CANONICAL CONTROLLERS AND REGULAR IMPLEMENTATION OF ND BEHAVIORS

Paula Rocha

Department of Mathematics, University of Aveiro, Campo de Santiago, 3810-193 Aveiro, Portugal
e-mail: `procha@mat.ua.pt`

Abstract: In this paper we study the solvability of a general nD partial control problem in the behavioral framework. This turns out to be characterized in terms of the solvability of another (full) control problem based on a canonical controller associated to the original problem. Moreover we investigate the performance of the canonical controller in achieving a given control objective and generalize the corresponding results previously obtained by other authors for the 1D case.

Keywords: behavior, control, interconnection, implementation

1. INTRODUCTION

The behavioral approach to control rests on the set theoretic interpretation of the basic idea that to control a system is to impose adequate additional restrictions to its variables in order to obtain a desired overall functioning pattern. In this context, there are two main situations to be considered: either all the system variables are available for control (i.e., are control variables) or only some of the variables are control variables. In order to distinguish these two cases, we respectively refer to *full control* and *(partial) control*.

The first results on solvability of full control problems for systems evolving over a time domain (1D systems) have been obtained in (Willems, 1997). In (Rocha and Wood, 2001), further results have been presented not only for 1D, but also for multidimensional (nD) systems (i.e., for systems evolving over n-dimensional domains).

As concerns partial control problems, the situation is somewhat more involved, since a direct characterization of solvability seems to be impossible. However, in (Belur and Trentelman, 2002) the solvability of a 1D partial control problem for a given plant has been related to the solvability of a suitable associated full control problem. In (Rocha, 2002) some preliminary results for the corresponding nD case have been obtained, by considering a special behavior that has also been introduced in (van der Schaft, 2002) and (Willems *et al.*, 2003) under the name of *canonical controller*.

In this paper we start by further investigating the solvability of nD partial control problems. More concretely, we establish a relation between the controllers that yield a desired control objective by partial control with the controllers that yield the associated canonical controller (now regarded as a control objective) by full control. This situation is different from the 1D case, but also allows to obtain necessary and sufficient conditions for the solvability of a partial control problem in terms of a full one.

In a second stage, we study the effectiveness of the canonical controller in solving the associated partial control problem - a question which has also been considered in (Willems *et al.*, 2003) for the 1D case - and generalize the corresponding 1D results to the nD case.

2. BEHAVIORAL CONTROL

To make the notions of full and partial control more precise we introduce the following notation. If a behavior $\mathcal{B}$ has variables z, we denote it by $\mathcal{B}_z$; moreover, if the variables z are partitioned as $z = (w, c)$, we define the w-behavior of $\mathcal{B}_{(w,c)}$ as $\mathcal{B}_w := \Pi_w \mathcal{B}_{(w,c)} := \{w \mid \exists c \text{ such that } (w, c) \in \mathcal{B}_{(w,c)}\}$; we also define $\mathcal{B}_c$ in an analogous way. On the other hand, given a behavior $\mathcal{B}_v$ we define its lifting to a (e, v)-behavior as $\mathcal{B}^*_{(e,v)} := \{(e, v) \mid e \text{ is free and } v \in \mathcal{B}_v\}$

In set theoretic terms, full control can be formulated as follows. If $\mathcal{P}_z$ is the behavior of the system to be controlled (the plant) and $\mathcal{C}_z$ is the set of all signals compatible with the additional restrictions to be imposed on z, i.e., the *full controller*, then the resulting controlled behavior is given by

$$\mathcal{P}_z \cap \mathcal{C}_z. \qquad (1)$$

This intersection is known as the *interconnection* of the behaviors $\mathcal{P}_z$ and $\mathcal{C}_z$. A desired controlled behavior $\mathcal{D}_z$ is said to be *implementable* (from $\mathcal{P}_z$) *by full control* if there exists a full controller $\mathcal{C}_z$ that *implements* it, i.e., such that its interconnection (1) with the plant behavior $\mathcal{P}_z$ results in $\mathcal{D}_z$.

As for partial control, assume now that the system variables z are partitioned as $z = (w, c)$, where c are the control variables and w the variables to be controlled. If $\mathcal{P}_{(w,c)}$ is the behavior of the system to be controlled and $\mathcal{C}_c$ is the *controller*, i.e., the set of all signals compatible with the additional restrictions to be imposed on the control variables c, then the resulting controlled (w, c)-behavior is given by the interconnection

$$\mathcal{P}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}. \qquad (2)$$

A desired behavior $\mathcal{D}_w$ for the variables to be controlled is said to be *implementable* (from $\mathcal{P}_{(w,c)}$) if there exists a controller $\mathcal{C}_c$ such that the w-behavior of the interconnection (2) is $\mathcal{D}_w$; in this case we say that $\mathcal{C}_c$ *implements* $\mathcal{D}_w$.

In the sequel we consider nD behaviors $\mathcal{B}_z$ that can be described by a set of linear partial difference or partial differential equations, i.e.,

$$\mathcal{B}_z = \ker H(\sigma_1, \ldots, \sigma_n) := \{z \in \mathcal{U} \mid Hz = 0\},$$

where $\mathcal{U}$ is the trajectory universe, the σ_i's are either the usual nD shifts or the elementary nD partial differential operators and $H(s_1, \ldots, s_n)$ is an nD polynomial matrix known as *representation* of $\mathcal{B}_z$. We refer to these behaviors as *kernel behaviors*. In case the variable z is partitioned as $z = (w, c)$, we consider the representation matrix H to be partitioned accordingly as $H = [R \ \ -M]$. This clearly corresponds to writing the equation $Hz = 0$ as $Rw = Mc$. Note that here, for the sake of simplicity, we have written H instead of $H(s_1, \ldots, s_n)$ and $H(\sigma_1, \ldots, \sigma_n)$. From now on, whenever it is clear from the context to which kind of object we are referring (nD polynomial matrix or nD partial difference / differential operator), we adopt this simplification.

Instead of characterizing $\mathcal{B}_z$ by means of a representation matrix H, it is also possible to characterize it by means of its *orthogonal module* $\mathrm{Mod}(\mathcal{B}_z)$, which consists of all the nD polynomial rows r such that $\mathcal{B}_z \subset \ker r$, and can be shown to coincide with the polynomial module generated by the rows of H.

Given two behaviors $\mathcal{B}^1_z$ and $\mathcal{B}^2_z$ their interconnection $\mathcal{B}^1_z \cap \mathcal{B}^2_z$ is said to be *regular* if

$$\mathrm{Mod}(\mathcal{B}^1_z) \cap \mathrm{Mod}(\mathcal{B}^2_z) = \{0\}. \qquad (3)$$

If $\mathcal{B}^1_z = \ker(H^1)$ and $\mathcal{B}^2_w = \ker(H^2)$, then $\mathcal{B}^1_z \cap \mathcal{B}^2_z = \ker(\begin{bmatrix} H^1 \\ H^2 \end{bmatrix})$ and it constitutes a regular interconnection if and only if

$$\mathrm{rank} \begin{bmatrix} H^1 \\ H^2 \end{bmatrix} = \mathrm{rank} H^1 + \mathrm{rank} H^2, \qquad (4)$$

where the ranks are taken as ranks of nD polynomial matrices.

A full controller $\mathcal{C}_z$ is called a *regular full controller*, if its interconnection (1) with the plant $\mathcal{P}_z$ is regular. A behavior $\mathcal{D}_z$ is *regularly implementable by full control* if it is implemented by a regular full controller. In turn, a controller $\mathcal{C}_c$ is said to be a *regular controller*, if the interconnection (2) is regular. In the same way, a behavior $\mathcal{D}_w$ is *regularly implementable* if it is implemented by a regular controller.

In this setting, the problem of (regular) full control can be stated as follows: given a plant $\mathcal{P}_z$ and a desired behavior (control objective) $\mathcal{D}_z$, design a (regular) full controller $\mathcal{C}_z$ that implements $\mathcal{D}_z$. The problem of (regular) control consists in: given a plant $\mathcal{P}_{(w,c)}$ and a control objective $\mathcal{D}_w$, find a (regular) controller $\mathcal{C}_c$ that implements $\mathcal{D}_w$.

3. THE CANONICAL CONTROLLER AND BEHAVIOR IMPLEMENTATION

The solvability of a given control problem is nothing else than the possibility of implementing the control objective from the given plant. Clearly, a behavior $\mathcal{D}_z$ is implementable from a plant $\mathcal{P}_z$ if and only if it is contained in the plant (in this case it suffices to take as controller $\mathcal{C}_z = \mathcal{D}_z$). However, this interconnection is regular only when $\mathcal{P}_z$ coincides with the whole universe for the signals z. The notion of implementation of nD behaviors by regular full control has been studied in (Rocha and

Wood, 2001) under the name of "achievability by regular interconnection". On the other hand, the implementation and regular implementation of 1D behaviors by partial control have been completely characterized in (Trentelman and Willems, 2002) and (Belur and Trentelman, 2002). As for partial control in the nD case, some preliminary results have been obtained in (Rocha, 2002).

Let $\mathcal{P}_{(w,c)}$ be an nD plant behavior with description

$$Rw = Mc \tag{5}$$

and $\mathcal{D}_w$ be a control objective. Define the hidden behavior $\mathcal{H}_w := \{w \mid (w,0) \in \mathcal{P}_{(w,c)}\}$. Clearly,

$$\mathcal{H}_w = \ker R. \tag{6}$$

The following theorem generalizes the 1D results on implementation by partial control (Trentelman and Willems, 2002).

Theorem 1. (Rocha, 2002) With the previous notation, the following statements are equivalent.

(1) $\mathcal{D}_w$ is implementable from $\mathcal{P}_{(w,c)}$.
(2) $\mathcal{H}_w \subset \mathcal{D}_w \subset \mathcal{P}_w$.

As concerns regular implementation, it is shown in (Belur and Trentelman, 2002) that in the 1D case a behavior $\mathcal{D}_w$ is regularly implementable from $\mathcal{P}_{(w,c)}$ (by partial control) if and only if it is regularly implementable from $\mathcal{P}_w$ by full control. Unfortunately, this no longer holds in the nD case, as illustrated in the following example.

Example 1. (Rocha, 2002) Let $\mathcal{P}_{(w,c)}$ be the 2D behavior described by the equation

$$w = \begin{bmatrix} \sigma_2 - 1 \\ 1 - \sigma_1 \end{bmatrix} c,$$

or equivalently, by

$$\begin{bmatrix} 1 & 0 & 1-\sigma_2 \\ 0 & 1 & \sigma_1 - 1 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and let $\mathcal{D}_w$ be the zero behavior. Define the controller $\mathcal{C}_c = \ker 1$ (which corresponds to forcing the control variable c to be equal to zero). Applying this controller to $\mathcal{P}_{(w,c)}$ corresponds to making the interconnection $\mathcal{P}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}$, given by the equation

$$\begin{bmatrix} 1 & 0 & 1-\sigma_2 \\ 0 & 1 & \sigma_1 - 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a regular interconnection, since

$$\text{rank} \begin{bmatrix} 1 & 0 & 1-s_2 \\ 0 & 1 & s_1 - 1 \\ 0 & 0 & 1 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 1-s_2 \\ 0 & 1 & s_1 - 1 \end{bmatrix} + \text{rank}[0 \;\; 0 \;\; 1].$$

Moreover, the associated w-behavior is obviously the zero behavior $\mathcal{D}_w$, showing that $\mathcal{D}_w$ is regularly implementable from $\mathcal{P}_{(w,c)}$ by partial control.

Consider now the behavior $\mathcal{P}_w$ associated with $\mathcal{P}_{(w,c)}$. It is not difficult to check that $\mathcal{P}_w = \ker[\sigma_1 - 1 \quad \sigma_2 - 1]$. If $\mathcal{D}_w$ were implementable from $\mathcal{P}_w$ by full control, the 2D polynomial matrix $[s_1 - 1 \quad s_2 - 1]$ would be completable to a unimodular matrix, which is not the case since it is not a zero-left-prime matrix. Thus we conclude that $\mathcal{D}_w$ is not implementable from $\mathcal{P}_w$ by full interconnection, showing that the 1D result does to generalize to nD behaviors.

Another distinction between the 1D and the nD cases is that whereas for 1D systems implementability by full control can be expressed in terms of controllability, the same does not happen for nD systems. Indeed, it is proven in (Rocha and Wood, 2001) and (Belur and Trentelman, 2002) that, in the 1D case $\mathcal{D}_w$ is regularly implementable from $\mathcal{P}_w$ by full control if and only if

$$\mathcal{D}_w + \mathcal{P}_w^{cont} = \mathcal{P}_w,$$

where $\mathcal{P}_w^{cont}$ is the controllable part of $\mathcal{P}_w$, i.e., the largest controllable sub-behavior or $\mathcal{P}_w$. This sum condition can be interpreted in terms of controllability both in the 1D and in the higher dimensional cases, see (Rocha and Wood, 2001). However, in the nD case, this condition is necessary but not sufficient for regular implementability by full control. We refer the reader to (Rocha and Wood, 2001) for further details.

In view of the foregoing considerations, we need to introduce new tools in order to analyze the problem of nD regular implementation. More concretely, we shall try to characterize regular implementation (by partial control) in terms of conditions on the control variable behavior, rather than by means of conditions on the behavior of the variables to be controlled. In order to do so, we next define the canonical controller associated to a given control problem. This controller has been considered in (Rocha, 2002), under a different designation, as well as in (van der Schaft, 2002) and (Willems *et al.*, 2003).

Definition 1. Let $\mathcal{P}_{(w,c)}$ be a given plant behavior and $\mathcal{D}_w$ a desired behavior (control objective). The *canonical controller* associated with $\mathcal{P}_{(w,c)}$ and $\mathcal{D}_w$ is defined as $\mathcal{C}_c^{can} := \{c \mid \exists w : (w,c) \in \mathcal{P}_{(w,c)} \text{ and } w \in \mathcal{D}_w\}$.

Thus, the canonical controller consists of all the control variable trajectories compatible with the desired behavior for the variables to be controlled.

We start by relating the implementation of $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$ (by partial control) with the implementation of the corresponding canonical controller from $\mathcal{P}_c := \Pi_c \mathcal{P}_{(w,c)}$ (the control variable behavior associated with $\mathcal{P}_{(w,c)}$) by full control. For that purpose we define the unobserved control variable behavior associated with $\mathcal{P}_{(w,c)}$ as $\mathcal{N}_c := \{c \mid (0,c) \in \mathcal{P}_{(w,c)}\}$.

Proposition 1. Given a plant behavior $\mathcal{P}_{(w,c)}$ and an implementable control objective $\mathcal{D}_w$, the following holds.

(1) If the controller $\mathcal{C}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by full control, then it implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$.
(2) If the controller $\tilde{\mathcal{C}}_c$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$, then the controller $\tilde{\mathcal{C}}_c + \mathcal{N}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by full control.

Proof
Let

$$Rw = Mc \tag{7}$$

be a representation of $\mathcal{P}_{(w,c)}$ and N be an nD polynomial matrix which is an minimal left annihilator (MLA) of R. Then, $\mathcal{P}_c = \ker NM$.

1. Assume that the controller $\mathcal{C}_c = \ker K$ implements $\mathcal{C}_c^{can}$ and apply this controller to the plant. This yields the (w,c)-behavior described by the equations:

$$\begin{cases} Rw = Mc \\ 0 \quad = Kc. \end{cases} \tag{8}$$

We next show that the corresponding w-behavior coincides with $\mathcal{D}_w$, which clearly implies that $\mathcal{C}_c$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$.

Suppose then that w^* belongs to the w-behavior induced by equations (8), i.e., that there exists c^* such that the pair (w^*, c^*) satisfies these equations. This implies that $c^* \in \mathcal{P}_c \cap \mathcal{C}_c = \mathcal{C}_c^{can}$ and hence, by the definition of the canonical controller, there exists $\bar{w} \in \mathcal{D}_w$ such that $(\bar{w}, c) \in \mathcal{P}_{(w,c)}$. Thus, by linearity, $(w^* - \bar{w}, 0) \in \mathcal{P}_{(w,c)}$, meaning that $w^* - \bar{w} \in \mathcal{H}_w$. Since $\mathcal{D}_w$ is by assumption implementable, $\mathcal{H}_w \subset \mathcal{D}_w$ and $w^* - \bar{w} \in \mathcal{D}_w$. Consequently also $w^* \in \mathcal{D}_w$ and therefore the w-behavior induced by equations (8) is contained in $\mathcal{D}_w$.

Conversely, suppose that $w^* \in \mathcal{D}_w$. Then obviously $w^* \in \mathcal{P}_w$ and hence there exists c^* such that $(w^*, c^*) \in \mathcal{P}_{(w,c)}$, i.e., such that

$$Rw^* = Mc^*.$$

By the definition of the canonical controller, this means that $c^* \in \mathcal{C}_c^{can}$. Since $\mathcal{C}_c^{can}$ is assumed to be implementable by $\mathcal{C}_c$, $\mathcal{C}_c^{can} \subset \mathcal{C}_c$ and therefore $c^* \in \mathcal{C}_c$, i.e.,

$$Kc^* = 0.$$

Thus, the pair (w^*, c^*) satisfies equations (8), which means that w^* is in the w-behavior induced by these equations. So, $\mathcal{D}_w$ is contained in that behavior. As mentioned before, this shows that $\mathcal{C}_c$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$.

2. Assume now that the controller $\tilde{\mathcal{C}}_c = \ker K$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$. Let $c^* \in \mathcal{C}_c^{can}$. This means that there exists w^* such that $(w^*, c^*) \in \mathcal{P}_{(w,c)}$ and $w^* \in \mathcal{D}_w$. This last condition implies that there exists $\bar{c} \in \tilde{\mathcal{C}}_c$ such that $(w^*, \bar{c}) \in \mathcal{P}_{(w,c)}$. Note that by the linearity of $\mathcal{P}_{(w,c)}$, $(0, c^* - \bar{c}) \in \mathcal{P}_{(w,c)}$; hence $c^* - \bar{c} \in \mathcal{N}_c$ and therefore (taking into account that $\bar{c} \in \tilde{\mathcal{C}}_c$) we have that $c^* \in \mathcal{N}_c + \tilde{\mathcal{C}}_c$. Thus, $\mathcal{C}_c^{can} \subset \mathcal{N}_c + \tilde{\mathcal{C}}_c$ and, since also $\mathcal{C}_c^{can} \subset \mathcal{P}_c$, $\mathcal{C}_c^{can} \subset (\mathcal{N}_c + \tilde{\mathcal{C}}_c) \cap \mathcal{P}_c$.

Conversely, assume that $c^* \in (\mathcal{N}_c + \tilde{\mathcal{C}}_c) \cap \mathcal{P}_c$. Then, there exist w^* and $\bar{c} \in \tilde{\mathcal{C}}_c$ such that $(w^*, c^*) \in \mathcal{P}_{(w,c)}$, $\bar{c} \in \tilde{\mathcal{C}}_c$ and $c^* - \bar{c} \in \mathcal{N}_c$. This implies that $(w^*, \bar{c}) \in \mathcal{P}_{(w,c)}$ and, since $\tilde{\mathcal{C}}_c$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$, $w^* \in \mathcal{D}_w$. Together with the fact that $(w^*, c^*) \in \mathcal{P}_{(w,c)}$, taking the definition of $\mathcal{C}_c^{can}$ into account, this allows to conclude that $c^* \in \mathcal{C}_c^{can}$. Therefore $(\mathcal{N}_c + \tilde{\mathcal{C}}_c) \cap \mathcal{P}_c \subset \mathcal{C}_c^{can}$. This finally proves that $\mathcal{C}_c^{can} = (\mathcal{N}_c + \tilde{\mathcal{C}}_c) \cap \mathcal{P}_c$, which amounts to say that $\mathcal{N}_c + \tilde{\mathcal{C}}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by full control. ■

Remark Note that, as a consequence of this proposition, if $\mathcal{N}_c = \{0\}$, then $\mathcal{C}_c$ implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$ if and only if it implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by full control.

It turns out that an analogous result to Proposition 1 holds true for regular implementation.

Proposition 2. Given a plant behavior $\mathcal{P}_{(w,c)}$ and an implementable control objective $\mathcal{D}_w$, the following holds.

(1) If the controller $\mathcal{C}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by regular full control, then $\mathcal{C}_c$ implements $\mathcal{D}_w$ regularly from $\mathcal{P}_{(w,c)}$.
(2) If the controller $\tilde{\mathcal{C}}_c$ implements $\mathcal{D}_w$ regularly from $\mathcal{P}_{(w,c)}$, then the controller $\tilde{\mathcal{C}}_c + \mathcal{N}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by regular full control.

Proof
Since the statements about implementation have already been proven in Proposition 1 it now suffices to prove the statements concerning regularity.

1. We have to show that $\mathrm{Mod}(\mathcal{P}_c) \cap \mathrm{Mod}(\mathcal{C}_c) = \{0\}$ (i.e., the regularity of $\mathcal{C}_c$ as a full controller applied to $\mathcal{P}_c$) implies $\mathrm{Mod}(\mathcal{P}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)}) = \{0\}$ (i.e., the regularity of $\mathcal{C}_c$ as a controller applied to $\mathcal{P}_{(w,c)}$). Let $r = [0 \quad \bar{r}] \in \mathrm{Mod}(\mathcal{P}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)})$ (note that since w is free in $\mathcal{C}^*_{(w,c)}$, the first components of r must be zero). Then, clearly, $\bar{r} \in \mathrm{Mod}(\mathcal{C}_c)$. Moreover, $\mathcal{P}_c \subset \ker \bar{r}$, and hence

$\bar{r} \in \mathrm{Mod}(\mathcal{P}_c)$. Therefore $\bar{r} \in \mathrm{Mod}(\mathcal{P}_c) \cap \mathrm{Mod}(\mathcal{C}_c)$. In this way, if $\mathrm{Mod}(\mathcal{P}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)})$ has a nonzero element $r = [0 \quad \bar{r}]$ with $\bar{r} \neq 0$ then also $\mathrm{Mod}(\mathcal{P}_c) \cap \mathrm{Mod}(\mathcal{C}_c)$ has a nonzero element $\bar{r}$, proving the desired implication.

2. Now, we must show that if $\mathrm{Mod}(\mathcal{P}_{(w,c)}) \cap \mathrm{Mod}(\tilde{\mathcal{C}}^*_{(w,c)}) = \{0\}$ then $\mathrm{Mod}(\mathcal{P}_c) \cap \mathrm{Mod}(\tilde{\mathcal{C}}_c + \mathcal{N}_c) = \{0\}$. Note that since $\mathrm{Mod}(\tilde{\mathcal{C}}_c + \mathcal{N}_c) = \mathrm{Mod}(\tilde{\mathcal{C}}_c) \cap \mathrm{Mod}(\mathcal{N}_c)$ and $\mathrm{Mod}(\mathcal{P}_c) \subset \mathrm{Mod}(\mathcal{N}_c)$, the condition to be proven is equivalent to $\mathrm{Mod}(\tilde{\mathcal{C}}_c) \cap \mathrm{Mod}(\mathcal{P}_c) = \{0\}$. Let $\tilde{r} \in \mathrm{Mod}(\tilde{\mathcal{C}}_c) \cap \mathrm{Mod}(\mathcal{P}_c)$, then, because $\tilde{r} \in \mathrm{Mod}(\mathcal{P}_c)$, $[0 \quad \tilde{r}] \in \mathrm{Mod}(\mathcal{P}_{(w,c)})$ and, because $\tilde{r} \in \mathrm{Mod}(\tilde{\mathcal{C}}_c)$, $[0 \quad \tilde{r}] \in \mathrm{Mod}(\tilde{\mathcal{C}}^*_{(w,c)})$. This means that $[0 \quad \tilde{r}] \in \mathrm{Mod}(\mathcal{P}_{(w,c)}) \cap \mathrm{Mod}(\tilde{\mathcal{C}}^*_{(w,c)})$. As previously, this yields the desired result. ∎

Remark Once more, as a consequence of Proposition 2, in case $\mathcal{N}_c = \{0\}$, $\mathcal{C}_c$ regularly implements $\mathcal{D}_w$ from $\mathcal{P}_{(w,c)}$ if and only if it implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ by regular full control.

As an immediate consequence of Proposition 2 we obtain our main result on the regular implementation of a given control objective.

Theorem 2. Let $\mathcal{P}_{(w,c)}$ be a given plant behavior and $\mathcal{D}_w$ a control objective. Assume further that $\mathcal{D}_w$ is implementable from $\mathcal{P}_{(w,c)}$. Then $\mathcal{D}_w$ is regularly implementable from $\mathcal{P}_{(w,c)}$ if and only if $\mathcal{C}_c^{can}$ is regularly implementable from $\mathcal{P}_c$ by full control.

Since the condition of regular implementation by full control can be checked by analyzing the kernel representations of the involved behaviors (Rocha and Wood, 2001), Theorem 2 does provide a way of checking regular implementation by partial control.

In the previous considerations, the canonical controller associated to a given control problem has been in a certain sense regarded as a "control objective" itself, whose ability to be implemented provides information on the possibility of implementing the true control objective. We now take a different perspective and consider the canonical controller in its most natural role, i.e., as being itself a controller. In this context, two questions obviously arise: does the canonical controller implement the control objective? if so, is this implementation regular? The answers to these questions are given below.

Theorem 3. Given a plant behavior $\mathcal{P}_{(w,c)}$, a control objective $\mathcal{D}_w$, let C_c^{can} be the associated canonical controller. Then, C_c^{can} implements $\mathcal{D}_w$ if and only of $\mathcal{D}_w$ is implementable.

Proof
The "only if" part of the statement is trivial. As for the "if" part, suppose that $\mathcal{D}_w$ is implementable, and let $\tilde{\mathcal{C}}_c = \ker K$ be a controller that implements this behavior. Then, by Proposition 1, the controller $\tilde{\mathcal{C}}_c + \mathcal{N}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$. If $Rw = Mc$ is a representation of $\mathcal{P}_{(w,c)}$ and N is a MLA of R, $\mathcal{N}_c = \ker M$ and $\mathcal{P}_c = \ker NM$. Therefore, the fact that $\tilde{\mathcal{C}}_c + \mathcal{N}_c$ implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$ means that $\mathcal{C}_c^{can}$ is the c-behavior induced by the following equations:

$$\begin{cases} NMc = 0 \\ c \quad = c_1 + c_2 \\ Kc_1 \quad = 0 \\ Mc_2 \quad = 0. \end{cases} \tag{9}$$

Consequently, applying the canonical controller to the plant $\mathcal{P}_{(w,c)}$ yields the restrictions:

$$\begin{cases} Rw \quad = Mc \\ NMc = 0 \\ c \quad = c_1 + c_2 \\ Kc_1 \quad = 0 \\ Mc_2 \quad = 0, \end{cases} \tag{10}$$

that can easily be shown to have the same w-behavior as

$$\begin{cases} Rw \quad = Mc_1 \\ Kc_1 = 0. \end{cases} \tag{11}$$

But this w-behavior is precisely $\mathcal{D}_w$, which proves that $\mathcal{C}_c^{can}$ indeed implements $\mathcal{D}_w$. ∎

Our last results concerns regular implementation by means of the canonical controller.

Theorem 4. Given a plant behavior $\mathcal{P}_{(w,c)}$, a control objective $\mathcal{D}_w$, let C_c^{can} be the associated canonical controller. Then, C_c^{can} regularly implements $\mathcal{D}_w$ if and only of $\mathcal{P}_c$ coincides with the whole c-trajectory universe, i.e., if and only if $\mathrm{Mod}(\mathcal{P}_c) = \{0\}$.

Proof
Assume that C_c^{can} regularly implements $\mathcal{D}_w$. Then, by Proposition 2, $\mathcal{C}_c^{can} + \mathcal{N}_c$ regularly implements $\mathcal{C}_c^{can}$ from $\mathcal{P}_c$. This implies that $\mathrm{Mod}(\mathcal{C}_c^{can} + \mathcal{N}_c) \cap \mathrm{Mod}(\mathcal{P}_c) = \{0\}$. But, as shown before, $\mathrm{Mod}(\mathcal{C}_c^{can} + \mathcal{N}_c) \cap \mathrm{Mod}(\mathcal{P}_c) = \mathrm{Mod}(\mathcal{C}_c^{can}) \cap \mathrm{Mod}(\mathcal{P}_c)$. As $\mathrm{Mod}(\mathcal{P}_c) \subset \mathrm{Mod}(\mathcal{C}_c^{can})$ (because $\mathcal{C}_c^{can} \subset \mathcal{P}_c$), we obtain that $\mathrm{Mod}(\mathcal{P}_c) = \{0\}$.

Conversely, if $\mathrm{Mod}(\mathcal{P}_c) = \{0\}$ then the canonical controller regularly implements itself from $\mathcal{P}_c$. By Proposition 2 this implies that $\mathcal{C}_c^{can}$ also implements $\mathcal{D}_w$ regularly. ∎

Corollary 1. The canonical controller is regular if and only if every controller is regular.

Proof
The "if" part is obvious. As for the "only if part", we start by noting that, given a controller $\mathcal{C}_c$, $\text{Mod}(\mathcal{P}_{(w,c)}) \cap \text{Mod}(\mathcal{C}^*_{(w,c)}) = \{r \mid r = [0 \;\; \bar{r}], \bar{r} \in \text{Mod}(\mathcal{C}_c) \cap \text{Mod}(\mathcal{P}_c)\}$. Assume now that the canonical controller is regular. Then, by the previous theorem, $\text{Mod}(\mathcal{P}_c) = \{0\}$ and consequently also $\text{Mod}(\mathcal{P}_{(w,c)}) \cap \text{Mod}(\mathcal{C}^*_{(w,c)}) = \{0\}$ for any given controller $\mathcal{C}_c$, which precisely means that the controller $\mathcal{C}_c$ is regular. This proves the desired result. ■

Theorems 3, 4 and Corollary 1 generalize the corresponding 1D results obtained in (Willems *et al.*, 2003) to the nD case.

4. CONCLUSION

We have considered nD partial control problems and started by establishing a relation between the controllers that yield a desired control objective with the controllers that yield the associated canonical controller (regarded as a control objective) by full control, both in the cases of simple and regular control. This has allowed to prove the equivalence between the (regular) implementation of a given control objective by partial control and the (regular) implementation of the corresponding canonical controller by full control. We then studied the effectiveness of the canonical controller in solving the associated control problem and generalized the corresponding 1D results in (Willems *et al.*, 2003) to the nD case.

ACKNOWLEDGMENTS

This research is partially supported by the *Unidade de Investigação Matemática e Aplicações* (UIMA), University of Aveiro, Portugal, through the *Programa Operacional "Ciência e Tecnologia e Inovação"* (POCTI) of the *Fundação para a Ciência e Tecnologia* (FCT), co-financed by the European Union fund FEDER.

5. REFERENCES

Belur, M.N and H.L. Trentelman (2002). Stabilization, pole placement and regular implementability. *IEEE Trans. Aut. Contr.* **47**, 735–744.

Rocha, P. (2002). Regular implementability of nd behaviors. In: *CD ROM Proceedings of the Fifteenth International Symposium on the Mathematical Theory of Networks and Systems, MTNS'02.* University of Notre Dame, South Bend, USA.

Rocha, P. and J. Wood (2001). Trajectory control and interconnection of 1d and nd systems. *SIAM J. Cont. and Optim.* **40 (1)**, 107–134.

Trentelman, H. and J.C. Willems (2002). Synthesis of dissipative systems using quadratic differential forms, part i. *IEEE Trans. Aut. Contr.* **47**, 53–69.

van der Schaft, A. (2002). Achievable behavior of general systems. *Systems and Control Letters* **49**, 141–149.

Willems, J.C. (1997). On interconnections, control and feedback. *IEEE Trans. Aut. Contr.* **42**, 326–339.

Willems, J.C., M.N. Belur, A.A. Julius and H. Trentelman (2003). The canonical controller and its regularity. In: *Proceedings of the 42nd IEEE Conference on Decision and Control.* Maui, Hawaii, USA.