ON EXTENDED CONTROLLER FORMS OF NONLINEAR DISCRETE-TIME SYSTEMS: THE SINGLE-INPUT CASE

S. Monaco [*] **, D. Normand-Cyrot** [**]

[*] *Dip. di Informatica e Sistemistica, Università di Roma "La Sapienza", Via Eudossiana 18, 00184 Rome, Italy*
[**] *Laboratoire des Signaux et Systèmes, CNRS-Supélec, Plateau de Moulon, 91190 Gif-sur-Yvette, France*

Abstract: In this paper, two types of extended controller forms are given for nonlinear discrete-time single-input dynamics controllable in first approximation. The study is set in the context of differential/difference representations of discrete-time dynamics.

Keywords: Nonlinear discrete-time systems, normal forms, series expansions.

1. INTRODUCTION

Up to what extend is it possible to simplify the nonlinearities of a given controlled dynamics through coordinates change and feedback and thus to define *extended normal forms*, is the question addressed in the present paper for linearly controllable discrete-time dynamics. The idea, launched in Krener (1984) in control theory finds its roots in Cartan's method of equivalence or Poincaré's normal forms (Wiggins (1990)). It has been more recently further developed and renewed (see Kang and Krener (1992), Guzella and Isidori (1993), Kang (1996) Tall and Respondek (2003) and the references therein), making reference to controllable or uncontrollable and/or unobservable systems so providing stabilizing strategies for dynamics exhibiting bifurcations. While the approach can be similarly developed for both cases of vector fields (differential dynamical systems) and maps (discrete-time systems), such a parallelism becomes difficult when dealing with forced dynamical systems. Even if many analogies can be set, specific studies are necessary.

In discrete time, most of the contributions are concerned with quadratic or cubic normal forms; this is enough to characterize important control properties: quadratic approximated feedback linearization under dynamic feedback in Barbot, Monaco, Normand-Cyrot (1997), stabilization of systems with uncontrollable modes or bifurcations in Hamzi, Barbot, Monaco and Normand-Cyrot (2001), observer design for systems with unobservable modes in Boutat-Baddas, Boutat, Barbot and Taleigne (2001). In Krener and Li (2002), quadratic and cubic normal forms are introduced for controllable or uncontrollable dynamics to classify discrete-time bifurcations. More recently, in Hamzi and Tall (2003), the structure of normal forms at a fixed degree m have been introduced.

The present work investigates normal forms of order m for nonlinear discrete-time single-input dynamics controllable in first approximation. With respect to previous work, the problem is presently set in the context of differential/difference representations of discrete-time dynamics proposed in Monaco and Normand-Cyrot (1998). It results that the normal forms obtained are different from those proposed in (Barbot, Monaco, Normand-Cyrot (1997), Hamzi, Barbot, Monaco and Normand-Cyrot (2001), Boutat-Baddas, Boutat, Barbot and Taleigne (2001), Krener and Li

(2002), Hamzi and Tall (2003)). Such a set up is chosen because, without requiring any extra assumption, it makes possible a quite complete answer to the problem and stresses a strong parallelism with continuous-time results (Kang and Krener (1992), Kang (1996), Tall and Respondek (2003)). More precisely, the study is addressed step-by-step, through homogeneous approximations of increasing degree of the Taylor-like expansions of the involved dynamics, coordinates changes and feedbacks. For each degree of approximation, say m, normal forms of degree m, containing all the nonlinear terms nonremovable under coordinates change and feedback (*resonance terms*), are characterized. Two types of normal forms are given depending if one privileges cancellation of the nonlinear terms in the controlled part of in the drift (dual normal form). Cancellation of the nonlinearities is carried out by solving a set of algebraic equations referred to as the *discrete-time homogeneous homological equations* in analogy with those of Poincaré for vector fields. These homogeneous normal forms, which can be seen as a generalization of the Brunovsky normal form, are referred to as *extended controller normal forms of degree m*. A prelimary study limited to the approximation at the second degree was proposed in Monaco and Normand-Cyrot (2004).

The paper is organized as follows. In Section 2, the context is defined and the problem set. In Section 3, two types of extended controller normal forms are given. It is shown that their m-th degree homogeneous parts are obtained solving a set of algebraic equations: *the discrete-time homogeneous homological equations*.

Notations - The state variables ζ and/or x belong to $\mathcal{X}$, an open set of R^n, which can be all R^n and the control variables v and/or u belong to $\mathcal{U}$, a neighborhood of zero in R. All the maps, vector fields and control systems are analytic on their domains of definition, infinitely differentiable admitting convergent Taylor series expansions.

A vector field on $\mathcal{X}$, analytically parameterized by u, $G(x,u) \in T_x\mathcal{X}$ defines a u-dependent differential equation of the form $\frac{dx^+(u)}{du} = G(x^+(u),u)$ where the notation $x^+(u)$ figurates that the state evolution is interpreted as a curve in R^n, parameterized by u. A R^n-valued mapping $F(.,u) : x \to F(x,u)$, defines a forced discrete-time dynamics while $F(.,0)$ and/or $F : x \to F(x)$ represent unforced evolutions.

Given a generic map on $\mathcal{X}$, its evaluation at x is denoted indifferently by "(x)" or "$\big|_x$". $J_xF|_{x=0} = \frac{d(F(x))}{dx}|_{x=0}$ indicates the jacobian of the function evaluated at $x=0$.

The upperscript $(.)^{[m]}$ stands for the homogeneous term of order m of the Taylor series expansion of the function or vector field into the parentheses. Analogously, $R^{[m]}(.)$ (resp. $R^{\geq m}(.)$) stand for the space of vector fields or functions whose components are polynomial (resp. formal power series) of degree m in the variables into the parentheses and $R^{\geq m}(.)$.

2. THE CONTEXT AND THE PROBLEM SETTLEMENT

Let a single-input discrete-time dynamics, $x \to F(x,u)$, controllable in first approximation around the equilibrium pair $(0,0)$, be represented following Monaco and Normand-Cyrot (1998) as two coupled differential/difference equations

$$\zeta^+ = F(\zeta) \tag{1}$$

$$\frac{d\zeta^+(v)}{dv} = G(\zeta^+(v),v)$$
$$= G_1(\zeta^+(v)) + \sum_{i\geq 1}\frac{v^i}{i!}G_{i+1}(\zeta^+(v));\ \zeta^+(0) = \zeta^+ \tag{2}$$

with $F(0) = 0$, $G_1(.) = G(.,v)|_{v=0}$, $G_1(0) \neq 0$ and for $i \geq 1$, $G_{i+1}(.) := \frac{\partial^i G(.,v)}{\partial v^i}|_{v=0}$.

Let us show that there is no loss of generality to consider such a differential/difference representation, (DDR).

• Provided completeness of the vector field $G(.,v)$, the flow associated with (2) is defined for any v, a nonlinear discrete-time dynamics in the usual form of a nonlinear difference equation $\zeta \to F(\zeta,v)$ can be recovered integrating (2) between 0 and $v(k)$ with initialization at $\zeta^+(0) = \zeta^+ = F(\zeta(k))$; we get

$$\zeta(k+1) = F(\zeta(k),v(k)) = \zeta^+(v(k))$$
$$= \zeta^+(0) + \int_0^{v(k)} G(\zeta^+(w),w)dw.$$

An explicit exponential representation of $F(.,v)$ can also be computed (see Monaco, Normand-Cyrot and Califano (2002)).

• Starting from a nonlinear difference equation $\zeta \to F(\zeta,v)$, the existence of a DDR (1-2) follows from the existence of $G(.,v)$ satisfying $G(.,v)|_{F(.,v)} = \frac{\partial F(.,v)}{\partial v}$; the invertibility of $F(.,0)$ is thus sufficient to prove that $G(.,v)$ can be locally uniquely defined as $G(.,v) := \frac{\partial F(.,v)}{\partial v}|_{F^{-1}(.,v)}$.

Due to the controllability assumption, there is no loss of generality to consider a DDR (1-2), modified under preliminary linear coordinates change and feedback to put its linear approximation

$$\zeta^+ = J_\zeta F|_{\zeta=0}\zeta = A\zeta \tag{3}$$

$$\frac{d\zeta^+(v)}{dv} = G_1(0) = B; \quad \zeta^+(0) = \zeta^+ \tag{4}$$

into the controllable form

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5}$$

Up to these comments, the present study is thus set, without loss of generality, on the representation below (referred to as $\Sigma^{[\infty]}$), with (A, B) as in (5)

$$\zeta^+ = F(\zeta) = A\zeta + \sum_{m=2}^{\infty} F^{[m]}(\zeta) \tag{6}$$

$$\begin{aligned} \frac{d\zeta^+(v)}{dv} &= G(\zeta^+(v), v) \\ &= B + \sum_{m=2}^{\infty}\sum_{i=1}^{m} \frac{v^{i-1}}{(i-1)!} G_i^{[m-i]}(\zeta^+(v)). \end{aligned} \tag{7}$$

Remark. The vector fields $(G_i^{[m-i]}; i \geq 2)$ model nonlinearities in the control variable. Setting $G_i^{[m-i]} = 0$ for $i \geq 2$, (7) reduces to $B + G_1^{[m-1]}(\zeta^+(v))$ and the results which can be obtained are the discrete-time counterpart of those obtained in the continuous-time case for input-affine dynamics as clarified in the sequel. $\triangleleft$

Remark. A differs from the Brunovsky form by the identity function and is thus invertible. This choice, which does not influence the generality of the approach or the results, is presently prefered to ensure local invertibility of F in (6). $\triangleleft$

It is now possible to formulate the general question asked in this paper: up to what extend is it possible to simplify the nonlinearities of $\Sigma^{[\infty]}$ and thus to achieve linearization through coordinates change and invertible feedback $\Gamma^{[\infty]}$. Do there exist for the nonlinear terms, higher degree canonical forms providing extensions to the Brunovsky or some controller linear normal form (A, B)?. Furthermore, how to find the coordinates change and feedback under which a dynamics can be transformed into these normal forms? The study is performed iteratively, by showing the result step-by-step, for each degree of approximation, through homogeneous manipulations of the series expansions involved.

3. EXTENDED CONTROLLER NORMAL FORMS

Two types of normal forms are given depending if one privileges cancellation of the nonlinearities in the drift F, or in the controlled vector field $G(., v)$.

Since the linear part of $\Sigma^{[\infty]}$ is already in normal form, we want to leave it invariant under coordinates change and feedback and therefore we use the group of formal transformations defined as

$$x = \zeta + \sum_{m=2}^{\infty} \phi^{[m]}(\zeta) \tag{8}$$

$$v = u + \sum_{m=2}^{\infty} \gamma^{[m]}(\zeta, u) \tag{9}$$

with $\gamma^{[m]}(., u) = \gamma_0^{[m]} + \sum_{i=1}^{m} \frac{u^i}{i!}\gamma_i^{[m-i]}$, and where the $\phi^{[m]}$ and $\gamma_i^{[m-i]}$'s, $i = (0, ..., m)$ are respectively R^n and R-valued mappings. We have

Theorem 1. The nonlinear discrete-time dynamics (6, 7) can be tranformed by a coordinates change and feedback of the form (8, 9) into a dynamics in one of the two following extended controller normal forms:

First type of normal form (dual normal form) - linearity of the drift - $\Sigma_{NFA}^{[\infty]}$

$$\begin{aligned} x^+ &= Ax \\ \frac{dx_1^+(u)}{du} &= \sum_{i=2}^{\infty} \frac{u^{i-1}}{(i-1)!} G_{i;1}(x^+(u)) \\ \frac{dx_2^+(u)}{du} &= x_n^+(u) Q_{2;n}(x_1^+(u), ..., x_n^+(u)) \\ &\quad + \sum_{i=1}^{\infty} \frac{u^i}{i!} G_{i+1;2}(x^+(u)) \\ &\dots \\ \frac{dx_{n-1}^+(u)}{du} &= \sum_{i=3}^{n} x_i^+(u) Q_{n-1;i}(x_1^+(u), ..., x_i^+(u)) \\ &\quad + \sum_{i=1}^{\infty} \frac{u^i}{i!} G_{i+1;n-1}(x^+(u)) \\ \frac{dx_n^+(u)}{du} &= 1 \end{aligned}$$

where $Q_{i;j}(x_1^+(u), ..., x_j^+(u))$ is a formal series defined by the formal summation

$$Q_{i;j}(x_1, ..., x_j) = \sum_{m=0}^{\infty} Q_{i;j}^{[m]}(x_1, ..., x_j).$$

Second type of normal form - linearity of G_1 - $\Sigma_{NFB}^{[\infty]}$

$$x_1^+ = x_1 + x_2 + \sum_{i=3}^{n} x_i^2 F_{1;i}(x_1, \dots, x_i)$$

$$\begin{aligned}
&\cdots\\
x_{n-2}^+ &= x_{n-2} + x_{n-1} + x_n^2 F_{n-2;n}(x_1,\dots,x_n)\\
x_{n-1}^+ &= x_{n-1} + x_n\\
x_n^+ &= x_n\\
\frac{dx_p^+(u)}{du} &= \sum_{i=1}^{\infty} \frac{u^i}{i!} G_{i+1;p}(x^+(u)); p = (1,...,n-1)\\
\frac{dx_n^+(u)}{du} &= 1
\end{aligned}$$

where $F_{i;j}(x_1,...,x_j)$ is a formal series defined by the formal summation

$$F_{i;j}(x_1,...,x_j) = \sum_{m=0}^{\infty} F_{i;j}^{[m]}(x_1,...,x_j).$$

Making reference to the notion of m-jets of a vector field which correspond to the first m-th degree terms in its Taylor expansion, the results in Theorem 1 characterize normal forms of m-jets for nonlinear discrete-time controlled dynamics. Feedback laws of the form (9) cannot cancel the nonlinearities in u of the form $\sum_{i=1}^{\infty} \frac{u^i}{i!} G_{i+1;p}$ for $(p = 1,...,n-1)$ in both $\Sigma_{NFB}^{[\infty]}$ and $\Sigma_{NFA}^{[\infty]}$. Setting $G_{i\geq 2} = 0$, discrete-time and continuous-time controller normal forms exhibit strongly comparable structures (see in the continuous-time case (Kang and Krener (1992),Kang (1996), Tall and Respondek (2003)).

The proof of Theorem 1 works out iteratively by showing that it is possible to achieve the result separately for each homogeneous approximation of degree m accordingly to Theorem 2 below.

3.1 Homogeneous controller normal forms

Homogeneous normal forms and transformation of degree m are defined ignoring the higher degree and lower degree terms except the linear ones. Given $\Sigma^{[\infty]}$ let, for any degree of approximation $m \geq 2$, $\Sigma^{[m]}$ be its homogeneous part of degree m around (A, B); i.e.

$$\zeta^+ = A\zeta + F^{[m]}(\zeta); \quad \zeta^+(0) = \zeta^+ \quad (10)$$

$$\frac{d\zeta^+(v)}{dv} = B + \sum_{i=0}^{m-1} \frac{v^i}{i!} G_{i+1}^{[m-i-1]}(\zeta^+(v)). \quad (11)$$

and analogously, let $\Gamma^{[m]}$ be the homogeneous part of degree m around the identity of (8-9). By construction, $\Gamma^{[m]}$ does not modify either the linear part (A, B) or the terms of degree $< m$ in $\Sigma^{[\infty]}$.

Remark. In the adopted formalism, $\Sigma^{[m]}$ is described by (10), the approximation of degree m of (6) around the linear evolution $A\zeta$ and (11), the homogeneous approximation of degree $m-1$ of (7) around B. Integrating (11) with respect to v and evaluating the result at (10) yields to a difference equation of the usual form which may contain terms of degree $> m$ in (ζ, v). This clearly explains why the normal forms introduced lateron differ from those obtained in related work set in the usual formalism (see Barbot, Monaco, Normand-Cyrot (1997), Krener and Li (2002), Hamzi and Tall (2003)). ◁

The following theorem shows that an homogeneous transformation of degree m is sufficient to cancel the nonlinear terms of the same degree and transforms the given dynamics into one of its normal forms of the same degree.

Theorem 2. For any degree $m \geq 2$ and neglecting higher degree terms, any homogeneous discrete-time dynamics (10, 11) can be tranformed under an homogeneous coordinates change and feedback into

$$x^+ = Ax + \bar{F}^{[m]}(x)$$

$$\frac{dx^+(u)}{du} = B + \bar{G}^{[m-1]}(x^+(u), u)$$

where the pair $(Ax+\bar{F}^{[m]}(x), B+\bar{G}^{[m-1]}(x^+(u), u))$ is in one of the extended controller normal forms $\Sigma_{NFA}^{[m]}$ or $\Sigma_{NFB}^{[m]}$, respectively defined as the homogeneous parts of degree m of $\Sigma_{NFA}^{[\infty]}$ or $\Sigma_{NFB}^{[\infty]}$.

As an homogeneous transformation of a given degree does not modify the lower degree terms, the following corollary is immediate.

Corollary 1. $\Sigma_{NFB}^{[\infty]}$ can be tranformed under coordinates change and feedback of the form (8-9) into

$$x^+ = Ax + \sum_{k=2}^{m-1} F^{[k]}(x) + \bar{F}^{[m]}(x) + O(x)^{>m}$$

$$\frac{dx^+(u)}{du} = B + \sum_{k=1}^{m-2} G^{[k]}(x^+(u), u) + \bar{G}^{[m-1]}(x^+(u), u) + O(x)^{>m-1}$$

where the pair $(Ax+\bar{F}^{[m]}(x), B+\bar{G}^{[m-1]}(x^+(u), u))$ is in one of the extended controller normal forms $\Sigma_{NFA}^{[m]}$ or $\Sigma_{NFB}^{[m]}$.

The proof of Theorem 2 is in Monaco and Normand-Cyrot (2004.a). Different proofs following the lines of the continuous-time case (Kang (1996), Tall and Respondek (2003)) could be performed, (see Barbot, Monaco, Normand-Cyrot (1997), Krener and Li (2002), Hamzi and Tall (2003)). Applying the results of Theorem 2 to each homogeneous part of degree m and then iteratively while increasing the degree, Theorem 1 follows. The results in Corollary 1 show that, neglecting terms of degree greater than m, the nonlinear discrete-time dynamics (6, 7) can be transformed into an extended controller form up to the degree m by a transformation containing terms of at most degree m.

We note that while m-th degree homogeneous normal forms are uniquely defined, the extended normal forms are not because homogeneous transformation of degree m which cannot change the terms of degree m can change terms of degree higher that m and thus the homogeneous normal forms of degree higher than m.

3.2 The homological equations

How to find such a coordinates change and feeback transformation is the problem addressed hereafter. Following the Poincaré's technique, we consider two dynamics which are identical up to their terms of degree m and ask the question how to find an homogeneous m-th transformation which puts the homogeneous m-th part of one dynamics into that of the other dynamics. To do so, let $\tilde{\Sigma}^{[m]}$ be the m-th homogeneous part of another dynamics $\tilde{\Sigma}^{[\infty]}$ of the form (10 - 11), we have

Theorem 3. There exists an homogeneous feedback transformation $\Gamma^{[m]}$ which brings $\Sigma^{[m]}$ into $\tilde{\Sigma}^{[m]}$ modulo terms in $R^{\geq m+1}(x,u)$, $\Sigma^{[m]}$ is feedback equivalent to $\tilde{\Sigma}^{[m]}$, if and only if there exist $(\phi^{[m]}, \gamma_i^{[m-i]}; i=(0,...,m))$ in (10, 11) satisfying the equations

$$\tilde{F}^{[m]}(x) - F^{[m]}(\zeta) = \phi^{[m]}(A\zeta) - A\phi^{[m]}(\zeta) + \gamma_0^{[m]}(\zeta)B \quad (12)$$

$$\sum_{i=0}^{m-1} \frac{u^i}{i!}(\tilde{G}_{i+1}^{[m-i-1]}(x) - G_{i+1}^{[m-i-1]}(\zeta)) = \quad (13)$$

$$\frac{d\phi^{[m]}(\zeta)}{d\zeta_n} + \sum_{i=0}^{m-1} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(A^{-1}\zeta - uA^{-1}B)B.$$

Proof The proof works out just writing down the action of $\Gamma^{[m]}$ over $\Sigma^{[m]}$. The transformation $\Gamma^{[m]}$ being composed with two parts, $\phi^{[m]}$ acts as a usual coordinates change so that $(F^{[m]}, G_i^{[m-i]})$ are simply transformed into $(\bar{F}^{[m]}, \bar{G}_i^{[m-i]})$ below

$$\bar{F}^{[m]}(x) = F^{[m]}(\zeta) + \phi^{[m]}(A\zeta) - A\phi^{[m]}(\zeta) \quad (14)$$

$$\bar{G}_1^{[m-1]}(x) = G_1^{[m-1]}(\zeta) + \frac{d\phi^{[m]}(\zeta)}{d\zeta}B \quad (15)$$

$$\bar{G}_i^{[m-i]}(x) = G_i^{[m-i]}(\zeta); \quad i=(2,...,m). \quad (16)$$

The feedback action further transforms (14) to (16) into

$$\tilde{F}^{[m]}(x) = \bar{F}^{[m]}(\zeta) + \gamma_0^{[m]}(\zeta)B$$
$$= F^{[m]}(\zeta) + \phi^{[m]}(A\zeta) - A\phi^{[m]}(\zeta) + \gamma_0^{[m]}(\zeta)B$$

$$\sum_{i=0}^{m-1} \frac{v^i}{i!}\tilde{G}_{i+1}^{[m-i-1]}(x) = \sum_{i=0}^{m-1} \frac{u^i}{i!}\bar{G}_{i+1}^{[m-i-1]}(\zeta)$$
$$+ \sum_{i=0}^{m-1} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(A^{-1}\zeta - vA^{-1}B)B$$
$$= \frac{d\phi^{[m]}(\zeta)}{d\zeta_n} + \sum_{i=0}^{m-1} \frac{u^i}{i!}G_{i+1}^{[m-i-1]}(\zeta)$$
$$+ \sum_{i=0}^{m-1} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(A^{-1}\zeta - vA^{-1}B)B$$

because, up to an error in $R^{\geq m}(\zeta)$

$$\zeta = A^{-1}\zeta^+(v) - vA^{-1}B$$
$$dv = du + \sum_{i=0}^{m-1} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(\zeta)du$$
$$= du + \sum_{i=0}^{m-1} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(A^{-1}(\zeta^+(v)) - vA^{-1}B)du$$

and up to an error in $R^{\geq m+1}(\zeta, v)$

$$\gamma^{[m]}(.,u)|_{u=0} = \gamma_0^{[m]}; \zeta^+(v)|_{v=\gamma_0^{[m]}} = \zeta^+(0) + \gamma_0^{[m]}B$$
$$x^+(v) = \zeta^+(v) + \phi^{[m]}(\zeta^+(v)).$$

In conclusion, $\Gamma^{[m]}$ brings the system $\Sigma^{[m]}$ into $\tilde{\Sigma}^{[m]}$ described by

$$x^+ = Ax + \tilde{F}^{[m]}(x)$$
$$= Ax + F^{[m]}(x) + \phi^{[m]}(Ax) - A\phi^{[m]}(x) + \gamma_0^{[m]}(x)B$$
$$\frac{dx^+(u)}{du} = B + \sum_{i=0}^{m-1} \frac{u^i}{i!}\tilde{G}_{i+1}^{[m-i-1]}(x^+(u))$$
$$= B + \frac{d\phi^{[m]}}{dx_n}(x^+(u)) + \sum_{i=0}^{m-1} \frac{u^i}{i!}G_{i+1}^{[m-i-1]}(x^+(u))$$
$$+ \sum_{i=1}^{m} \frac{u^i}{i!}\gamma_{i+1}^{[m-i-1]}(A^{-1}x^+(u) - uA^{-1}B)B \quad (17)$$

which achieves to prove the equalities (12, 13). ◁

Remark. We deduce from (17) that

$$\tilde{G}_1^{[m-1]}(x) = G_1^{[m-1]}(x) + \frac{d\phi^{[m]}(x)}{dx_n} + \gamma_1^{[m-1]}(A^{-1}x)B \quad (18)$$

while for $i \geq 2$, $\tilde{G}_i^{[m-i]}$ and $G_i^{[m-i]}$ differ from their last component only but the computation of $\tilde{G}_i^{[m-i]}$ in terms of $G_i^{[m-i]}$ involve preliminarily the expansion with respect to u of $\gamma_i^{[m-i]}(A^{-1}x - uA^{-1}B)$. ◁

Recalling that complete cancellation of the nonlinearities through coordinates change and feedback corresponds to achieve *full feedback linearization*, *m-th degree homogeneous linear feedback equivalence* corresponds to complete cancellation under $\Gamma^{[m]}$ of the terms of degree m in (10) and of degree $m-1$ in (11); when this is not achievable, the remaining terms define the so-called *m-th degree homogeneous normal forms*. The following result is an immediate consequence of the equalities (12) and (13) in Theorem 2.

Proposition 1. $\Sigma^{[m]}$ is full feedback linearizable at the degree m if and only if there exist $(\phi^{[m]}, \gamma_i^{[m-i]}; i = (0, ..., m))$ in (8, 9) satisfying the following equations

$$-F^{[m]}(\zeta) = \phi^{[m]}(A\zeta) - A\phi^{[m]}(\zeta) + \gamma_0^{[m]}(\zeta)B \quad (19)$$

$$-\sum_{i=0}^{m-1} \frac{v^i}{i!} G_{i+1}^{[m-i-1]}(\zeta) = \frac{d\phi^{[m]}(\zeta)}{d\zeta_n} + \sum_{i=0}^{m-1} \frac{v^i}{i!} \gamma_{i+1}^{[m-i-1]}(A^{-1}\zeta - vA^{-1}B)B \quad (20)$$

Equations (19-20) are referred to as the *m-th degree discrete-time homogeneous homological equations.*

4. CONCLUSION

Two types of extended controller normal forms have been given for nonlinear discrete-time dynamics represented as coupled differential/ difference equations. Re-interpreting the existence of normal forms as the property of feedback linearizability up to a certain degree of approximations, further results can be given as detailed in a forthcoming paper Monaco and Normand-Cyrot (2004.a).

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