} P_1.$$

Note that by construction $\Gamma_{true}(u, y)$ is normalized.

This means that if the true error probability differs from the nominal value, the bound (32) leads to a corresponding bound on the risk-neutral cost.

The cost observables $L(u)$ and N are both diagonal with respect to the basis $|-1\rangle$, $|1\rangle$. Thus we have the spectral formulas

$$L(u) = (1 + c(u)) P_{-1} + c(u) P_1, \quad N = 1 P_{-1} + 0 P_1.$$

Consider the terminal cost observable N. Then $\langle \omega, N\rangle = \omega_{11}$, and $\langle \omega, \tilde{N})\rangle = g_2(1|N, \omega)\omega_{11}$ where $\tilde{N} = N g_2(N|N, \omega)$; cf. Theorem 4.4. If we let

$$g_2(1|N, \omega) = \tilde{n}_2, \quad g_2(0|N, \omega) = \frac{1 - \tilde{n}_2 \omega_{11}}{\omega_{22}},$$

for some $\tilde{n}_2 \geq 0$, then

$$g_2(1|N, \omega)\langle \omega, P_{-1}\rangle + g_2(0|N, \omega)\langle \omega, P_1\rangle = 1$$

and so (30) is satisfied. Note that $\tilde{n}_2$ can be chosen arbitrarily subject to the constraint $0 \leq \tilde{n}_2 \leq \frac{1}{\omega_{11}}$. Also $\langle \omega, \tilde{N}\rangle = \tilde{n}_2 \omega_{11}$.

Now consider the running cost observable $L(u)$. Since $L(0) = N$, $\langle \omega, L(0) \rangle = \omega_{11}$, and we set

$$g_1(1|L(0), \omega) = \tilde{n}_1, \quad g_1(0|L(0), \omega) = \frac{1 - \tilde{n}_1 \omega_{11}}{\omega_{22}},$$

for some $0 \leq \tilde{n}_1 \leq \frac{1}{\omega_{11}}$.

Now

$$\langle \omega, L(1) \rangle = (1 + p)\omega_{11} + p\omega_{22}.$$

Let $\tilde{p} \geq 0$, and define

$$g_1(1 + p\,|L(1), \omega) = \frac{1 + \tilde{p}}{1 + p},$$

$$g_1(p\,|L(1), \omega) = \frac{1 - \frac{1+\tilde{p}}{1+p}\omega_{11}}{\omega_{22}}$$

Then (30) is satisfied, and

$$\langle \omega, \tilde{L}(1) \rangle = (1 + \tilde{p})\omega_{11} + p(1 - \frac{1 + \tilde{p}}{1 + p}\omega_{11}).$$

where

$$\tilde{L}(1) = L(1) g_1(L(1)|L(1), \omega).$$

From the expressions (31) and (34), the quantity $\mathfrak{C}(\tilde{L}||L)$ can be calculated in terms of $\tilde{n}_1$ and $\tilde{n}_2$ which leads to a corresponding bound on the risk neutral cost. Hence we have performance bounds for a class of cost functions and a class of uncertainty.

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