ON PRICING IN THE SOUTH AFRICAN RENEWABLE COMMODITIES MARKET

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Abstract: In this contribution, we discuss and apply a stochastic model that may be used for the pricing of renewable commodities like (white and yellow) maize, wheat and sunflowers. More specifically, we conclude that an appropriate model for determining the spot price of the aforementioned commodities is the mean-reverting model. An important feature of this model is that it reflects reality in the marketplace by making allowances for spot prices and convenience yields to be mean-reverting. In order to illustrate these ideas we provide simulations and numerical examples using data from the South African renewable commodities market. When using the mean-reverting model, we find that the behaviour of the spot prices for the simulations and real data bear a close resemblance to each other.

Keywords: Option Pricing; Renewable Commodities; Stochastic Modelling.

1. INTRODUCTION

In this paper, we apply a pricing model to determine the behaviour of spot prices in the renewable commodities market (see, for instance, Pring, 1985; Shuldiner, 1996 and Telser, 1958). We are particularly interested in commodity markets that trade (white and yellow) maize, sunflower and wheat. The choice of pricing model is motivated by our desire to avoid the over- and underpricing of commodities that both lead to the mispricing of options. The inherent uncertainty associated with these prices makes a stochastic approach to pricing options, futures and forward contracts in such markets inevitable. In this regard, there exists mean reversion in both prices and the convenience yield. Here prices will not grow at a constant rate as is the case in stock markets, but will eventually return to an average value over time. When prices are assumed to be mean-reverting it leads to a new set of problems. We have to determine the level to which it reverts called the equilibrium price. Also we are interested in the speed of mean reversion which depends on the commodity being studied.

A mathematical model that is a candidate for accomplishing the pricing of the aforementioned commodities is the classical Black-Scholes model and its variations (see Black and Scholes, 1973 and Lence and Hayes, 2002). However, this model has its limitations when applied to the commodities market in terms of mean reversion and volatility. Instead, in this article, we choose a mean-reverting model (compare with Schwartz, 1997 and Schwartz, 1998) for the stochastic modelling of prices. In reality, both the spot price of the commodity and the convenience yield are mean reverting. If there is only mean reversion in convenience yield we might expect the futures prices would eventually reflect a normal cost-of-carry market, but then those prices would stay at high levels. Thus we can make the assumption that the price levels is also a mean reverting process. Also, in this paper, we illustrate the merits of the mean-reverting model for option pricing by

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developing numerical examples and simulations from the South African commodities market. We can derive a formula for the determination of the futures price by mainly using Ito's Lemma, which we can then use to price futures and forward contracts.

The current section is introductory in nature while the rest of the paper is structured in the following way. In Section 2 we give an explanation of the mean-reverting model for spot price behavior. The third section describes the main properties of a commodities market with particular emphasis on the South African situation. In Section 4 we supply numerical examples and simulations with data being sourced from the South African renewable commodities market.

2. MEAN-REVERTING PRICING MODEL

In this section, we provide a brief description of the mean-reverting models that may be used for the option pricing of renewable commodities. More specifically, we describe two mean-reverting factor models for spot commodity prices (compare with Schwartz, 1997 and Schwartz, 1998). The first is the one-factor model (see Subsection 2.1) in which we assume that the logarithm of the spot price is stochastic and follows a mean reverting process of Ornstein-Uhlenbeck type. The second model is a two-factor model (see Subsection 2.2) where we add a second stochastic variable, the convenience yield. The convenience yield can be seen as the flow of services accruing to the holder of the spot commodity but not to the holder of a futures contract. This is also assumed to be mean reverting. These models allow for closed form solutions for futures prices and for a linear relation between the logarithm of futures prices and the underlying factors.

2.1 *Mean-Reverting One-Factor Model*

We assume that the spot price of the commodity follows the stochastic process

$$\begin{aligned} dS(t) &= S(t)[k(m - \ln(S(t))]dt \\ &+ S(t)\sigma dW(t) \end{aligned} \tag{1}$$

Now we set $X_1 = lnS(t)$ and we obtain

$$f(X_1) = \ln(X_1),\ f'(X_1) = \frac{1}{X_1}, \quad f''(X_1) = \frac{-1}{X_1^2}$$

with

$$\begin{aligned} dX_1(t) &= \frac{1}{S(t)} + \frac{1}{2}(\frac{-1}{S(t)^2})S(t)^2\sigma^2 dt \\ &= k(m - X_1(t))dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \\ &= [k(m - X_1(t)) - \frac{1}{2}\sigma^2]dt + \sigma dW(t) \\ &= [(km - \frac{1}{2}\sigma^2) - kX_1(t)]dt + \sigma dW(t) \\ &= k(a - X_1(t))dt + \sigma dW(t). \end{aligned} \tag{2}$$

$$a = m - \frac{\sigma^2}{2k}$$

where

S : Spot Price;
k : Speed of Mean-Reversion;
m : The Total Expected Return on the Spot Commodity;
σ : Volatility of the Spot Prce;
dW : Increment to the Std Brownian Motion.

The spot price, or equivalently the log of the spot price, plays the role of an underlying state variable upon which contingent claims can be written. Under standard assumptions, the dynamics of the Ornstein-Uhlenbeck process under the equivalent martingale measure can be rewritten as

$$dX_1(t) = k(a^* - X_1(t))dt + \sigma dW(t), \tag{3}$$

where $a^* = a - l$, l is the market price of risk (assumed constant). From (3), the conditional distribution of X at time T under the equivalent martingale measure is normal with mean and variance given by

$$\mathbf{E}_0[X_1(T)] = e^{-kT}X(0) + (1 - e^{-kT})a^*$$

and

$$Var_0[X_1(T)] = \frac{\sigma^2}{2k}(1 - e^{-2kT}), \tag{4}$$

respectively. By assuming a constant interest rate, we can state that the futures price of the commodity with maturity T is the expected spot price of the commodity at time T under the equivalent martingale measure. Then

$$\begin{aligned} F(S,T) &= \mathbf{E}\big[S(T)\big] \\ &= \exp(\mathbf{E}_0\big[X_1(T)\big] + \frac{1}{2}Var_0\big[X_1(T)\big]) \end{aligned} \tag{5}$$

and it follows that

$$F(S,T) = \exp\{e^{-kT}lnS + (1 - e^{-kT})a^*$$

$$+\frac{\sigma^2}{4k}(1-e^{-2kT})\}. \tag{6}$$

This equation is equivalent in log form to

$$\begin{aligned}\ln F(S,T) = e^{-kT} lnS + (1-e^{-kT})a^* \\ +\frac{\sigma^2}{4k}(1-e^{-2kT}).\end{aligned} \tag{7}$$

By using Ito‘s Lemma again to find a relationship between F and the stochastic process of S, we obtain the partial differential equation

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + k(m - l - \ln S)SF_S - F_T = 0 \tag{8}$$

with boundary condition $F(S,0) = S$.

2.2 *Mean-Reverting Two-Factor Model*

In the mean-reverting two-factor model, the stochastic variables are the commodity spot price and the instantaneous convenience yield which we will denote by $X_2(t)$. We now have the two stochastic processes

$$\begin{aligned}dS(t) = S(t)\big[(c_1 + a_{12}X_2(t))\big]dt \\ + S(t)m_{11}dW_1(t) + S(t)m_{12}dW_2(t)\end{aligned} \tag{9}$$

and

$$\begin{aligned}dX_2(t) = [a_{21}X_1(t) + a_{22}X_2(t) + c_2]dt \\ + m_{21}dW_1(t) + m_{22}dW_2(t),\end{aligned} \tag{10}$$

where W_1 and W_2 are independent Brownian motions. If we set $X_1(t) = ln(S(t))$ then

$$\begin{aligned}dX_1(t) &= \frac{1}{S(t)}dS(t) \\ &\quad +\frac{1}{2}\left(\frac{-1}{S(t)^2}\right)(m_{11}^2 + m_{12}^2)dt \\ &= (c_1 + a_{12}X_2(t))dt \\ &\quad + m_{11}dW_1(t) + m_{12}dW_2(t) \\ &\quad -\frac{1}{2}(m_{11}^2 + m_{12}^2)dt \\ &= [a_{12}X_2(t) + c_1 \\ &\quad -\frac{1}{2}(m_{11}^2 + m_{12}^2)]dt \\ &\quad + m_{11}dW_1(t) + m_{12}dW_2(t).\end{aligned} \tag{11}$$

We choose $a_{12} = 0$ and $c_1 = r - X_2(t)$ which is the risk adjusted drift of the commodity price process with r being the interest rate assumed to be constant in this model. Now we set

$$\sigma = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix} \text{ and } z = \begin{bmatrix} W_1(t) \\ W_2(t)\end{bmatrix}$$

If $\sigma_1 = \begin{bmatrix} m_{11} & m_{12} \\ 0 & 0\end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} 0 & 0 \\ m_{21} & m_{22}\end{bmatrix}$ we obtain from (9) and (10) that

$$dS(t) = (r - X_2(T))S(t)dt + \sigma_1 S(t)dz(t) \tag{12}$$

$$dX_2(t) = [k(m - X_2(t)) - l]dt + \sigma_2 dz(t) \tag{13}$$

with

$$dW_1(t)dW_2(t) = pdt.$$

$S:$ Spot Price;
$X_2:$ Convenience Yield;
$r:$ Interest Rate;
$k:$ Speed of Mean-Reversion;
$m:$ The Total Expected Return on the Spot Commodity;
$l:$ Market Price of Convenience Yield Risk;
$\sigma_1:$ Volatility of the Spot Price;
$\sigma_2:$ Volatility of the Convenience Yield;
$dW:$ Increment to the Std Brownian Motion.

Here l is assumed to be constant. Again applying Ito‘s Lemma we are able to deduce the following partial differential equation

$$\begin{aligned}\frac{1}{2}\sigma_1^2 S^2 F_{SS} + \sigma_1\sigma_2 pSF_{SX_2} \\ +\frac{1}{2}\sigma_2^2 S^2 F_{X_2X_2} + (r - X_2)SF_S \\ +(k(a - X_2) - l)F_{X_2} - F_T = 0\end{aligned} \tag{14}$$

with boundary condition $F(S, X_2, 0) = S$. The solution to (14) is presented in Jamshidian and Fein (1990) and Bjerksund (1991), with the futures price being given by

$$\begin{aligned}F(S, X_2, T) = S\exp[-X_2\frac{1-e^{-kT}}{k} \\ +A(T)],\end{aligned} \tag{15}$$

where

$$\begin{aligned}A(T) = (r - \hat{a} + \frac{1}{2}\frac{\sigma_2^2}{k^2} - \frac{\sigma_1\sigma_2 p}{k})T \\ +\frac{1}{4}\sigma_2^2\frac{1-e^{-2kT}}{k^3} \\ +(\hat{a}k + \sigma_1\sigma_2 p - \frac{\sigma_2^2}{k})\frac{1-e^{-kT}}{k^2}\end{aligned} \tag{16}$$

with $\hat{a} = a - \frac{l}{k}$. The formula (15) may be rewritten in log form as

$$\ln F(S, X_2, T) = \ln S - X_2\frac{1-e^{-kT}}{k} \tag{17}$$

$$+ A(T). \tag{18}$$

3. PRICING OF RENEWABLE COMMODITIES

In this section, we will give a description of a commodities market in general and then provide information about the South African situation. We consider some of the main factors affecting the market and provide a discussion on how to determine the constants needed for the simulation of the spot prices.

3.1 *Pricing in Commodities Markets*

Futures markets have existed for many centuries with the earliest being the rice market in Japan. In the futures market there is a great deal of investment tools to help an investor who is seeking to constitute a portfolio that is profitable, however, there is also a great amount of risk involved in this market. Next we discuss the main factors regarding price behavior.

Some of the main factors that influence the spot commodity prices are supply-and-demand levels, cost of production and exchange rates. A high supply rate for example would have a downward effect on the prices and on the other hand, high demand levels will cause an increase in the prices.

Mean reversion is the tendency for prices to stabilize to a "mean" value. We observe that, in our case, there is evidence that the spot price reverts to an average value over time. When we determined k, the speed of mean reversion, we found a negative value which may be an indication of mean-reversion (see, for instance, Blanco and Soronow, 2001 and Bessembinder, Coughenour, Segium and Smoller, 1995). In Blanco and Soronow (2001), the principle of mean reversion is encapsulated in the equation

$$S(t+1) - S(t) = k(a - S(t)) + \sigma\varepsilon(t),$$

with $S(t)$ the spot commodity price at time t, a the long-run mean price and k the speed of mean reversion. ε is the random shock to the spot commodity price from time t to time $t+1$. The first term on the right, is the drift term or mean reversion component and the second term is the random component. We see that the former is influenced by the difference between the long run mean price and the spot commodity price, and also the speed of mean reversion. We can easily see that if $S(t)$ is smaller than a, the mean reversion component will be positive and will push the spot price upward and vice versa.

The volatility, σ, is the expected average change in the spot price for every time interval over the whole period. In this paper, we tried the square-root-of-time rule to give us a value for σ, (see Black and Scholes, 1973) but this was not realistic.

3.2 *Pricing in SAn Commodities Market*

The futures market in South Africa is administered by South African Futures Exchange (SAFEX). Futures were not traded until 1987 in South Africa when Rand Merchant Bank (RMB) introduced contracts and also in 1988 when they introduced a long bond future. South Africa is a developing country with farmers not having the resources to accommodate large fluctuations in the cost of production. At present, the Black-Scholes equation is used to price commodities. This paper suggests that a switch to the mean-reverting model may be a prudent step in improving the price behavior predictions.

In South Africa, the commodities market is influenced a great deal by the exchange rate of the Rand (South African currency) to the other major currencies. This is partly because South Africa export a great amount of the harvest. When the Rand is weaker against the Dollar for instance it will push up the prices of the commodities. Figures 1 and 2 show the spot prices for white maize and the Rand-Dollar exchange rate, respectively, for the period October 1997 to August 2004. When the exchange rate is high, as it was around 2001-2002, so is the spot price of the commodities. This is of course the case for wheat and sunflower as well. From the figures we can also see that the spot prices tend to revert back to an average value.

Fig. 1. Maize Spot Prices

We can see that the effect of the exchange rate is felt a few weeks later and also that the price takes a few months longer to revert back to the average.

Cost of production is linked directly to the exchange rates. If the cost of production is to high, it will have a negative effect on production.

Despite the fact that the convenience yield depends on some of the above mentioned factors,

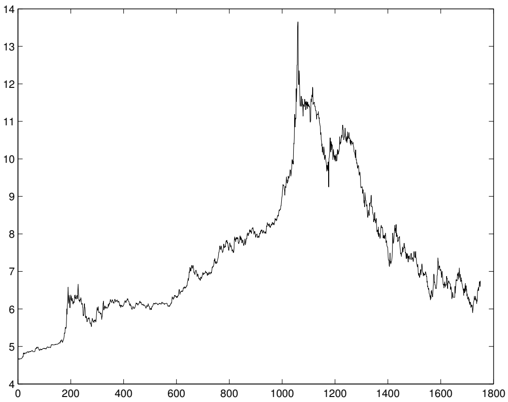

Fig. 2. Effects of Rand-Dollar Exchange Rate

we will only consider the effect of the convenience yield on the prices. In Schwartz (1997) he found that if the convenience yield was taken to be constant, the pricing model was not as effective as the two-factor model that assumed a stochastic convenience yield. One viewpoint is that we should consider the commodity to be an asset that pays a dividends yield to the owner. This dividends yield is then regarded as the convenience yield $X_2(t)$.

4. NUMERICAL EXAMPLES AND SIMULATIONS

In this section, we make use of the mean-reverting model that was described in Subsection **??** to determine the path of the prices for wheat, maize and sunflowers. In particular, we consider the commodities market in South Africa. We used information found on www.grainsa.co.za.

4.1 *South African Commodities Market Data*

In our illustrative examples, we used the data on the spot price of white maize over a one month period (August 2004). We note that there was 20 trading days in this month. Firstly, we calculated the change in the price levels and then change expressed as percentages. We calculate the standard deviation for the change of prices and multiplied this by the square root of 20 to obtain the volatility. We then estimated mean reversion and its parameters by regressing absolute price changes on the previous prices. With the slope being negative we see that there is evidence of mean reversion. The speed of mean reversion is the negative of the slope. The long run mean is the intercept of the data divided by the speed of mean reversion. The volatility is the residual standard deviation divided by the long run mean.

Table 1. Market Data for White Maize

Date	S	$\Delta\ S$ (%)	$\Delta\ S(R)$	Prev. S
2/8/04	886	-	-	-
3/8/04	931	4.96	45	886
4/8/04	940	0.96	9	931
5/8/04	922	-1.93	-18	940
6/8/04	935	1.40	13	922
7/8/04	935	0.00	0	935
8/8/04	916	-2.05	-19	935
9/8/04	941	2.69	25	916
10/8/04	917	-2.58	-24	941
11/8/04	949	3.43	32	917
16/8/04	949	0.00	0	949
17/8/04	928	-2.24	-21	949
18/8/04	942	1.50	14	928
19/8/04	928	-1.50	-14	942
20/8/04	932	0.43	4	928
23/8/04	963	3.27	31	932
24/8/04	985	2.26	22	963
25/8/24	999	1.41	14	985
26/8/04	980	-1.92	-19	999
27/8/04	987	0.71	7	980

4.2 *Application of the One-Factor Model*

In the numerical example above, we see that the volatility calculated with the square-root-of-time rule as opposed to using mean reversion parameters, differ substantially. We simulated the path of the spot commodity price over one month. Of course the choice of the volatility and the other factors used is a study on its own with the values used here not being unique.

We used the standard deviation of the spot prices as the volatility, 29 % .The standard deviation of the change in prices expressed as a percentage is 2.19 %. The square root of 20 is 4.4721. Thus we get the volatility by multiplying 2.19 % by 4.4721, which is 9.81 %. If we calculate the slope, intercept and residual standard deviation for the values in the second table, we obtain -35.61 %, 340.403 and 18.706, respectively. Furthermore, the long run mean is found to be 955.874. To determine the path of the spot price from (2), we only need m, the total expected return on the spot commodity. We defined a as the long run mean log price. From the volatility, σ, and speed of mean reversion, k, determined above, we conclude that the total expected return on the spot commodity $m = 6.69$.

4.3 *Application of the Two-Factor Model*

As we asserted before, the two-factor model assumes a stochastic convenience yield, $X_2(t)$. The convenience yield can be interpreted as the flow of services accruing to the holder of the spot commodity but not to the owner of a futures contract (see Schwartz, 1997). In order to implement the two-factor model, we require the following values from equations (9) and (10). The values for m, k and a are obtained from the one-factor model for the one month data. Equation (10) is used to determine the path of the convenience yield, $X_2(t)$ which is assumed to be stochastic. We also need σ_1 and σ_2, that are the volatilities of the spot prices and the convenience yield, respectively. Next we simulate (10) to obtain values for $X_2(t)$ by using k and a as with the one-factor model. Subsequently, we used the same technique to determine the volatility for the convenience yield as for the spot prices of white maize. When we substitute the values for $X_2(t)$, we are able to determine the volatility as before and obtain the value 0.1 for σ_2.

4.4 *Simulations vs Real Data*

We used the Euler-Maruyama method to numerically solve (1) with the constants given in Subsection 4.1. Using these parameters in (1), we get the following path for the price of white maize for the specified time periods. We then simulated the path using (9) and (10). From the figures below we see that the simulated path is almost the same as the true value path.

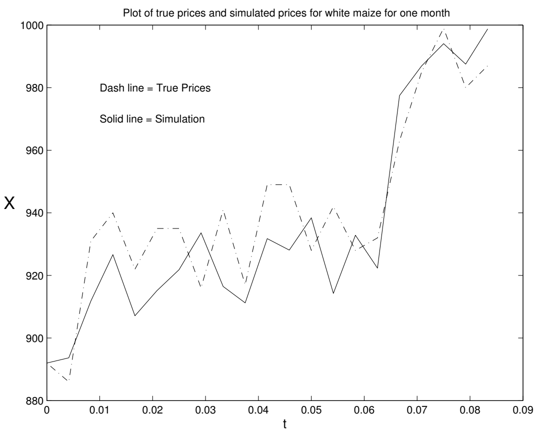

Fig. 3. White Maize Spot Prices (One-Factor)

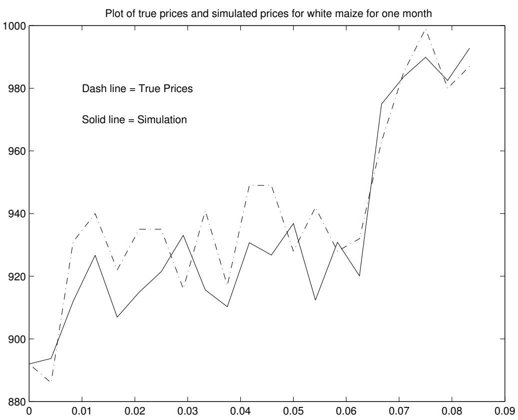

Fig. 4. White Maize Spot Prices (Two-Factor)

In the first figure we simulated the spot price using (1). Also, in the second figure we used (9) in conjunction with (10). It is clear that the simulated values are closely related to the true values.

References

Bessembinder, H., J.F. Coughenour, P.J. Segium and M.M. Smoller (1995). Mean Reversion in Equilibrium Asset Prices: Evidence from the Futures Term Structure. *Journal of Finance*, **50**, 361-375.

Bjerksund, P. (1991). Contingent Claims Evaluation when the Convenience Yield is Stochastic: Analytical Results. Working Paper, Norwegian School of Economics and Business Administration.

Black, F. and M. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **3**, 637-654.

Blanco, C. and D. Soronow, (2001) Mean Reverting Processes. *Commodities Now*, 68-72.

Jamshidian, S.H. and D.J. Fein (1990). Closed-Form Solutions for Oil Futures and European Options in the Gibson-Schwartz Model: A Note. Merril Lynch Capital Markets.

Lence, S.H. and D.J. Hayes (2002). Option Pricing on Renewable Commodity Markets. Working Paper.

Pring, M.J. (1985). The McGraw-Hill Handbook of Commodities and Futures, McGraw-Hill, USA.

Schwartz, E.S. (1997). The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging. *Journal of Finance*, **52**, 923-973.

Schwartz, E.S. (1998). Valuing Long-Term Commodity Assets. *Journal of Energy Finance and Development*, **3**, 85-99.

Shuldiner, A. and G.X. Norkus (1996). "Buying Prices" on the Commodity Futures Market. *Hotel and Restaurant Administration Quarterly.*

Telser, L. (1958). Futures Trading and the Storage of Cotton and Wheat. *Journal of Political Economy*, **66**, 233-255.