MINIMAX LQG CONTROL FOR UNCERTAIN SYSTEMS WITH A NORMALISED COPRIME FACTOR UNCERTAINTY STRUCTURE ⋆

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Abstract: The paper extends the recently developed robust LQG control design methodology to stochastic uncertain systems with a normalised coprime factor uncertainty structure. To consider this class of model uncertainties, a special technique of probability measure transformations is developed. As a by-product, other types of the model uncertainty, for example, those reflecting passivity of the uncertainty, are captured.

Keywords: Stochastic systems, minimax stochastic control, robust control, normalised coprime factor uncertainty, risk-sensitive control

1. INTRODUCTION

The minimax stochastic optimization framework recently proposed in (Petersen *et al.* 2000, Ugrinovskii and Petersen 1999, Ugrinovskii and Petersen 2001) has led to the development of a new robust control design methodology termed minimax LQG design. This methodology enables the design of controllers which combine the performance properties of LQG controllers and robustness characteristics of H_∞ controllers.

The foundations of minimax LQG control theory can be found in the theory of large deviations (Dupuis and Ellis 1997) and risk-sensitive control (Whittle 1990). The minimax LQG approach to the robust control design makes use of a stochastic minimax game-type formulation of the robust control problem in which the uncertainty is modeled in terms of probability distributions rather than time-varying disturbance signals. This leads to an uncertain system model in which system dynamics are described by a stochastic differential equation in an uncertainty probability space, and the probability laws of admissible uncertainties are restricted to belong to a specified set of probability distributions. The derivation of a suitable mathematical description of the admissible set of probability distributions constitutes a major step in the application of the minimax LQG control design method. As with all robust control techniques exploiting the worst-case design paradigm, a control system performance which can be achieved through utilizing a minimax optimal LQG controller largely depends on how well the chosen mathematical description of the set of admissible uncertain perturbations represents the uncertainty in the physical system under consideration.

The mathematical description of the class of admissible stochastic uncertain systems introduced in (Petersen *et al.* 2000, Ugrinovskii and Petersen 1999, Ugrinovskii and Petersen 2001) uses the notion of relative entropy to characterize the size of the uncertain disturbances in the system. Specifically, the magnitude of the disturbances is measured using the relative en-

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tropy between admissible probability distributions and the reference probability distribution of a white noise disturbance, both defined on samples of a Brownian motion process. This reflects a common situation where the uncertain perturbation represents a superposition of the disturbance signal and a white noise signal. This uncertainty structure allows one to conveniently express bounds on the magnitude of admissible uncertain perturbations in the form of a constraint on the corresponding relative entropy (Petersen *et al.* 2000, Ugrinovskii and Petersen 1999). The advantage of the relative entropy uncertainty description is that it allows one to convert the underlying robust control design problem into a partially observed risk sensitive stochastic control problem; the latter problem is known to admit a tractable solution (Bensoussan and Schuppen 1985, Pan and Başar 1996). The equivalence between risk-sensitive control problems and stochastic minimax games (James *et al.* 1994, Dai Pra *et al.* 1996) makes this conversion possible.

The existing results on minimax LQG control focus on model uncertainty presented in a Linear Fractional Transformation (LFT) form. The models falling into this class include for instance, H_∞-norm bounded unmodeled dynamics. At the same time, the relative entropy constraint uncertainty description has been less successful in accounting for some other types of uncertain dynamics such as those in which the uncertainty has a normalised coprime factor structure; e.g., see (McFarlane and Glover 1990).

The objective of this paper is to develop an extension of the existing minimax LQG optimal control approach whereby uncertainty with a normalised coprime factor uncertainty structure can be accounted for. To this end, we develop a special technique of probability measure transformations which allows us to consider this class of model uncertainties. As a by-product, some other types of the model uncertainty, for example, those reflecting passivity of the uncertainty, are captured. Also, the proposed technique naturally allows for the presence of uncertainty feedforward in the cost functional being considered.

2. MINIMAX LQG CONTROL PROBLEM FOR SYSTEMS WITH NORMALISED COPRIME FACTOR UNCERTAINTY STRUCTURE

Consider a plant whose transfer function matrix from the control input u to the controlled output z_2 is expressed in the normalised coprime factor form,

$$z_2 = (M - \Delta_M)^{-1}(N + \Delta_N)u.$$

Here $G(s) = M(s)^{-1}N(s)$ is a normalised coprime factor representation of the nominal plant transfer function matrix $G(s)$ (McFarlane and Glover 1990).

The block diagram representing this uncertainty structure is shown in Figure 1. In this figure, $\Delta_N(s)$,

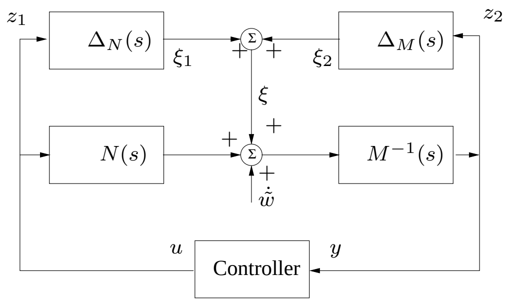

Fig. 1. An uncertain system with normalised coprime factor uncertainty structure.

$\Delta_M(s)$ denote the normalised coprime factor uncertainties, and $\dot{w}$ is the system noise. Using the notation shown in the figure, a state-space realization of this uncertain system can be written in the following form:

$$\begin{aligned}\dot{x} &= Ax + B_1 u + B_2(\xi + \dot{w}), \qquad (1)\\ z &=: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix} x + \begin{bmatrix} I \\ D \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} (\xi + \dot{w}),\\ \xi &:= \xi_1 + \xi_2 = \begin{bmatrix} \Delta_N & \Delta_M \end{bmatrix} z;\end{aligned}$$

e.g., see (McFarlane and Glover 1990). We now make the following standard assumptions. Assume that the system noise $\dot{w}$ is white noise with zero mean and covariance matrix $W = W' > 0$ and assume the following bounds on the uncertainties:

$$\begin{bmatrix} \Delta_N'(-j\omega) \\ \Delta_M'(-j\omega) \end{bmatrix} W^{-1} \begin{bmatrix} \Delta_N(j\omega) & \Delta_M(j\omega) \end{bmatrix} < \bar{W}^{-1} := \begin{bmatrix} G & H \\ H' & W^{-1} \end{bmatrix} \quad \forall \omega \in (-\infty, \infty). \qquad (2)$$

Let $T > 0$ be a constant which will denote the finite time horizon considered throughout the paper. Using the Parseval identity and taking the expectation, condition (2) can be re-written in the form of an integral quadratic constraint

$$\mathbf{E}\int_0^T \big(\|\bar{z}(t)\|^2_{W^{-1}} + 2(H'z_1(t) + W^{-1}Z(t))'\xi(t)\big)\,dt > -T; \qquad (3)$$

$$\bar{z} = \begin{bmatrix} z_1 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix} x + \begin{bmatrix} I \\ D \end{bmatrix} u. \qquad (4)$$

where $\|a\|_{W^{-1}} := (a'W^{-1}a)^{1/2}$.

The robust control problem of interest is to design an output feedback controller for the uncertain system (1) subject to a constraint of the form (3). A common approach to achieving this objective is to design a controller that delivers an acceptable level of guaranteed H_∞ performance. In the presence of random disturbances, a corresponding operator norm induced by the signal norm $\left(\mathbf{E}\int_0^T \|\cdot\|^2 dt\right)^{1/2}$ may be used as a measure of performance. This leads us to introduce the following minimax optimization problem

$$\inf_{u(\cdot)} \sup_{\xi(\cdot)} \mathbf{E} \int_0^T (\|Z(t)+\xi(t)\|^2_{W^{-1}} - \gamma^2 \|\xi(t)\|^2_{W^{-1}})dt, \quad (5)$$

In (5) the supremum is taken over the set of admissible uncertainties subject to the constraint (3), and the infimum is taken over a class of linear output feedback controllers utilizing measurements of the controlled variable Z. The solution of this problem is in the main focus of this paper. The derivation of the solution will make use of a technique developed for solving a general class of similar stochastic robust control problems (Ugrinovskii and Petersen 1999, Ugrinovskii and Petersen 2001). In the next section, we will present a rigorous formulation of the stochastic minimax optimization problem corresponding to the problem (5). An important distinction between this problem and those considered in (Ugrinovskii and Petersen 1999, Ugrinovskii and Petersen 2001) relates to the structure of the uncertainty set, and also to the presence of the uncertainty feedforward term in the cost function.

3. DEFINITIONS

3.1 Uncertain signals and probability measures

Consider a complete probability space $(\Omega, \mathcal{F}, P)$. In this probability space, define independent Wiener processes $w(\cdot) \in \mathbf{R}^q$, $v(\cdot) \in \mathbf{R}^p$ with covariance matrices $W(t)$, $\Sigma(t)$; $W(t), \Sigma(t) \geq \rho I > 0$ for all $t \in [0,T]$. The sample space Ω is defined as $\Omega = \mathbf{R}^n \times C([0,T], \mathbf{R}^q) \times C([0,T], \mathbf{R}^p)$ and is equipped with the filtration $\{\mathcal{F}_t, \ t \geq 0\}$ generated by the mappings $\{\Pi_t, t \geq 0\}$ where $\Pi_0(x, w(\cdot), v(\cdot)) = x$ and $\Pi_t(x, w(\cdot), v(\cdot)) = (w(t), v(t))$ for $t > 0$. P is a Wiener measure defined on sets in the complete filtration $\{\mathcal{F}_t, t \geq 0\}$. The expectation with respect to P will be denoted $\mathbf{E}$.

In the probability space $(\Omega, \mathcal{F}, P)$, consider system dynamics driven by the Wiener processes $w(\cdot)$, $v(\cdot)$,

$$\begin{aligned} dx(t) &= (A(t)x(t) + B_1 u(t))dt + B_2(t)dw(t), \quad (6)\\ dy(t) &= Z(t)dt + dw(t) + \beta dv(t),\\ Z(t) &= C_2(t)x(t) + D_2(t)u(t),\\ z(t) &= C_1(t)x(t) + D_1(t)u(t). \end{aligned}$$

In (6), $x(t) \in \mathbf{R}^n$ is the state, $z(t) \in \mathbf{R}^p$ is the uncertainty output, $Z(t) \in \mathbf{R}^q$ is the controlled output, and $y(t) \in \mathbf{R}^q$ is the measured output. The initial condition $x(0) = x_0 \colon \Omega \to \mathbf{R}^n$ is a Gaussian random variable with mean $\check{x}_0$ and variance $Y_0 > 0$. It is assumed that x_0 and $(w(t), v(t))$ are independent.

In a practical problem, it is natural to assume that noisy measurements of the controlled output Z are taken directly. However, for the sake of mathematical convenience, we will be using the process $y(t)$ in the derivation of a solution to the robust control problem under consideration. In particular, it will be convenient to define feasible output feedback control laws using a stochastic differential equation driven by $y(t)$,

$$\begin{aligned} d\hat{x} &= A_c(t)\hat{x} + B_c(t)dy(t); \quad (7)\\ u(t) &= K_c(t)\hat{x}(t). \end{aligned}$$

All coefficients in equations (6), (7) are assumed to be deterministic bounded sufficiently smooth matrix valued functions defined on $[0,T]$.

To introduce disturbances into the system (6), consider a perturbation probability measure Q which is absolutely continuous with respect to the nominal Wiener probability measure P, $Q \ll P$. Following (Dai Pra *et al.* 1996), a pair of progressively measurable processes $\xi(t)$, $\nu(t)$ adapted to $\{\mathcal{F}_t, t \geq 0\}$ and a Wiener process $\{\tilde{w}(t), \tilde{v}(t), \mathcal{F}_t, t \geq 0\}$ are associated with Q, so that under Q

$$\begin{bmatrix} \tilde{w}(t) \\ \tilde{v}(t) \end{bmatrix} = \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} - \int_0^t \begin{bmatrix} \xi(s) \\ \nu(s) \end{bmatrix} ds.$$

As in (Petersen *et al.* 2000, Ugrinovskii and Petersen 1999), the model uncertainty is formulated in terms of a collection of probability measures $\mathcal{P}$. We consider all probability measures Q such that

$$h(Q\|P) < \infty. \quad (8)$$

In equation (8), $h(Q\|P)$ denotes the *relative entropy* between a probability measure Q and the reference probability measure P (Dupuis and Ellis 1997):

$$h(Q\|P) \colon = \begin{cases} \mathbf{E}^Q \log\left(\dfrac{dQ}{dP}\right) & \text{if } Q \ll P \text{ and} \\ & \log\left(\dfrac{dQ}{dP}\right) \in L_1(dQ), \\ +\infty & \text{otherwise.} \end{cases}$$

For $Q \in \mathcal{P}$, using localizations one can express the relative entropy between Q and P as

$$h(Q\|P) = \frac{1}{2}\mathbf{E}^Q \int_0^T (\|\xi(t)\|^2_{W^{-1}} + \|\nu(t)\|^2_{\Sigma^{-1}})dt; \quad (9)$$

see (Dai Pra *et al.* 1996). Hence, the satisfaction of condition (8) included in the definition of the set $\mathcal{P}$ implies that $\mathbf{E}^Q \int_0^T (\|\xi(t)\|^2_{W^{-1}} + \|\nu(t)\|^2_{\Sigma^{-1}})dt < \infty$.

We further restrict the class of uncertain perturbations and consider disturbances for which

$$\mathbf{E}^Q \int_0^T \left(\|x(t)\|^2 + \|u(t)\|^2\right) dt < \infty. \quad (10)$$

This technical assumption will allow us to perform probability measure transformations needed in order to derive a suitable representation of the stochastic uncertain system under consideration. Also, for the sake of simplicity we will restrict attention to perturbed probability measures Q which are associated with the pair of processes $(\xi(\cdot), \nu(\cdot))$, such that $\nu(t) = 0$ a.s.. The set of probability measures which meet this requirement and for which condition (10) holds, will be denoted $\mathcal{P}_0$. We will also say that the corresponding measurable process $\xi(\cdot)$ belongs to $\mathcal{P}_0$.

As explained in Section 2, the following constraint can be used to quantify the size of the uncertainty with a normalised coprime factor structure.

Definition 1. Given a constant $d > 0$ and a quadratic form $F_0(z,u) = z'S_{11}z + 2z'S_{12}u + u'S_{22}u$, an uncertainty $\xi(\cdot) \in \mathcal{P}_0$ is said to be an admissible uncertainty if the following integral quadratic constraint holds:

$$\mathbf{E}^Q \int_0^T (F_0(z(t),u(t)) + 2z'(t)H\xi(t))\, dt \geq -d. \tag{11}$$

In (11), $x(\cdot)$, $z(\cdot)$ are defined by equation (6) considered in the probability space $(\Omega, \mathcal{F}, Q)$. We denote the set of admissible uncertainties by Ξ_d.

3.2 The minimax control problem

Introduce the process

$$\eta^\dagger(t) := e^{-\int_0^t Z(s)'W^{-1}(s)dw(s) - \frac{1}{2}\|Z(s)\|^2_{W^{-1}}ds}. \tag{12}$$

Here, $Z(t)$ is the controlled output of the closed loop system (6), (7). It can be shown that $\eta^\dagger(t)$ is a martingale. Therefore, a new probability measure $P^\dagger$ can now be defined on events of $\mathcal{F}_t$ by letting $P^\dagger(d\omega) = \eta^\dagger(t)P(d\omega)$. Furthermore, since $P\left(\int_0^T \|Z(t)\|^2_{W^{-1}}dt < \infty\right) = 1$ and $\mathbf{E}\eta^\dagger(T) = 1$, then the probability measures P and $P^\dagger$ are equivalent, $P \sim P^\dagger$ (Liptser and Shiryayev 1977).

From this discussion, $Q \ll P^\dagger$ for $Q \in \mathcal{P}$. Therefore, the relative entropy between Q and $P^\dagger$ is well defined. In this paper we are concerned with the constrained optimization problem associated with the system (6), considered in the probability space $(\Omega, \mathcal{F}, Q)$, $Q \in \mathcal{P}_0$, the cost functional

$$J^\gamma(u,Q) = \frac{1}{2}\mathbf{E}^Q \int_0^T F_1(x(t),u(t))dt + h(Q\|P^\dagger) - \gamma^2 h(Q\|P), \tag{13}$$

$$F_1(x,u) := x'N_{11}x + 2x'N_{12}u + u'N_{22}u,$$

and the constraint (11). In this optimization problem we seek to find a control law $u(\cdot)$ of the form (7) which minimizes the worst case of the cost functional $J^\gamma(u,Q)$:

$$V^\gamma := \inf_{u(\cdot)} \sup_{Q \in \Xi_d} J^\gamma(u(\cdot),Q), \tag{14}$$

where the maximizing player input is an admissible probability measure $Q \in \Xi_d$.

For uncertainties $Q \in \mathcal{P}_0$, the cost functional $J^\gamma(u,Q)$ can be represented in a more explicit form. Indeed, the value of $h(Q\|P)$ has been given in (9). Also, $h(Q\|P^\dagger)$ can be readily computed for $Q \in \mathcal{P}_0$. Hence,

$$J^\gamma(u,Q) = \frac{1}{2}\mathbf{E}^Q \int_0^T (F_1(x(t),u(t)) + \|Z(t) + \xi(t)\|^2_{W^{-1}} - \gamma^2\|\xi(t)\|^2_{W^{-1}})\, dt. \tag{15}$$

Note that in the special case $F_1 = 0$, the cost functional (15) reduces to the cost functional of the optimization problem (5).

4. MINIMAX CONTROL UNDER STOCHASTIC UNCERTAINTY CONSTRAINTS

As in references (Petersen *et al.* 2000, Ugrinovskii and Petersen 1999, Ugrinovskii and Petersen 2001), the derivation of a solution to the constrained minimax optimization problem (14) will use a special stochastic version of the Lagrange multiplier technique (Luenberger 1969). It will be used to convert the constrained optimization problem (14) into a similar optimization problem without constraints. This unconstrained optimization problem is defined in terms of the system (6) and the following cost functional

$$J^{\gamma,\theta}(u,Q) := J^\gamma(u,Q) + \frac{\theta}{2}\mathbf{E}^Q \int_0^T (F_0(z,u) + 2z'(t)H\xi(t))\, dt, \tag{16}$$

where $\theta > 0$ is a Lagrange multiplier. In the unconstrained optimization problem, we seek to find

$$V^{\gamma,\theta} := \inf_u \sup_{Q \in \mathcal{P}_0} J^{\gamma,\theta}(u,Q). \tag{17}$$

Lemma 1. If the set $\Gamma := \{\gamma > 0, \theta > 0 \colon V^{\gamma,\theta} < \infty\}$ is not empty, then the value (14) is finite, and one can find γ such that

$$V^\gamma \leq \inf_{\theta > 0}(V^{\gamma,\theta} + \frac{1}{2}\theta d). \tag{18}$$

□

From Lemma 1, it follows that the guaranteed cost controller achieving an upper bound on the value of the minimax optimization problem (14) can be obtained by minimizing the value of the optimization problem (17). We therefore focus on the problem (17).

Let us define

$$F^{\gamma,\theta} = \frac{1}{\gamma^2 - 1}\left(F_1(x,u) + \theta F_0(z,u) + \|Z\|^2_{W^{-1}}\right) + \frac{1}{(\gamma^2-1)^2}\|Z + \theta WHz\|^2_{W^{-1}}. \tag{19}$$

Also, let us introduce a fictitious output of the system (6), (7) $z^{\gamma,\theta}$ and an associated noise process $w^{\gamma,\theta}$,

$$z^{\gamma,\theta} := -\frac{1}{\gamma^2 - 1}(Z + \theta WHz); \tag{20}$$

$$w^{\gamma,\theta}(t) := \int_0^t z^{\gamma,\theta}(s)ds + w(t). \tag{21}$$

It will be convenient to change the probability measure on $(\Omega, \mathcal{F})$ in order to transform $w^{\gamma,\theta}(\cdot)$ into a Wiener process. Since it can be shown that

$$\eta^{\gamma,\theta}(t) := e^{-\int_0^t (z^{\gamma,\theta}(t))' W^{-1}(s) dw(s)} \times e^{-\frac{1}{2}\|z^{\gamma,\theta}(t)\|^2_{W^{-1}} ds} \qquad (22)$$

is a martingale, then we define a new probability measure $P^{\gamma,\theta}$ by letting $P^{\gamma,\theta}(d\omega) = \eta^{\gamma,\theta}(t)P(d\omega)$. Girsanov's Theorem yields that in $(\Omega, \mathcal{F}, P^{\gamma,\theta})$, the process $\{(w^{\gamma,\theta}(t), v(t)), \mathcal{F}_t, t \geq 0\}$ is a Wiener process. Also, from Theorem 7.1 of (Liptser and Shiryayev 1977), $P \sim P^\dagger \sim P^{\gamma,\theta}$.

Lemma 2. For any $Q \in \mathcal{P}_0$,

$$\frac{J^{\gamma,\theta}(u,Q)}{\gamma^2 - 1} = \frac{1}{2}\mathbf{E}^Q \int_0^T F^{\gamma,\theta}(x,u)dt - h(Q\|P^{\gamma,\theta}). \qquad (23)$$

□

Lemma 2 allows us to convert the original constrained minimax optimization problem (14) into a risk sensitive optimal control problem. This will be achieved by using the duality relationship between free energy and relative entropy given in (Dai Pra *et al.* 1996, Dupuis and Ellis 1997).

First we observe from (21), that in the probability space $(\Omega, \mathcal{F}, P^{\gamma,\theta})$, the system (6) has the form

$$\begin{aligned} dx(t) &= (A(t)x(t) + B_1(t)u(t) - B_2(t)z^{\gamma,\theta}(t))dt \\ &\quad + B_2(t)dw^{\gamma,\theta}(t), \qquad (24) \\ Z(t) &= C_2(t)x(t) + D_2(t)u(t), \\ z(t) &= C_1(t)x(t) + D_1(t)u(t), \\ z^{\gamma,\theta} &= -\frac{1}{\gamma^2 - 1}(Z + \theta W H z), \\ dy(t) &= (Z(t) - z^{\gamma,\theta}(t))dt + dw^{\gamma,\theta}(t) + \beta dv(t). \end{aligned}$$

Associated with the system (24), consider a risk sensitive cost functional

$$\Im_T(u(\cdot)) = \mathbf{E}^{P^{\gamma,\theta}} e^{\frac{1}{2}\int_0^T F^{\gamma,\theta}(x(t),u(t))dt}, \qquad (25)$$

where $F^{\gamma,\theta}$ is defined in (19). We will use a shorthand notation for the coefficients of $F^{\gamma,\theta}$

$$F^{\gamma,\theta}(x,u) = x' R_{\gamma,\theta}(t)x + 2x'\Upsilon_{\gamma,\theta}(t)u + u'U_{\gamma,\theta}(t)u.$$

The result of reference (Dai Pra *et al.* 1996) applied to the system (24) and to the cost functional (25), states that for each admissible control $u(\cdot)$,

$$\sup_{Q \in \mathcal{P}^{\gamma,\theta}} \left[\frac{1}{2}\mathbf{E}^Q \int_0^T F^{\gamma,\theta}(x(t),u(t))dt - h(Q\|P^{\gamma,\theta})\right] = \log \Im_T(u(\cdot)). \qquad (26)$$

Here, $\mathcal{P}^{\gamma,\theta}$ denotes a convex set of all probability measures Q for which $h(Q\|P^{\gamma,\theta}) < \infty$. Clearly, $\mathcal{P}_0 \subseteq \mathcal{P}^{\gamma,\theta}$, therefore we conclude that an output feedback control law solving the stochastic risk sensitive optimal control problem

$$\inf_{u(\cdot)} \Im_T(u(\cdot)) \qquad (27)$$

will deliver a guaranteed bound on the upper value of the unconstrained optimization problem (17). Solutions to the risk-sensitive optimal control problem (27) have been obtained in (Pan and Başar 1996, Bensoussan and Schuppen 1985). Hence, linking the results of those references with those of Lemma 1 via the duality relation (26), we arrive at a guaranteed bound on the upper value of the minimax control problem (14).

The risk-sensitive control technique was developed in the above references for the case where the control input does not have a feedforward connection to the measured output of the system. This is not the case with the system (24). Therefore in order to apply the results of (Pan and Başar 1996, Bensoussan and Schuppen 1985) to the system (24), we augment the measurement process into an extended state dynamics equation as follows. Consider the system

$$\begin{aligned} d\tilde{x}(t) &= (\tilde{A}(t)\tilde{x}(t) + \tilde{B}_1 u(t))dt + \tilde{B}_2(t)d\mu^{\gamma,\theta}(t), \\ d\tilde{y} &= \tilde{C}_2\tilde{x} + \tilde{D}_2 d\mu^{\gamma,\theta}(t). \qquad (28) \end{aligned}$$

Here, the following notation has been used

$$\begin{aligned} \tilde{x} &= \left[\, x' \;\; Z' - (z^{\gamma,\theta})' \,\right]', \\ \mu^{\gamma,\theta}(t) &= \left[\, (W^{-1/2}w^{\gamma,\theta}(t))' \;\; (\Sigma^{-1/2}v(t))' \,\right]', \\ \tilde{A} &= \begin{bmatrix} A + \frac{1}{\gamma^2-1}B_2(C_2 + \theta W H C_1) & 0 \\ \frac{\gamma^2}{\gamma^2-1}C_2 + \frac{\theta}{\gamma^2-1}WHC_1 & 0 \end{bmatrix}, \\ \tilde{B}_1 &= \begin{bmatrix} B_1 + \frac{1}{\gamma^2-1}B_2(D_2 + \theta W H D_1) \\ \frac{\gamma^2}{\gamma^2-1}D_2 + \frac{\theta}{\gamma^2-1}WHD_1 \end{bmatrix}, \\ \tilde{B}_2 &= \left[\, \hat{B}_2 \;\; 0 \,\right] W^{1/2}, \qquad \hat{B}_2 = \begin{bmatrix} B_2 \\ I \end{bmatrix}, \\ \tilde{C}_2 &= \left[\, 0 \;\; I \,\right], \qquad \tilde{D}_2 = \left[\, W^{1/2} \;\; \beta\Sigma^{1/2} \,\right]. \qquad (29) \end{aligned}$$

The variance matrix of the random variable $\tilde{x}_0$ will be denoted $\tilde{Y}_0$. Clearly, this matrix is uniquely determined by the parameters Y_0, $\check{x}_0$ of the initial Gaussian distribution of $x(0) = x_0$.

The solution method developed in (Pan and Başar 1996) can now be applied to the risk-sensitive optimal control problem (27), (28). The solution to this problem is given in terms of a pair of the Riccati differential equations.

Assumption 1. There exist constants $\gamma > 1$, $\theta > 0$, such that the following conditions hold:

(i) There exists a symmetric solution $Y(t)$ to the following filter type Riccati differential equation

$$\dot{Y} = (\tilde{A} - \hat{B}_2 W\Phi\tilde{C}_2)Y + Y(\tilde{A} - \hat{B}_2 W\Phi\tilde{C}_2)' - Y(\tilde{C}_2'\Phi\tilde{C}_2 - R_{\gamma,\theta})Y + \Psi, \quad Y(0) = \tilde{Y}_0, \quad (30)$$

such that $Y(t) \geq c_0 I$ for some $c_0 > 0$ and for all $t \in [0,T]$; here $\Phi := (W + \beta^2\Sigma)^{-1}$,

$$\Psi := \begin{bmatrix} \beta^2 B_2 W\Phi\Sigma B_2' & \beta^2 B_2 W\Phi\Sigma \\ \beta^2\Sigma\Phi W B_2 & \beta^2\Sigma\Phi W \end{bmatrix}.$$

(ii) $R_{\gamma,\theta} - \Upsilon_{\gamma,\theta}U_{\gamma,\theta}^{-1}\Upsilon_{\gamma,\theta}' \geq 0$ and furthermore there exists a symmetric nonnegative definite solution $X(t)$ to the standard H_∞ control type Riccati differential equation

$$\begin{aligned} &\dot{X} + X(\tilde{A} - \tilde{B}_1 U_{\gamma,\theta}^{-1}\Upsilon_{\gamma,\theta}) \\ &+(\tilde{A} - \tilde{B}_1 U_{\gamma,\theta}^{-1}\Upsilon_{\gamma,\theta})'X + (R_{\gamma,\theta} - \Upsilon_{\gamma,\theta}U_{\gamma,\theta}^{-1}\Upsilon_{\gamma,\theta}') \\ &-X(\tilde{B}_1 U_{\gamma,\theta}^{-1}\tilde{B}_1' - \hat{B}_2 W\hat{B}_2')X = 0, \qquad (31) \\ &X(T) = 0. \end{aligned}$$

(iii) For each $t \in [0,T]$, the matrix $I - Y(t)X(t)$ has only positive eigenvalues.

The solutions to Riccati differential equations (30), (31) will define the optimal controller in the risk-sensitive optimal control problem. Indeed, for any $\gamma > 1$, $\theta > 0$ satisfying Assumption 1, consider the controller

$$\begin{aligned} d\check{x}(t) = &\Big[\tilde{A} + YR_{\gamma,\theta} - (Y\tilde{C}_2' + \hat{B}_2 W)\Phi\tilde{C}_2 \\ &- (\tilde{B}_1 + Y\Upsilon_{\gamma,\theta})U_{\gamma,\theta}^{-1}(\tilde{B}_1'X + \Upsilon_{\gamma,\theta}') \\ &\times(I - YX)^{-1}\Big]\check{x}dt \\ &+(Y\tilde{C}_2' + \hat{B}_2 W)\Phi d\tilde{y}(t), \qquad (32) \\ \check{x}(0) = &\check{x}_0, \\ u^*(t) = &-U_{\gamma,\theta}^{-1}(t)(\tilde{B}_1'(t)X(t) + \Upsilon_{\gamma,\theta}(t)) \\ &\times[I - Y(t)X(t)]^{-1}\check{x}(t). \qquad (33) \end{aligned}$$

The following result is a straightforward consequence of the output feedback risk sensitive optimal control result of (Pan and Başar 1996, Theorem 2) and the duality result of (Dai Pra *et al.* 1996).

Lemma 3. Suppose Assumption 1 is satisfied. Then the set Γ is not empty. Furthermore, for any γ, θ satisfying Assumption 1,

$$\begin{aligned} V^{\gamma,\theta} \leq (\gamma^2 - 1)\Big\{&\check{x}_0'X(0)(I - Y_0X(0))^{-1}\check{x}_0 \\ + \int_0^T &\mathrm{tr}\Big[YR_{\gamma,\theta} + (Y\tilde{C}_2'(t) + \hat{B}_2(t)W)\Phi \\ \times\ &(\tilde{C}_2(t)Y + W\hat{B}_2'(t))X(I - YX)^{-1}\Big]dt\Big\}. \quad (34) \end{aligned}$$

□

From Lemma 3, the main result now follows.

Theorem 1. Assume that there exist constants $\gamma > 1$, $\theta > 0$ such that Assumption 1 is satisfied. Then the set Γ is non-empty. Furthermore, suppose the pair (γ^*, θ^*) attains the infimum

$$\inf_{\gamma,\theta}(\tilde{V}^{\gamma,\theta} + \frac{1}{2}\theta d). \qquad (35)$$

where $\tilde{V}^{\gamma,\theta}$ denotes the expression on the right-hand side of (34); the infimum is taken over $\gamma > 1$, $\theta > 0$ satisfying Assumption 1. The corresponding control input $u^*(\cdot)$ defined by (33), is an output feedback control that guarantees the upper bound on the worst case of the cost functional (13) in the constrained stochastic minimax optimization problem (14) subject to the stochastic integral quadratic constraint (11). □

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