OPTIMAL SWITCHED LAW OF SWITCHED LINEAR SYSTENS BASED ON CONVERGENCE DIRECTION

LI Jun-sheng, GAO Li-qun

(School of Information Sciences & Engineering, Northeaster University, Shenyang 110004,China)
Correspondent: LI Jun-sheng, E-mail: emailljs312 @sohu.com.

Abstract: A study of optimal switched strategy of switched linear systems in periodic switching is presented here. On the premise that linear switched systems are convergent, optimal switched law is searched to make systems converge at origin in the optimal movement direction. To determine the optimal switched law only requires resolving matrix equation, and the construction of the equation is very simple and precise. Once the optimal switched law is determined, its calculation and configuration form are simple and straightforward. Our simulation examples show that the presented method is simple and effective. At the same time, switched signals designed by the current approach are also very simple. It is in favor of practice application in engineering. The method concluded here can be directly adopted in many kinds of control system design, and it has greate value for both theory and practice.

Keywords: switched linear systems; periodical switching sequence; asymptotical stabilization; switched law; Gymnastics Robot; Control strategy

1. INTRODUCTION

A switched linear system is a special kind of hybrid system. In general, it mostly studies the stability of a system (Philippos , *et al.* 1989; Michael, 1994; J Malmborg, *et al.* 1996). It also addresses some other problems, such as design of switched law(Frommer, *et al.*1998; Wang, *et al.*1998), stability of switched linear system with constant coefficients(M A Wichs, *et al.*1994) and quadratic stability(J Malmborg, *et al.*1996). Moreover, time optimal control and the performances and complexity of switched control strategy of hybrid systems are also studied(Riedinger, *et al.*1999; Kullarni, *et al.*1997).

McGeer, T. (1990) and Goswami, *et al.* (1997) had studied some mathematic models of switched systems and presented a switched control problem of motion system based on compass gait biped. Further more, based on their study, Mark W.Spopng presented the switched stability control problem for Gymnastics Robot in 2000(Mark 2000).

In the movement process of Gymnastics Robot, two control strategies are adopted: the swing-up controller and the balancing controller.

Under the control of swing-up controller, Robot Gymnastics fulfill every motion planned. After 20 seconds, it automatically switches to be controlled under balancing controller. Then the system achieves stabilization under the control of balancing controller. Demand of stability was hardly satisfied only by using swing-up controller to control it. Therefore a switched system was introduced; two controllers were switched to improve the stability performance of system.

Gymnastics Robot is a switched system having two subsystems. First, it fulfills every swing-up under control of swing-up controller, after 20 seconds, it is automatically switched to balancing controller to make the system stable again, then switched to swing-up controller. The switching process above is repeated, then a switched system is formed.

The switched system is continually switched between the two controllers to reach an ideal and stable motion state.

Then the stable control problem for Gymnastics Robot is effectively resolved.

A switched problem between two subsystems can be abstracted from the control model of Gymnastics Robot.

Consider a switched linear system that consists of two subsystems

$$\dot{X} = A_i X,\ i = 1,2,\ \mathbf{X} \in R^n,$$
$$\mathbf{A}_i \in R^{n\times n}, \mathbf{X}_0 = \mathbf{X}(0) \in R^n \quad (1)$$

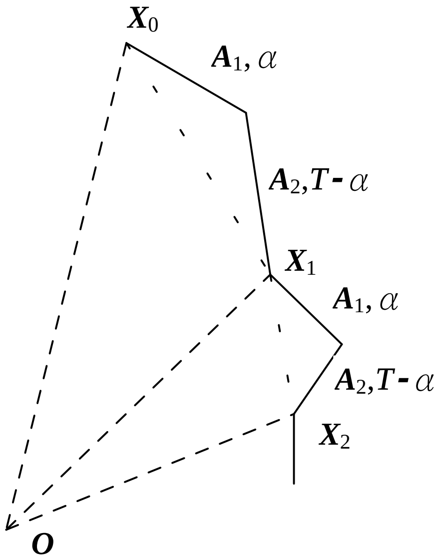

Figure 1. The state movement track of switched linear system

Where A_1, A_2 are system matrices of the subsystems. And X is the system state vector.

Then the switched law is

$$S=\{X_0:(a,A_1),(T-a,A_2);\mathbf{L}\} \tag{2}$$

Where T is switching period length, therefore it is called as periodic switching strategy. When adopting this strategy, the system (1) will start to run from t=0, at first it will run for a time a according to $\dot{X}=A_1X$, and then run for a time ($T-a$) according .

To $\dot{X}=\mathbf{A_2X}$ to finish a switching period and then repeat the process above from $t=T$, and will switch cyclically in this mode. If the switching period T has been given, a period switching strategy is determined by time interval a and the running sequence of the subsystems in the period. Therefore, the design of period switching strategy is also the selection of a and the determination of the running sequences of the subsystems in the period. Commonly, if system (1) can achieve asymptotical stability by switching, the value a and the switching sequences aren't exclusive, so a problem for optimal switching occurs.

The system starts to run from the initial state point X_0, then it reaches X_1 after a movement period.

Marking the angle between vectors $\overrightarrow{X_0O}$ and $\overrightarrow{X_0X_1}$ as $<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>$, then the angle denotes movement direction of the system in the first cycle period.

The smaller the angle; the more it can converge at origin O. That is, the system is under the optimal state in the convergence process.

This paper presents a method about how to seek the optimal value a that is marked as a^*, and optimal switched law based on convergence direction is designed.

2.PREPARATIVE WORK

Lemma 1 Suppose $X_0\in R^n$, $D(a)\in R^{n\times n}$ are matrices about variable a , then

$$\frac{d}{da}[X_0^T D(a)X_0]=X_0^T[\frac{d}{da}D(a)]X_0 \tag{3}$$

Proof: For instance, a second-order system is shown as follow. Assume

$$D(a)=\begin{bmatrix} d_{11}(a) & d_{12}(a)\\ d_{21}(a) & d_{22}(a)\end{bmatrix},\ X_0=\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}$$

then

$$\frac{d}{da}D(a)=\begin{bmatrix}\frac{d}{da}d_{11}(a) & \frac{d}{da}d_{12}(a)\\ \frac{d}{da}d_{21}(a) & \frac{d}{da}d_{22}(a)\end{bmatrix}$$

and

$$X_0^T D(a)X_0=\begin{bmatrix}x_{01} & x_{02}\end{bmatrix}\begin{bmatrix} d_{11}(a) & d_{12}(a)\\ d_{21}(a) & d_{22}(a)\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}$$

$$=d_{11}(a)x_{01}^2+d_{12}(a)x_{01}x_{02}+d_{21}(a)x_{01}x_{02}+d_{22}(a)x_{02}^2$$

so

$$\frac{d}{da}[X_0^T D(a)X_0]$$

$$=\frac{d}{da}d_{11}(a)x_{01}^2+\frac{d}{da}d_{12}(a)x_{01}x_{02}$$

$$+\frac{d}{da}d_{21}(a)x_{01}x_{02}+\frac{d}{da}d_{22}(a)x_{02}^2$$

$$=\begin{bmatrix}x_{01} & x_{02}\end{bmatrix}\begin{bmatrix}\frac{d}{da}d_{11}(a) & \frac{d}{da}d_{12}(a)\\ \frac{d}{da}d_{21}(a) & \frac{d}{da}d_{22}(a)\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}$$

$$=X_0^T[\frac{d}{da}D(a)]X_0.$$

3.PRIMARY RESULTS

The system starts to run from the initial point X_0, then converges at origin.

Consider the angle $<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>$ between vectors $\overrightarrow{X_0O}$ and $\overrightarrow{X_0X_1}$ in the first cycle period. The angle is computed as below.

$$\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>)=\frac{\overrightarrow{X_0O}\cdot\overrightarrow{X_0X_1}}{\left|\overrightarrow{X_0O}\right|\left|\overrightarrow{X_0X_1}\right|} \tag{4}$$

due to

$$\overrightarrow{X_0X_1}=\overrightarrow{X_0O}-\overrightarrow{X_1O}$$

$$\left|\overrightarrow{X_0O}\right|=(X_0^T X_0)^{\frac{1}{2}}$$

$$\left|\overrightarrow{X_0X_1}\right|=[(X_0-X_1)^T(X_0-X_1)]^{\frac{1}{2}}$$

put it into the above formula (4), the angle of convergence direction is:

$$\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>) = \frac{X_0^T(X_0-X_1)}{(X_0^TX_0)^{\frac{1}{2}}[(X_0-X_1)^T(X_0-X_1)]^{\frac{1}{2}}} \quad (5)$$

Via the two subsystems $\dot{X}=A_1X$ and $\dot{X}=A_2X$, the switched system starts to run from the initial state point X_0 to X_1.

The relationship between X_1 and X_0 is $X_1=e^{A_2(T-a)}e^{A_1a}X_0$. Assume $D(a)=e^{A_2(T-a)}e^{A_1a}$, then

$$X_1=D(a)X_0 \quad (6)$$

and then

$$\begin{aligned}&\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>)\\ &=\frac{X_0^T(X_0-X_1)}{(X_0^TX_0)^{\frac{1}{2}}[(X_0-X_1)^T(X_0-X_1)]^{\frac{1}{2}}}\\ &=\frac{X_0^T(X_0-D(a)X_0)}{(X_0^TX_0)^{\frac{1}{2}}[(X_0-D(a)X_0)^T(X_0-D(a)X_0)]^{\frac{1}{2}}}\\ &=\frac{X_0^T(I-D(a))X_0}{(X_0^TX_0)^{\frac{1}{2}}[X_0^T(I-D(a))^T(I-D(a))X_0]^{\frac{1}{2}}}\end{aligned} \quad (7)$$

So we can draw a conclusion thereinafter.

Theorem 1 If a switched linear system converges at origin according to the principle that the direction is minimal, a only need to satisfy equation below.

$$\begin{aligned}&2X_0^T\frac{d(I-D(a))}{da}X_0\cdot X_0^T(I-D(a))^T(I-D(a))X_0\\ &=X_0^T(I-D(a))X_0\cdot X_0^T\frac{d(I-D(a))^T(I-D(a))}{da}X_0\end{aligned} \quad (8)$$

Proof

$$\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>) = \frac{X_0^T(I-D(a))X_0}{(X_0^TX_0)^{\frac{1}{2}}[X_0^T(I-D(a))^T(I-D(a))X_0]^{\frac{1}{2}}}$$

According to lemma 1, we can get

$$\begin{aligned}&\frac{d}{da}\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>)\\ &=\frac{X_0^T\frac{d(I-D(a))}{da}X_0\cdot[X_0^T(I-D(a))^T(I-D(a))X_0]^{\frac{1}{2}}}{(X_0^TX_0)[X_0^T(I-D(a))^T(I-D(a))X_0]}\\ &-\frac{\frac{1}{2}X_0^T(I-D(a))X_0\cdot[X_0^T(I-D(a))^T(I-D(a))X_0]^{-\frac{1}{2}}}{(X_0^TX_0)[X_0^T(I-D(a))^T(I-D(a))X_0]}\\ &\cdot X_0^T\frac{d(I-D(a))(I-D(a))}{da}X_0\end{aligned}$$

The value $\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>)$ is requested to be minimal, the equation below need to be satisfied:

$$\frac{d}{da}\cos(<\overrightarrow{X_0O},\overrightarrow{X_0X_1}>)=0$$

Note that $X_0^TX_0$, $X_0^T(I-D(a))^T(I-D(a))X_0$ are both of single dimension variable. Therefore the formula (8) is got via cleaning up them.

4.SIMULATION EXAMPLES

Consider a switched linear system consists of two second-order subsystems, matrices of the subsystem are respectively $A_1=\begin{bmatrix}1&0\\0&-2\end{bmatrix}$, $A_2=\begin{bmatrix}-4&0\\0&1\end{bmatrix}$, the initial point $X_0=\begin{bmatrix}10\\10\end{bmatrix}$; let's seek the value a of the optimal switched law.

Proposition 1 (Liberzon, *et al.*1999). If the matrix $aA_1+(T-a)A_2$ is stable, a switched linear system is asymptotically stable under some switched law.

According to Proposition 1, a system is asymptotically stable, and it need

$$aA_1+(T-a)A_2=\begin{bmatrix}5a-4T&0\\0&-3a+T\end{bmatrix} \quad (9)$$

is a stable matrix, so $5a-4T<0$, $-3a+T<0$. If assume $T=1$, then $0.33<a<0.8$. When the value a of the optimal switched law satisfies the equation below.

$$\begin{aligned}&2X_0^T\frac{d(I-D(a))}{da}X_0\cdot X_0^T(I-D(a))^T(I-D(a))X_0\\ &=X_0^T(I-D(a))X_0\cdot X_0^T\frac{d(I-D(a))^T(I-D(a))}{da}X_0\end{aligned} \quad (10)$$

Where $D(a)=e^{A_2(T-a)}e^{A_1a}$, correlative data are

$$\begin{aligned}&e^{A_1a}=\begin{bmatrix}e^a&0\\0&e^{-2a}\end{bmatrix}=e^{A_1^Ta}\\ &e^{A_2(T-a)}=\begin{bmatrix}e^{4a-4T}&0\\0&e^{-a+T}\end{bmatrix}\\ &D(a)=e^{A_2(T-a)}e^{A_1a}=\begin{bmatrix}e^{5a-4T}&0\\0&e^{-3a+T}\end{bmatrix}\\ &\quad\triangleq\begin{bmatrix}M&0\\0&N\end{bmatrix}\\ &I-D(a)=\begin{bmatrix}1-e^{5a-4T}&0\\0&1-e^{-3a+T}\end{bmatrix}\\ &\quad=(I-D(a))^T\\ &(I-D(a))^T(I-D(a))\\ &\quad=\begin{bmatrix}(1-e^{5a-4T})^2&0\\0&(1-e^{-3a+T})^2\end{bmatrix}\\ &\quad=\begin{bmatrix}(1-M)^2&0\\0&(1-N)^2\end{bmatrix}\end{aligned}$$

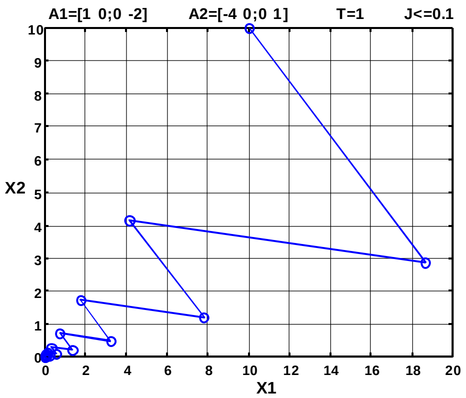

Figure2. The track of the system movement state (a =0.625).

$$\frac{d}{da}(\boldsymbol{I}-\boldsymbol{D}(a)) = \begin{bmatrix} -5e^{5a-4T} & 0 \\ 0 & 3e^{-3a+T} \end{bmatrix} = \begin{bmatrix} -5M & 0 \\ 0 & 3N \end{bmatrix}$$

$$\frac{d}{da}(\boldsymbol{I}-\boldsymbol{D}(a))^T\ (\boldsymbol{I}-\boldsymbol{D}(a)) = \begin{bmatrix} -10M(1-M) & 0 \\ 0 & 6N(1-N) \end{bmatrix}$$

Put these data into equation (10), we can obtain an equation of a .

$$5M^2 - 8M^2N - 2MN + 8MN^2 - 3N^2 = 0 \qquad (11)$$

Where $M = e^{5a-4T}$, $N = e^{-3a+T}$. Assume switching period T =1, then a^* =0.625. The optimal switched law is

$$S = \{X_0; (0.625, A_1), (0.375, A_2); ......\} \qquad (12)$$

If the system runs by the optimal switched law designed, the track of the system state is shown as Fig.2.

5.CONCLUSION

The optimal switched law of switched linear systems was presented in the paper. The computing process for a^* was also given. The problem for design of optimal switched law was effectively resolved. The optimal switched law presented here indicates that the convergence direction of the system state point is minimal. We may also consider obtaining the optimal in other form; thereby a multiple optimal switched law can be realized.

When the value a^* is determined, under the optimal switched law, the switched linear system converges at origin according to the principle that the convergence direction is minimal. Suppose a system runs to the origin via cyclical switching for n times in this mode and reaches stabilization, then the state of the system is

$$\boldsymbol{X}_{2n} = e^{A_2(T-a)}\boldsymbol{X}_{2n-1} = \boldsymbol{D}^n(a)\boldsymbol{X}_0$$

When it enters d adjacent field of the origin, then

$$\|\boldsymbol{X}_{2n}\| = (\boldsymbol{X}_0^T\boldsymbol{D}^{Tn}(a)\boldsymbol{D}^n(a)\boldsymbol{X}_0)^{\frac{1}{2}} \le d \qquad (13)$$

where $\boldsymbol{D}(a) = e^{A_2(T-a)}e^{A_1a}$, hence we can determine the cyclical switching times n, that is, it costs Tn long for the system to enter to the stable region via n times of cyclical switching.

Designing the switched law by using the method presented in this paper is in favor of engineering application, and the signal is very simple; the conclusion drawn here can be directly adopted in many kinds of control system design, and has great value for both theory and practice.

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