ANALYSIS AND NONLINEAR CONTROL OF IMPLICIT DISCRETE-TIME DYNAMIC SYSTEMS

Johann Holl [*,**] **Kurt Schlacher** [*,**]

[*] *Institute of Automatic Control and Control Systems Technology, Johannes Kepler University of Linz, Austria*
[**] *Christian Doppler Laboratory for Automatic Control of Mechatronic Systems in Steel Industries*

Abstract: This contribution is concerned with the observability and accessibility analysis of implicit discrete-time dynamic systems. The presented approach is motivated by a geometric representation of discrete-time systems and the crucial observation that the Lie group investigations known for implicit continuous-time dynamic systems is also appropriate in the discrete-time scenario. The obtained formal method to state conditions for local observability and accessibility allows can be done by computer algebra. Furthermore, a nonlinear discrete-time controller design is discussed by considering the class of input-to-state linearizable continuous-time control systems. This approach is based on the calculation of a control sequence such that the system trajectories considered at equidistant time steps coincide with those of a linear discrete-time one.

Keywords: Discrete-time systems, Nonlinear control systems, Implicit systems, Observability, Accessibility

1. INTRODUCTION

In physics and engineering most dynamic processes and systems are continuous in time and described mathematically by a system of differential equations. Nevertheless, the class of discrete-time dynamic systems arises quite naturally in modelling processes with changes in the dependent variables at equidistant time steps such as for example in financial or in economic problems. In control theory discrete-time dynamic systems appear as the time discretization of continuous-time systems with regard to a digital implementation of control laws. Especially for linear and time-invariant systems this approach is state of the art since the last two decades. Therefore, this contribution is focused on the study of dynamic systems described by a set of n_e implicit difference equations and n_y output functions of the form

$$0 = f^{i_e}\left(k, z_k, z_{k+1}\right) \ , \quad i_e = 1, \ldots, n_e \tag{1a}$$

$$y_k^{\alpha_y} = c^{\alpha_y}\left(k, z_k\right) \ , \quad \alpha_y = 1, \ldots, n_y \tag{1b}$$

in the independent variable $k \in \mathbb{Z}$ denoting the discrete time kT_s with some fixed (sample) time T_s and the dependent variables z^{α_z}, $\alpha_z = 1, \ldots, n_z$. For an investigation of observability and accessibility properties of implicit continuous-time systems the Lie group approach turned out to be an appropriate one, (Schlacher *et al.*, 2002). It is shown that this geometric method is also valid for implicit discrete-time systems.

This paper is organized as follows. In Section 2 some notational conventions and mathematical basics of transformation groups are discussed very briefly. The observability and accessibility investigations of implicit dynamic systems by considering invariants of transformation groups acting on the solution of the system are motivated in Section 3 by the special case of explicit dynamic systems. Consequently, in Section 4 the general implicit case is treated. A nonlinear discrete-time control design approach for a class of continuous-time dynamic systems is presented in Section 5.

2. MATHEMATICAL PRELIMINARIES

Throughout this contribution the language of manifolds and bundles is used (the reader is kindly referred to, e.g., (Boothby, 1986), (Saunders, 1989) and the references therein for a detailed treatment of these topics) since they offer a geometric interpretation of nonlinear dynamic systems, system analysis, and controller synthesis. For notational convenience and in order to keep formulas short index notation of tensor calculus and the Einstein sum convention is arranged.

The observability and accessibility investigations treated in the following sections make fundamental use of Lie groups, therefore the definitions, see, e.g., (Olver, 1993), are summarized very briefly. A Lie group is a smooth manifold $\mathcal{G}$ that is also a group and both the composition and the inversion map are smooth. A transformation group acting on a manifold $\mathcal{M}$ is determined by a Lie group $\mathcal{G}$ and a smooth map $\Phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that

$$\Phi_e(x) = x \ , \tag{2a}$$
$$\Phi_{g \circ h}(x) = \Phi_g \circ \Phi_h(x) \ , \tag{2b}$$
$$\Phi_{g^{-1}} \circ \Phi_g(x) = x \tag{2c}$$

is satisfied with the neutral element e of $\mathcal{G}$ and for all $g, h \in \mathcal{G}$ as well as $x \in \mathcal{M}$. Further, let us assume a subset $\mathcal{S} \subset \mathcal{M}$ that is a solution set of a system of equations. A symmetry group is a local group of transformations such that

$$\Phi_g(\mathcal{S}) \subseteq \mathcal{S}$$

holds for all $g \in \mathcal{G}$, meaning that solutions of the system are mapped to other solutions. A function $I : \mathcal{M} \to \mathbb{R}$ is called an invariant of the transformation group Φ if and only if

$$I(x) = I(\Phi_g(x))$$

is met for all $g \in \mathcal{G}$ and $x \in \mathcal{M}$.

For the following, transformation groups Φ_ε with one real parameter $\varepsilon \in \mathbb{R}$ are of special interest. The vector field

$$v = v^i(x)\,\partial_i \ , \quad \partial_i = \frac{\partial}{\partial x^i} \ , \tag{3a}$$
$$v^i(x) = \left(\partial_\varepsilon \Phi_\varepsilon^i(x)\right)\Big|_{\varepsilon=0} \tag{3b}$$

is called the infinitesimal generator of the group Φ_ε and is an element of the set of all smooth sections on the tangent bundle $\mathcal{T}(\mathcal{M})$. By means of a Taylor series expansion of a one–parameter transformation group Φ_ε it is obvious that the vector field $v = v^i\partial_i$ generates the group action at least locally. In local coordinates the expression

$$\Phi_\varepsilon^i(x) = x^i + \varepsilon v^i(x) + \frac{\varepsilon^2}{2} v\left(v^i\right)(x) + O\left(\varepsilon^3\right) \tag{4}$$

is obtained and the Lie series follows under the assumption of convergence of the Taylor series.

These definitions are the basis for the investigation of the observability and accessibility problem of the system (1) where it is the key observation that the geometric picture of continuous-time systems (see, e.g., (Schlacher and Zehetleitner, 2003)) can be extended to the discrete-time case. For a motivation of this analysis the case of explicit dynamic systems is discussed preliminarily.

3. EXPLICIT SYSTEMS

A nonlinear explicit discrete-time dynamic system is assumed to be given in the form

$$x_{k+1}^{\alpha_x} = f^{\alpha_x}(k, x_k, u_k) \ , \quad \alpha_x = 1, \ldots, n_x \tag{5a}$$
$$y_k^{\alpha_y} = c^{\alpha_y}(k, x_k, u_k) \ , \quad \alpha_y = 1, \ldots, n_y \tag{5b}$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$, $u_k \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$, $y_k \in \mathcal{Y} \subseteq \mathbb{R}^{n_y}$ denote the state, input and output variables, respectively. In order to give a geometric interpretation of the dynamic system (5) the trivial bundles $\mathcal{E}_\mathcal{X} = (\mathcal{B} \times \mathcal{X}, \mathrm{pr}_1, \mathcal{B})$, $\mathcal{E}_\mathcal{U} = (\mathcal{B} \times \mathcal{U}, \mathrm{pr}_1, \mathcal{B})$ with the discrete set $\mathcal{B} \subseteq \mathbb{Z}$, the smooth n_x–dimensional state manifold $\mathcal{X}$, and the smooth n_u–dimensional input manifold $\mathcal{U}$ are considered. Furthermore, adapted coordinates $(k, x_k^{\alpha_x})$, $(k, u_k^{\alpha_u})$ are introduced at least locally for $\mathcal{E}_\mathcal{X}$ and $\mathcal{E}_\mathcal{U}$, respectively. Note that the base set $\mathcal{B}$, which can be interpreted as a zero–dimensional discrete manifold, is clearly not connected but all fibers of the bundles $\mathcal{E}_\mathcal{X}$, $\mathcal{E}_\mathcal{U}$ are smooth manifolds and assumed to be diffeomorphic to a typical fiber. With means of the fibered product $\mathcal{E}_\mathcal{X} \times_\mathcal{B} \mathcal{E}_\mathcal{U}$ and the bundle $\mathcal{E}_\mathcal{E} = (\mathcal{E}_\mathcal{X} \times_\mathcal{B} \mathcal{E}_\mathcal{U}, \mathrm{pr}_1 \times_\mathcal{B} \mathrm{pr}_1, \mathcal{B})$ it follows quite straightforward that the functions f and c of system (5) are mappings of the form $f : \mathcal{E}_\mathcal{E} \to \mathcal{X}$ and $c : \mathcal{E}_\mathcal{E} \to \mathcal{Y}$. Furthermore, solutions of (5a) are sections of $\mathcal{E}_\mathcal{X}$. With this geometric picture of system (5) all basics for the study of observability and accessibility properties are at our disposal.

3.1 Observability

For the following it is assumed that a symmetry group $\Phi_\varepsilon : \mathcal{E}_\mathcal{E} \to \mathcal{E}_\mathcal{E}$ can be found which is acting on the state variables of system (5) only,

$$(k, \overline{x}_k, u_k) = \Phi_\varepsilon(k, x_k, u_k) = \Phi_{k,\varepsilon} \ , \tag{6}$$

such that the output functions $c_k^{\alpha_y}$ are invariants of $\Phi_{k,\varepsilon}$. If there exists such a non-trivial transformation group, the dynamic system is not observable and one can find a diffeomorphism $\varphi_k : (x_k^{\alpha_x}, u_k^{\alpha_u}) \rightarrow (r_k^{\alpha_r}, s_k^{\alpha_s}, u_k^{\alpha_u})$, $\alpha_r = 1, \ldots, n_r \geq 1$, $\alpha_s = 1, \ldots, n_x - n_r$ such that (5) takes the form

$$r_{k+1}^{\alpha_r} = \bar{f}^{\alpha_r}(k, r_k, u_k) \quad (7a)$$

$$s_{k+1}^{\alpha_s} = \bar{f}^{n_r+\alpha_s}(k, r_k, s_k, u_k) \quad (7b)$$

$$y_k^{\alpha_y} = \bar{c}^{\alpha_y}(k, r_k, u_k) \ . \quad (7c)$$

To test whether such a non-trivial transformation group exists the infinitesimal invariance criterion with the infinitesimal generator of the symmetry group $v|_{x_k} = v_k = (\partial_\varepsilon \Phi_{k,\varepsilon})|_{\varepsilon=0} = X^{\alpha_x}(k, x_k, u_k)\partial_{\alpha_x} = X_k^{\alpha_x}\partial_{\alpha_x}$ which reads as

$$v_k\left(c_k^{\alpha_y}\right) = X_k^{\beta_x}\partial_{\beta_x}c_k^{\alpha_y} = \left\langle \mathrm{d}c_k^{\alpha_y}, v_k \right\rangle = 0 \quad (8)$$

is investigated. Furthermore, a system of difference equations is invariant under a group of transformation if and only if the flow ϕ_k^f of (5) and the action of the transformation group commute,

$$\Phi_{k,\varepsilon} \circ \phi_k^f = \phi_k^f \circ \Phi_{k,\varepsilon} \ ,$$

and consequently, the infinitesimal criterion

$$\sigma\left(X_k^{\alpha_x}\right) = X_{k+1}^{\alpha_x} = X_k^{\beta_x}\partial_{\beta_x}f_k^{\alpha_x} \quad (9)$$

follows. Since the invariance criterion (8) has to be fulfilled for all time steps additional restrictions by applying the time shift operator σ to (8) and substitution of the commutator criterion (9) are obtained. In a first step this procedure results in $\langle \omega_1, v_k \rangle = 0$ with $\omega_1 = \partial_{\alpha_x} c_{k+1}^{\alpha_y} \partial_{\beta_x} f_k^{\alpha_x} \mathrm{d}x^{\beta_x}$. The repetition of these operations leads finally to the algebraic restrictions

$$\langle \omega_i, v_k \rangle = 0 \ , \quad i = 0, \ldots, n_x - 1 \ , \quad (10)$$

using the abbreviations for the 1–forms

$$\omega_0 = \partial_{\alpha_x} c_k^{\alpha_y} \mathrm{d}x^{\alpha_x}$$
$$\omega_i = \partial_{\alpha_x} c_{k+i}^{\alpha_y} \partial_{\beta_x} f_{k+i-1}^{\alpha_x} \partial_{\gamma_x} f_{k+i-2}^{\beta_x} \ldots \partial_{\eta_x} f_k^{\zeta_x} \mathrm{d}x^{\eta_x} \ ,$$
$$i = 1, \ldots, n_x - 1$$

and for the partial derivatives

$$\partial_{\alpha_x} c_{k+i}^{\alpha_y} = \partial_{x_{k+i}^{\alpha_x}} c_{k+i}^{\alpha_y} \ .$$

Thus, for a given input sequence u_k a test on observability follows by checking whether the 1–forms $\{\omega_i\}$ are linear independent. If this is the case, only the trivial infinitesimal generator v_k exists and the system (5) is locally observable. Furthermore, it is also straightforward to ask whether there exists an input sequence such that the given system (5) is locally observable.

3.2 Accessibility

For the investigation of the accessibility property of (5) a set of one–parameter Lie groups acting on the dependent variables x_k, u_k is assumed and furthermore, let

$$\Phi_\varepsilon : (k, x_k, u_k) \rightarrow (k, \overline{x}_k, \overline{u}_k)$$

with infinitesimal generator $v|_{x_k} = v_k = X_k^{\alpha_x}\partial_{\alpha_x} + U_k^{\alpha_u}\partial_{\alpha_u}$ be a subset of the considered Lie groups. If this subset owns a common non-trivial invariant function $I(k, x_k) = I_k$, then the system (5) is not accessible and there exists a diffeomorphism $\varphi_k : (x_k^{\alpha_x}, u_k^{\alpha_u}) \rightarrow (r_k^{\alpha_r}, s_k^{\alpha_s}, u_k^{\alpha_u})$, $\alpha_r = 1, \ldots, n_r$, $\alpha_s = 1, \ldots, n_x - n_r \geq 1$ such that (5) takes the form

$$r_{k+1}^{\alpha_r} = \bar{f}^{\alpha_r}(k, r_k, s_k, u_k) \quad (11a)$$

$$s_{k+1}^{\alpha_s} = \bar{f}^{n_r+\alpha_s}(k, s_k) \ . \quad (11b)$$

In order to investigate the existence of an invariant I_k again the infinitesimal criterion is analyzed which follows as

$$v_k(I_k) = \langle \omega_k, v_k \rangle = 0 \ , \quad \omega_k = \omega_{k,\alpha_x}(k, x_k)\,\mathrm{d}x^{\alpha_x} \ . \quad (12)$$

Since the considered transformation groups have to be symmetry groups also the infinitesimal invariance criterion reads as

$$\sigma\left(X_k^{\alpha_x}\right) = \partial_{\beta_x} f_k^{\alpha_x} X_k^{\beta_x} + \partial_{\alpha_u} f_k^{\alpha_x} U_k^{\alpha_u} \ . \quad (13)$$

Furthermore, from $\sigma(\langle \omega_k, v_k \rangle) - \langle \omega_k, v_k \rangle = 0$ which must hold for any choice of X_k and U_k one derives

$$\omega_{k+1,\alpha_x}\partial_{\beta_x} f_k^{\alpha_x} - \omega_{k,\beta_x} = 0 \ ,$$
$$\omega_{k+1,\alpha_x}\partial_{\alpha_u} f_k^{\alpha_x} = 0 \ .$$

Under the assumption that $\left[\partial_{\beta_x} f_{k+i}^{\alpha_x}\right]$ is non-singular and $\Delta = \mathrm{span}\{\bar{w}_j\}$ is the involutive closure of the vector fields

$$w_i = \left(\partial_{\eta_x} f_k^{\zeta_x}\right)^{-1} \ldots \left(\partial_{\beta_x} f_{k+i}^{\alpha_x}\right)^{-1} \partial_{\alpha_u} f_{k+i}^{\alpha_x} \partial_{\eta_x}$$

the algebraic restrictions of (12) and (5) are obtained as

$$\langle \omega_k, \bar{w}_j \rangle = 0 \ , \quad j = 0, \ldots, n_x - 1 \ .$$

If $\dim(\Delta) = n_x$ holds, then for ω_k there exists only the trivial solution which implies that no invariant function I_k of the considered subset of Lie groups can be found and consequently, the system is locally strongly accessible. Therefore, the test on local strong accessibility of (5) follows by checking the dimension of the distribution Δ. It has also to be mentioned that in the case of a singular Jacobi-matrix $\left[\partial_{\beta_x} f_{k+i}^{\alpha_x}\right]$ separate investigations are necessary.

4. IMPLICIT SYSTEMS

In this section the analysis of observability and accessibility properties of discrete-time dynamic systems is generalized to the case of implicit systems. It is worth mentioning that for implicit dynamic systems the distinction between state and input variables is in general not valid a priori as it is for explicit systems and therefore, a new coordinate $z \in \mathcal{Z} \subseteq \mathbb{R}^{n_z}$, cp. (1), is used for describing implicit dynamic systems. It will be shown in the following that the implicit system (1) is locally equivalent to an explicit one under a certain regularity condition, cp., e.g., (Schlacher *et al.*, 2002) for continuous-time systems. For the presented formal approach, however, it is not necessary to have the associated explicit system at our disposal but a certain normal form which for continuous-time systems is called formally integrable.

For illustrating the corresponding geometric picture of the dynamic system (1) the trivial bundle $\mathcal{E}_{\mathcal{Z}} = (\mathcal{B} \times \mathcal{Z}, \mathrm{pr}_1, \mathcal{B})$ is introduced where the base manifold $\mathcal{B} \subseteq \mathbb{Z}$ is again a zero–dimensional discrete manifold and $\mathcal{Z}$ indicates an n_z–dimensional smooth manifold. With the trivial bundle $\mathcal{E}_{\mathcal{I}} = (\mathcal{E}_{\mathcal{Z}} \times \mathcal{Z}, \mathrm{pr}_1, \mathcal{E}_{\mathcal{Z}})$ and adapted coordinates $(k, z_k^{\alpha_z})$ for $\mathcal{E}_{\mathcal{Z}}$ and $(k, z_k^{\alpha_z}, \sigma(z_k^{\alpha_z}))$ for $\mathcal{E}_{\mathcal{I}}$ it follows that the geometric picture of (1a) is that of a submanifold $\mathcal{S}$ of $\mathcal{E}_{\mathcal{I}}$.

Confining the considerations to systems which are described by a regular submanifold $\mathcal{S} \subset \mathcal{E}_{\mathcal{I}}$ and which do not contain any hidden constraints the system (1a) can be rewritten as a coupled system of difference and algebraic equations

$$0 = \bar{f}^{\alpha_x}(k, z_k, z_{k+1}) \ , \quad \alpha_x = 1, \ldots, n_x \tag{14a}$$
$$0 = \bar{f}^{n_x+\alpha_s}(k, z_k) \ , \quad \alpha_s = 1, \ldots, n_e - n_x \tag{14b}$$

at least locally where the functions of the set $\left\{\bar{f}^{\alpha_x}, \sigma\left(\bar{f}^{\alpha_s}\right)\right\}$ are functionally independent with respect to z_{k+1} on $\mathcal{S}$ and the output functions $y_k^{\alpha_y} = c^{\alpha_y}(k, z_k)$ remain unchanged. In the case that the system (1) contains additional hidden constrains, the corresponding normal form (14) is obtained by repeatedly applying the time shift operator to the algebraic equations followed by substitution of the difference equations, cp. again, e.g., (Schlacher *et al.*, 2002) for continuous-time systems. This procedure stops, if no additional constraints are obtained and the system is equivalent to (1). The system in the form (14) is now equivalent to an explicit one by a local diffeomorphism $\psi_k : (z_k^{\alpha_z}) \to (x_k^{\alpha_x}, s_k^{\alpha_s}, u_k^{\alpha_u})$,

$$x_k^{\alpha_x} = \psi_k^{\alpha_x} \ , \quad \alpha_x = 1, \ldots, n_x \tag{15a}$$
$$s_k^{\alpha_s} = \psi_k^{n_x+\alpha_s} \ , \quad \alpha_s = 1, \ldots, n_e - n_x \tag{15b}$$
$$u_k^{\alpha_u} = \psi_k^{n_e+\alpha_u} \ , \quad \alpha_u = 1, \ldots, n_z - n_e \tag{15c}$$

where in particular $s_k^{\alpha_s} = \psi_k^{n_x+\alpha_s} = \bar{f}^{n_x+\alpha_s}$ is met, such that the system reads in the new coordinates as

$$\tilde{f}^{\alpha_x}(k, x_k, x_{k+1}, u_k, u_{k+1}) = 0 \tag{16a}$$
$$s_{k+i}^{\alpha_s} = 0 \ , \quad i \geq 0 \tag{16b}$$

with the algebraic restrictions in the trivial form. Since by construction of the coordinate transformation ψ the matrix $\left[\partial_{x_{k+1}^{\alpha_x}} \tilde{f}^{\beta_x}\right]$ is non-singular, the explicit form

$$x_{k+1}^{\alpha_x} = \hat{f}^{\alpha_x}(k, x_k, u_k, u_{k+1}) \tag{17a}$$
$$s_{k+i}^{\alpha_s} = 0 \ , \quad i \geq 0 \tag{17b}$$

follows immediately by the implicit function theorem. It has to be mentioned that in general besides the input $u_k^{\alpha_u}$ the system (17) depends on the time shifted input $u_{k+1}^{\alpha_u}$ also.

For the further investigations of the observability and accessibility properties it is assumed that the implicit system is given in the form (14) which does not mean any restriction of generality as mentioned before.

4.1 Observability

In analogy to the observability analysis of explicit systems a symmetry group with an infinitesimal generator $v_k = Z^{\alpha_z}\partial_{\alpha_z}$ is assumed. Since the symmetry group may only act on the state variables and additionally the functions $s_k^{\alpha_s}$ of (15) have to be invariants of this symmetry group the restrictions

$$v_k\left(\psi_k^{n_e+\alpha_u}\right) = \left\langle \mathrm{d}\psi_k^{n_e+\alpha_u}, v_k \right\rangle = 0 \tag{18a}$$
$$v_k\left(\psi_k^{n_x+\alpha_s}\right) = \left\langle \mathrm{d}\psi_k^{n_x+\alpha_s}, v_k \right\rangle = 0 \tag{18b}$$

have to be met with the diffeomorphism (15). The infinitesimal criterion for a local non-observable system follows in a straightforward manner as

$$v_k\left(c_k^{\alpha_y}\right) = \left\langle \mathrm{d}c_k^{\alpha_y}, v_k \right\rangle = 0 \ . \tag{19}$$

In contrast to the analysis of explicit systems the conditions for the invariance of the flow of the implicit system (1) under the transformation group can not be stated explicitly. For this task in general nonlinear equations have to be solved. By studying whether there exists a non-trivial solution for v_k such that (18) and (19) hold a criterion for observability of implicit dynamic systems is obtained.

4.2 Accessibility

As shown for explicit systems the accessibility considerations are based on finding an invariant function I_k of a subset of Lie groups. The infinitesimal generator of these Lie groups reads for implicit systems as $v_k = Z^{\alpha_z}\partial_{\alpha_z}$. Since the functions $s_k^{\alpha_s}$ of (15) must be invariants of the considered subset of Lie groups the restrictions

$$v_k\left(\psi_k^{n_x+\alpha_s}\right) = \left\langle \mathrm{d}\psi_k^{n_x+\alpha_s}, v_k\right\rangle = 0 \tag{20}$$

have to be fulfilled for the corresponding infinitesimal generator v_k. The problem of strong accessibility is solved by considering the infinitesimal criterion for the invariant function I_k,

$$v_k(I_k) = \langle \omega_k, v_k\rangle = 0 \ , \tag{21}$$

as well as the invariance of the flow of the implicit system (1) under the transformation group and finding all hidden constraints.

5. NONLINEAR CONTROL

The theory of nonlinear control offers comprehensive controller design methods for nonlinear continuous-time systems. However, these nonlinear control laws are commonly implemented on a digital processor in a quasi-continuous-time manner by assuming that the sample time is sufficiently small. E.g., in (Nijmeijer and van der Schaft, 1991) a controller design is discussed where it is assumed that the nonlinear dynamic system is given in a discrete-time form.

In this contribution it is the intention to present a nonlinear discrete-time controller design approach which is based on the knowledge of the continuous-time control system. In particular, we want to consider the class of nonlinear systems that is exactly input-to-state linearizable by a static feedback, cp., e.g., (Isidori, 1995). In a first step, this approach is investigated for single-input single-output control systems.

5.1 Single-input single-output systems

Let us consider a nonlinear continuous-time dynamic system of the form

$$\dot{x}^{\alpha_x} = f^{\alpha_x}(t, x, u) \ , \quad \alpha_x = 1, \ldots, n \tag{22a}$$

$$y = c(t, x) \tag{22b}$$

which has a relative degree $r = n$ for the considered output y. With a diffeomorphic coordinate transformation

$$z^{\alpha_x} = \varphi^{\alpha_x}(t, x) \tag{23}$$

and a nonlinear control law

$$u = \mu(t, z, v) \ , \tag{24}$$

with new input v, the system (22) is therefore static feedback equivalent to a linear and time-invariant one.

By applying the diffeomorphic coordinate transformation (23) to the nonlinear system (22) the system reads in the new coordinates z as

$$\dot{z}^{\alpha_x} = \bar{f}^{\alpha_x}(t, z, u) \tag{25a}$$

$$y = \bar{c}_{\alpha_x} z^{\alpha_x} \tag{25b}$$

where the output function is linear and time-invariant. It is now the objective of this approach to find a discrete-time control sequence such that the trajectories z^{α_x} of (25a) evaluated at the discrete time steps kT_s coincide with the trajectories $z_k^{\alpha_x}$ of a desired linear and time-invariant discrete-time control system

$$z_{k+1}^{\alpha_x} = a_{\beta_x}^{\alpha_x} z_k^{\beta_x} + b^{\alpha_x} v_k \ , \quad \alpha_x = 1, \ldots, n \tag{26}$$

with input v_k and sample time T_s. For this purpose let us assume an n–times faster sample time $T_{s,f}$ and a control sequence $(u_{k,0}, \ldots, u_{k,n-1})$, with $u_{k,j}$ the on the time interval $[\tau_j, \tau_{j+1})$, $\tau_j = kT_s + jT_{s,f}$, constant control input of (25a). In order to meet the desired equivalence of (25a) and (26) at the discrete time steps kT_s the n conditions

$$z^{\alpha_x}(T_2) = \phi_{\tau_n,\tau_{n-1}}^{\alpha_x}(u_{k,n-1}) \circ \ldots \circ \phi_{\tau_2,\tau_1}(u_{k,1}) \circ \phi_{\tau_1,\tau_0}(u_{k,0}, z(T_1)) \tag{27}$$

follow, where the abbreviations $T_1 = kT_s$, $T_2 = (k+1)T_s$ are used and $\phi_{\tau_{j+1},\tau_j}^{\alpha_x}(u_{k,j}, z(\tau_j))$ denotes the flow of the nonlinear system (25) valid on the time interval $[\tau_j, \tau_{j+1})$ with control input $u_{k,j}$ and initial value $z(\tau_j)$, see figure 1. Since, in general, for a nonlinear system the

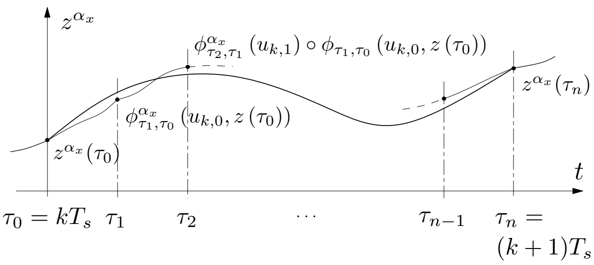

Fig. 1. About discrete-time control of single-input single-output continuous-time systems under consideration of two different sample times.

flow ϕ^{α_x} can not be calculated explicitly an approximation, e.g., a Lie series approximation, cp. equ. (4), can be used and the control sequence $(u_{k,0}, \ldots, u_{k,n-1})$ follows by solving the n nonlinear algebraic equations (27). This calculation can be done at least numerically. It is worth mentioning that the control sequence has to be computed at time steps kT_s exclusively and only the hold element has to run with the fast sample time $T_{s,f}$, see figure 2. Furthermore, this control design approach passes over to the input-to-state linearization of continuous-time systems by taking the limit $T_s \to 0$.

Since the obtained closed loop system is linear and time-invariant an extensive theory for discrete-time control design is available. Especially, the concept of trajectory planning and trajectory

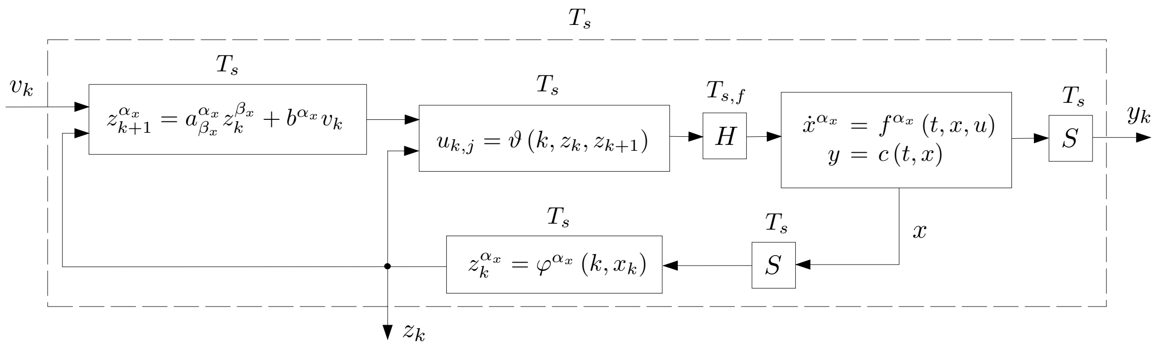

Fig. 2. Scheme of the single-input single-output discrete-time control concept.

tracking, see, e.g., (Fliess *et al.*, 1995), can therefore be applied in discrete-time in a straightforward manner. Furthermore, it has to be mentioned that this approach is also applicable to implicit systems since the corresponding formally integrable system is at least locally equivalent to

$$\dot{x}^{\alpha_x} = f^{\alpha_x}(t, x, u, v)$$
$$\dot{u} = v \; .$$

5.2 Multi-input multi-output systems

Considering a multi-input multi-output system of the form

$$\dot{x}^{\alpha_x} = f^{\alpha_x}(t, x, u) \; , \quad \alpha_x = 1, \ldots, n_x \qquad (28a)$$
$$y^{\alpha_y} = c^{\alpha_y}(t, x) \; , \quad \alpha_y = 1, \ldots, n_y \qquad (28b)$$

which shows for the considered outputs y^{α_y} the property that it is exact input-to-state linearizable. Furthermore, it is assumed that the number of control inputs is equal to the number of outputs, $n_u = n_y$. The approach discussed for single-input single-output systems can be extended in such a way that the fast sample time $T_{s,f}$ is chosen r_{max}–times faster than the sample time T_s where r_{max} follows as

$$r_{max} = \max_{i=1,\ldots,n_y}(r_i)$$

and r_i are the elements of the relative degree $r = \left(r_1, \ldots, r_{n_y}\right)$. Analogously, by means of a diffeomorphic coordinate transformation $z^{\alpha_x} = \varphi^{\alpha_x}(t, x)$ one obtains the corresponding conditions (27) for a multi-input multi-output system such that the system is at discrete-time steps equivalent to a linear and time-invariant discrete-time one but in contrast to the single-input single-output case $(r_{max} n_u - n_x)$ control inputs can be chosen arbitrary.

6. CONCLUSIONS

This contribution deals with a Lie group approach for the observability and accessibility analysis of discrete-time dynamic systems described by implicit difference equations. The fundamental observation that the geometric picture of the Lie group approach for continuous-time systems is also appropriate in the discrete-time case is the key for these investigations. Furthermore, a nonlinear discrete-time control design for continuous-time dynamic systems is presented. It is the objective of this design approach to find a control sequence such that the closed loop system is equivalent to a desired linear and time-invariant discrete-time control system. In order to solve this problem two related sample times are used.

REFERENCES

Boothby, W.M. (1986). *An Introduction to Differentiable Manifolds and Riemannian Geometry.* 2nd ed.. Academic Press Inc.

Fliess, M., J. Levine, P. Martin and P. Rouchon (1995). Flatness and defect of non-linear systems: Introductory theory and examples. *Internat. J. Control* **61**, 1327–1361.

Isidori, A. (1995). *Nonlinear Control Systems.* 3rd ed.. Springer.

Monaco, S. and D. Normand-Cyrot (2001). Issues on Nonlinear Digital Control. *European Journal of Control* (7), 160–177.

Nijmeijer, H. and A.J. van der Schaft (1991). *Nonlinear Dynamical Control Systems.* Springer.

Olver, P.J. (1993). *Application of Lie Groups to Differential Equations.* 2nd ed.. Springer.

Rudolph, J. (2003). *Flatness Based Control of Distributed Parameter Systems.* Shaker.

Saunders, D.J. (1989). *The Geometry of Jet Bundles.* Cambridge University Press.

Schlacher, K., A. Kugi and K. Zehetleitner (2002). A lie group approach for nonlinear dynamic systems described by implicit ordinary differential equations. *Proceedings in Mathematical Theory of Networks and Systems (MTNS).*

Schlacher, K. and K. Zehetleitner (2003). Computer algebra algorithms for control related tests of implicit dynamic systems. *Proceedings in Applied Mathematics and Mechanics (PAMM)* **3**(1), 140–143.