STATE-SPACE SOLUTION TO STOCHASTIC H_∞-OPTIMIZATION PROBLEM WITH UNCERTAINTY

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Abstract: Robust stochastic anisotropy-based H_∞-optimization problem for discrete linear time-invariant (LTI) systems with structured parametric uncertainty is considered. It is shown that the problem can be reduced to mixed H_2/H_∞-like problem. The resulting control problem involves the minimization of anysotropic and H_∞ norms of the system. Explicit state-space formulas are also obtained for the optimal controller. The problem covers the standard H_2/H_∞ optimization problem and H_∞-optimization problem as two limiting cases.

Keywords: Stochastic H_∞ - optimization, parametric uncertainty, entropy.

1. INTRODUCTION

Well-known H_2- and H_∞-theories of optimization for LTI systems are based on the H_2- and H_∞-norms as performance criteria. The H_2-control theory assumes that the input disturbance is the Gaussian white noise. The H_∞-control theory assumes it to be a square-summarable signal. As a consequence, use of H_2-optimal controllers in feedback loop leads to poor functioning of a closed-loop system if strongly colored random noise is fed to the input. On the other hand, H_∞-optimal controllers are conservative if the input disturbance is white or slightly colored noise.

Stochastic approach to H_∞-optimization for discrete LTI systems was proposed in (Semyonov *et al.*, 1994). This approach exploits an input signal "colourness" characteristic introduced in (Vladimirov *et al.*, 1995) and called mean anisotropy. Anisotropy norm (Diamond P. *et al.*, 2001) of the closed-loop transfer function is proposed to be the performance criterion. The controller design problem with such performance criterion was solved in (Vladimirov *et al.*, 1996). Stochastic (anisotropy-based) H_∞-optimal controllers are located "between" H_2-optimal and H_∞-optimal controllers. Moreover, H_2- and H_∞-optimal controllers are the limiting cases of anisotropy-based controllers when mean anisotropy of input signal tends to zero or to infinity, respectively. The anisotropy-based optimization problem with mean anisotropy level a will be referred to AB_a-problem.

The problem of robust state feedback H_∞-control design for class of LTI systems with parametric uncertainty was solved in (Xie *et al.*, 1991).

In this paper we formulate and solve the robust anisotropy-based stochastic H_∞ optimization problem for discrete LTI systems with parametric uncertainty. It is shown that the problem can be reduced to mixed H_2/H_∞-like problem (Doyle *et al.*, 1994). The resulting control problem involves the minimization of anisotropic and H_∞-

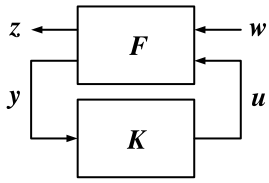

Fig. 1. Closed-loop system

norms of the system. Explicit state-space formulas are also obtained for the optimal controller. The standard H_2/H_∞-optimization problem and H_∞-optimization problem are included as two limiting cases.

2. PROBLEM STATEMENT

We consider the open-loop linear discrete-time-invariant causal system F:

$$\left.\begin{aligned} x_{k+1} &= (A + F_1\Omega_k E_1)\,x_k \\ &\quad + (B_0 + F_2\Phi_k E_2)w_k \\ &\quad + (B_2 + F_3\Psi_k E_3)u_k, \\ z_k &= C_1 x_k + D_{12}u_k, \\ y_k &= C_2 x_k + D_{21}w_k, \end{aligned}\right\} \quad (2.1)$$

where $-\infty < k < \infty$, $\|x_{-\infty}\| < +\infty$, $\|\cdot\|$ denotes the Euclidean norm, $x_k \in \mathbb{R}^n$ is the state, $z_k \in \mathbb{R}^{p_1}$ is the controlled signal, $u_k \in \mathbb{R}^{m_2}$ is the control, $w_k \in \mathbb{R}^{m_1}$ is the disturbance, $y_k \in \mathbb{R}^{p_2}$ is the observation; A, B_0, B_2, C_1, C_2, D_{12}, D_{21},E_1, E_2, E_3, F_1, F_2, F_3 are known matrices of appropriate dimensions; Ω_k, Φ_k, Ψ_k are unknown matrix functions corresponding to unknown parameters, which satisfies

$$\Omega_k^T\Omega_k \le I,\ \Phi_k^T\Phi_k \le I,\ \Psi_k^T\Psi_k \le I, \quad (2.2)$$

where I denotes the identity matrix of appropriate dimension. Denote by $\Delta_k = diag\left(\Omega_k,\ \Phi_k,\ \Psi_k\right)$ all uncertainties in the system. Then (2.2) reduces to $\Delta_k^T\Delta_k \le I$.

The closed-loop transfer function from input W to output Z in figure 1 is given by lower LFT:

$$\mathcal{L}(F,K) = F_{11} + F_{12}K(I - F_{22}K)^{-1}F_{21}.$$

We will use formally defined upper LFT with respect to the matrix Δ:

$$\mathcal{U}(F,\Delta) = F_{22} + F_{21}\Delta(I - F_{11}\Delta)^{-1}F_{12}.$$

Denote by $\mathbf{G}(0,I)$ the class of discrete m-dimensional Gaussian white noise with zero expectation and unit covariance matrix.

Assume that W is a stationary Gaussian sequence whose mean anisotropy is upper-bounded by a known nonnegative parameter a. This means that $W = G \otimes V$, $V \in \mathbf{G}(0,I)$ and $G \in \mathcal{G}_a$, where

$$\mathcal{G}_a \equiv \left\{G \in H_2^{m_1\times m_1} : \overline{A}(G) \le a\right\},$$

and the mean anisotropy of W is defined as

$$\overline{A}(G) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\left(\frac{m}{\|G\|_2^2}\,\widehat{G}(\omega)(\widehat{G}(\omega))^*\right)d\omega.$$

Let us consider the corresponding class of signals

$$\begin{aligned} &W_a = \{w_k \in l_2 \,:\, w_k = Gv_k, \\ &\text{where } V \in \mathbf{G}(0,I),\ G \in \mathcal{G}_a\}\,. \end{aligned}$$

We will use the power norm of a signal u:

$$\begin{aligned} \|u\|_{\mathcal{P}} &= \left(\lim_{N\to\infty}\frac{1}{2N}\sum_{k=-N}^{N}\|u_k\|^2\right)^{1/2} \\ &= \sqrt{\operatorname{Trace} R_{uu}(0)}, \end{aligned}$$

where $R_{uu}(n)$ denotes the auto-correlation function of sequence U. The Fourier transform of $R_{uu}(n)$ is called the spectral density and is denoted by S_{uu}. The signals having bounded power norm and bounded spectrum are referred to as BP-signals and BS-signals, respectively.

The uncertainty Δ_k is called admissible if $\Delta_k \in RH_\infty$ and the system $\mathcal{U}(F,\Delta_k)$ is internally stable. A controller K is called strictly causal if control u_k depends on any instant k only on the preceding observations $y_j, j < k$. A controller K is called admissible if it is strictly causal and it internally stabilizes the closed-loop system on figure 1. The set of all admissible controllers for the given system F is denoted by $\mathbf{K}$ and the set of all admissible uncertainties Δ_k for the system F is denoted by $\mathbb{D}$.

The anisotropic norm of an arbitrary causal system $F \in H_\infty^{p_1\times m_1}$ is

$$|\!|\!|F|\!|\!|_a \equiv \sup_{G\in\mathcal{G}_a}\frac{\|FG\|_{\mathcal{P}}}{\|G\|_{\mathcal{P}}}. \quad (2.3)$$

The robust anisotropy-based stochastic H_∞-optimization problem is formulated as follows:

Problem 1. For the given system (2.1) and input mean anisotropy level $a \ge 0$, find an admissible controller $K \in \mathbf{K}$ that minimizes the maximal value of a-anisotropic norm of the closed-loop transfer function $\mathcal{L}(F,K)$

$$\sup_{\Delta_k\in\mathbb{D}} |\!|\!|\mathcal{L}(F,K)|\!|\!|_a \searrow \inf\,. \quad (2.4)$$

The last criterion is identical to

$$\sup_{\Delta_k\in\mathbb{D}}\sup_{G\in\mathcal{G}_a}\|z\|_{\mathcal{P}}^2 \searrow \inf,\ K \in \mathbf{K}. \quad (2.5)$$

The problem described above we will be referred to as RASHO problem.

The following standard assumptions on system (2.1) will be used throughout the text.

Assumptions

(A)

$$D_{12}^T C_1 = 0,$$
$$D_{12}^T D_{12} = I;$$

(B) the system F is stabilizable and detectable;
(C) $p_1 < m_1$;
(D) the matrix D_{21} in (2.1) has full row rank;
(E) the matrix D_{12} in (2.1) has full column rank.

3. EMBEDDING INTO THE GENERALIZED ANISOTROPY-BASED H_∞-PROBLEM

In this section we reduce the RASHO problem to an anisotropy-based H_∞-optimization problem for a certain system. Let us consider auxiliary system $\widetilde{F}$

$$\left.\begin{aligned} x_{k+1} &= Ax_k + B_0 w_k + B_2 u_k + B_1 \eta_k, \\ z_k &= C_1 x_k + D_{12} u_k, \\ y_k &= C_2 x_k + D_{21} w_k, \end{aligned}\right\} \quad (3.6)$$

where $B_1 \equiv (F_1, F_2, F_3)$ and all the matrices are the same as in (2.1). System (3.6) has three inputs. Bounded spectrum signal w_k is fed to the first one, bounded power signal η_k is fed to the second one, and control signal u_k is fed for the third one. Further, the first and the second input of system (3.6) will be referred as BS-input and BP-input, respectively. Original system (2.1) coincides with system (3.6) if $\eta_k = col(\eta_{1k}, \eta_{2k}, \eta_{3k})$, where

$$\eta_k = \Delta_k \begin{pmatrix} E_1 x_k \\ E_2 w_k \\ E_3 u_k \end{pmatrix}. \quad (3.7)$$

Since $w_k \in l_2$ and $u_k \in l_2$, we have $x_k \in l_2$, and, consequently, $\eta_k \in l_2$. From (2.2) it follows

$$\begin{aligned} \|\eta_{1k}\| &\le \|E_1 x_k\|, \\ \|\eta_{2k}\| &\le \|E_2 w_k\|, \\ \|\eta_{3k}\| &\le \|E_3 u_k\|. \end{aligned} \quad (3.8)$$

The set of all η_k satisfying (3.8) is denoted by $I\!D_\eta$. The equality (3.7) means that η_k causally depends on w_k. It may be either $S_{w_1 w_0} \equiv 0$ or there exists $\mathcal{M}(s) \in H_2$ such that $\eta(s) = \mathcal{M}(s) w(s)$. Let us formulate new optimization

Problem 2. For given system (3.6) and input mean anisotropy level $a \ge 0$, find an admissible controller $K \in \mathbf{K}$ that minimizes the maximal value of a-anisotropic norm of closed-loop system transfer function $\mathcal{L}(\widetilde{F}, K)$

$$\sup_{\eta_k \in I\!D_\eta} \|\mathcal{L}(\widetilde{F}, K)\|_a \searrow \inf. \quad (3.9)$$

Let us find the controller $K(s)$ in the following form:

$$\begin{aligned} \xi_{k+1} &= \widehat{A} \xi_k + \widehat{B} y_k, \\ u_k &= \widehat{C} \xi_k. \end{aligned} \quad (3.10)$$

Then $\zeta_k \equiv col(x_k, \xi_k)$ is the state of the system $\mathcal{L}(\tilde{F}, K)$. Denote

$$Q \equiv \begin{pmatrix} C_1^T C_1 + \gamma_1^2 E_1^T E_1 & 0 \\ 0 & \widehat{C}^T (I + \gamma_3^2 E_3^T E_3) \widehat{C} \end{pmatrix},$$
$$\Gamma \equiv diag\left\{\gamma_1^2 I, \gamma_2^2 I, \gamma_3^2 I\right\},$$
$$S_0 \equiv \gamma_2^2 E_2^T E_2,$$

where $\gamma_i, i = \overline{1,3}$, are some scalar values and let $\gamma = col\{\gamma_1, \gamma_2, \gamma_3\}$. Define $\Theta \equiv \sum_{k=-\infty}^{\infty} (\zeta_k^T Q \zeta_k - \eta_k^T \Gamma \eta_k + w_k^T S_0 w_k)$.

Problem 3. Let us fix $\gamma_i, i = \overline{1,3}$. For given system (3.6) and input mean anisotropy level $a \ge 0$ find an admissible controller $K \in \mathbf{K}$ that minimizes

$$J(K, \gamma) \equiv \sup_{w \in W_a} \sup_{\eta_k} \Theta$$

Theorem 1. Let K_γ be an admissible controller, which minimizes the cost function $J(K, \gamma)$ for some fixed $\gamma \ne 0$. Then K_γ minimizes the cost

$$J_1(K, \gamma) \equiv \sup_{\eta_k \in I\!D_\eta} \sup_{w \in W_a} \|z\|_{\mathcal{P}}^2$$

and

$$\inf_{K \in \mathbf{K}} \sup_{\Delta_k \in I\!D} \|\mathcal{L}(\mathcal{U}(F, \Delta_k), K)\|_a = \inf_{\gamma} \inf_{K \in \mathbf{K}} J_1(K, \gamma).$$

This means that we can reduce the RASHO problem to a new problem which are called the mixed AB_a/H_∞-problem. The criterion for the new "generalized" problem is

$$\inf_K \sup_{w \in W_a} \sup_{\eta_k} \Theta. \quad (3.11)$$

To solve problem 3, new mixed H_2/H_∞-like method is proposed.

4. SADDLE-POINT TYPE CONDITION OF OPTIMALITY IN THE MIXED AB_α/H_∞-PROBLEM

The problem (3.11) is a minimax problem; hence game-theory approach may be appropriate. Saddle point of the game is a triplet (K^*, G_0^*, G_1^*) such that the following inequality holds:

$$J(K^*, G_0, G_1) \le J(K^*, G_0^*, G_1^*) \le J(K, G_0^*, G_1^*).$$

Let us consider the following sets:

$$\mathbf{K}_* \equiv \mathrm{Arg} \min_{K \in \mathbf{K}} \mathcal{T}, \quad (4.12)$$

$$\mathbf{G}_{0*} \equiv \mathrm{Arg} \max_{G_0 \in \mathcal{G}_0, \|G_0\|_2 = 1} \mathcal{T}, \quad (4.13)$$

$$\mathbf{G}_{1*} \equiv \mathrm{Arg} \max_{G_1 \in RH_\infty^{m_3 \times m_3}, \|G_1\| = 1} \mathcal{T}, \quad (4.14)$$

$$\mathcal{T} = \left\| \mathcal{L}(\mathcal{U}(\widetilde{F}, \Delta_k), K) \begin{bmatrix} G_0 & 0 \\ 0 & G_1 \end{bmatrix} \right\|_2 .$$

The set (4.12) is formed by the controllers which are solutions of the mixed AB_a/H_∞-optimization problem corresponding to the assertion that the input W of the closed-loop system $\mathcal{L}(\widetilde{F}, K)$ is generated by known generating filter $G_0 \in \mathcal{G}_a$, i.e. $W = G_0 \otimes V$. The input η is generated by a known generating filter $G_1 \in RH_\infty$, i.e. $col(\eta_k) = G_1 \otimes W$. Appropriately, the set (4.13) is formed by the worst-case input generating filters with bounded anisotropy for a given controller $K \in \mathbf{K}$ and filter $G_1 \in RH_\infty^{m_3 \times m_3}$. Similarly, the set (4.14) is formed by the worst-case input generating filter with unbounded anisotropy for given controller $K \in \mathbf{K}$ and filter $G_0 \in \mathcal{G}_a$.

If the assumption holds, the set (4.12) consists of a unique I/O-operator.

Lemma 1. If the controller K is a fixed point of the following map

$$K \in \mathbf{K}_* \left(\widetilde{F}, \mathbf{G}_{0*}, \mathbf{G}_{1*} \right) , \tag{4.15}$$

then it is a solution to problem (2.4).

5. WORST-CASE BP-INPUT DISTURBANCE SCENARIO VS FINITE-DIMENSIONAL CONTROLLER IN THE PRESENCE OF ARBITRARY BS-INPUT

The closed-loop system $\mathcal{L}(\widetilde{F}, K)$ has the following state-space realization:

$$\mathcal{L}(\widetilde{F}, K) = \left[\begin{array}{cc|cc} A & B_2\widehat{C} & B_0 & B_1 \\ \widehat{B}C_2 & \widehat{A} & \widehat{B}D_{21} & 0 \\ \hline C_1 & D_{12}\widehat{C} & 0 & 0 \\ C_2 & 0 & D_{21} & 0 \end{array} \right] \equiv \left[\begin{array}{c|cc} A_t & B_t & F_t \\ \hline C_t & 0 & 0 \\ * & * & * \end{array} \right] . \tag{5.16}$$

Theorem 2. Let $\gamma > \|\mathcal{L}(\widetilde{F}, K)\|_\infty$. Then

$$\sup_{\eta_k} \Theta \le \text{Trace} \left\{ B_t^T (I + 2A_t)^T (P + PF_t \Pi^{-1} F_t^T P) B_t + S_0 \right\} ,$$

where $\Pi = \Gamma - F_t^T P F_t \ge 0$ and $P \in R^{n \times n}$ is an admissible solution of the discrete algebraic Riccati equation

$$\begin{aligned} &A_t^T P A_t - P + \\ &+ A_t^T P F_t \Pi^{-1} F_t^T P A_t + Q = 0. \end{aligned} \tag{5.17}$$

The worst case input scenario is

$$\widetilde{\eta}_k = \Pi^{-1} F_t^T P (A_t x_k + B_t w_k) \tag{5.18}$$

and the matrix $A_t + F_t \Pi^{-1} F_t^T P A_t$ is stable.

The worst case disturbance scenario $\widetilde{\eta}_k$ can be generated from the BS-signal w_k by the shaping filter $\widetilde{G}_1$, whose internal state is a copy of the system state ζ_k, i.e. its realization is

$$\widetilde{G}_1 = \left[\begin{array}{c|c} \widetilde{\mathcal{P}} A_t & \widetilde{\mathcal{P}} B_t \\ \hline \Pi^{-1} F_t P A_t & \Pi^{-1} F_t P B_t \end{array} \right] , \tag{5.19}$$

where $\widetilde{\mathcal{P}} = I + F_t \Pi^{-1} F_t P$.

6. WORST-CASE BS-INPUT DISTURBANCE SCENARIO VS FINITE-DIMENSIONAL CONTROLLER FOR THE WORST-CASE BP-INPUT

In this section our goal is to find worst-case shaping filter generating a signal w_* that maximizes BS-disturbance gain over all Gaussian sequences, from Gaussian white noise $\mathbf{G}(0, I)$. Direct calculation gives

$$\mathcal{L}(\widetilde{F}, K) \begin{bmatrix} I & 0 \\ 0 & \widetilde{G}_1 \end{bmatrix} = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & D_w \\ * & * \end{array} \right] , \tag{6.20}$$

where

$$A_w = \widetilde{\mathcal{P}} A_t, \quad B_w = \Xi B_t,$$
$$C_w = \begin{bmatrix} C_1 & D_{12}\widehat{C} \end{bmatrix}, \quad D_w \equiv 0.$$

It is clear that

$$J = \sup_{w_k \in W_a} \left\| \mathcal{L}(\widetilde{F}, K) \begin{bmatrix} I & 0 \\ 0 & \widetilde{G}_1 \end{bmatrix} \right\|_{\mathcal{P}}^2 ,$$

where $\widetilde{G}_1$ satisfies (5.19). The problem at the right-hand side of the last equality can be solved by using anisotropic technique developed in (Vladimirov *et al.*, 1996). The frequency description of the worst shaping filter $\widetilde{G}_0$ is given by the following proposition.

Theorem 3. (Diamond *et al.*2001). Let the system $F \in H_\infty^{p \times m}$ and the filter $G \in H_\infty^{m \times m}$ satisfy

$$\widehat{G}(\omega)(\widehat{G}(\omega))^* = \left(I_m - q\widehat{F}^*(\omega)\widehat{F}(\omega) \right)^{-1} , \tag{6.21}$$

for $q = \overline{A}^{\,-1}(G)$. Then $G(s)$ belongs to the set of worst-case input generating filters (4.13).

Let $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \in I\!R^{m_1 \times 2n}$ be a matrix such that $A + BL$ is asymptotically stable, and let $\Sigma \in I\!R^{m \times m}$ be a positive definite symmetric matrix. Consider the generating filter G_0 with the input V and output W governed by the equations (3.6), combined with

$$\begin{aligned} w_k &= L_1 x_k + L_2 \xi_k + \Sigma^{1/2} v_k \\ &= L\zeta_k + \Sigma^{1/2} v_k. \end{aligned} \tag{6.22}$$

It is straightforward to verify that

$$G(s) = \left[\begin{array}{c|c} A_w + B_w L & B_w \Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right]. \qquad (6.23)$$

Lemma 2. (Diamond *et al.* 2001). For given asymptotically stable system

$$\left[\begin{array}{c|c} A & B \\ \hline * & * \end{array}\right],$$

the mean anisotropy of the sequence $W = G \otimes V$ generated by asymptotically stable filter (6.23) is equal to

$$\overline{A}(G) = -\frac{1}{2} \ln\det\left(\frac{m_1\,\Sigma}{\mathrm{Trace}\,(L\widetilde{Y}L^T + \Sigma)}\right),$$

where $\widetilde{Y} \in I\!R^{n\times n}$ is the controllability gramian of G satisfying the Lyapunov equation

$$\widetilde{Y} = (A + BL)\widetilde{Y}(A + BL)^T + B\Sigma B^T. \quad (6.24)$$

We consider the following Riccati equation for the matrix $R \in I\!R^{2n\times 2n}$

$$R = A_w^T R A_w + q C_w^T C_w + L_w^T \Sigma^{-1} L_w, \quad (6.25)$$

$$L = (\Sigma B_w^T R A_w + q D_w^T C_w), \quad (6.26)$$

$$\Sigma = (I_{m_1} - B_w^T R B_w)^{-1}. \quad (6.27)$$

A solution R of (6.25)–(6.27) is called *admissible* if R is symmetric, Σ is positive-definite and $A + BL$ is asymptotically stable. Note that for any $q \in [0, \|F\|_\infty^{-2})$, the equation above has a unique admissible solution, which is positive and semidefinite.

The formulas for the worst-case shaping filter are based on the following theorem.

Theorem 4. (Diamond *et al.* 2001). Let the system (6.20) be asymptotically stable, and the matrices L and Σ correspond to the admissible solution R of Riccati equation (6.25)–(6.27), where parameter $q \in [0, \|F\|_\infty^{-2})$ is the solution of equation

$$a = -\frac{1}{2} \ln\det\left(\frac{m_1\,\Sigma}{\mathrm{Trace}\,\left(L\tilde{Y}L^T + \Sigma\right)}\right), \quad (6.28)$$

and $\tilde{Y}$ satisfies (6.24). Then generating filter (6.23) satisfies (6.21). In that case, a-anisotropic norm (2.3) of the system F is given by

$$|\!|\!| F |\!|\!|_a = \left(\frac{1}{q}\left(1 - \frac{m_1}{\mathrm{Trace}\,(L\tilde{Y}L^T + \Sigma)}\right)\right)^{1/2}.$$

7. STATE ESTIMATING FORMULAS

Denote by $\mathcal{F}_k^Y$ a σ-algebra of random events induced by the history $\{y_j\}_{j\le k}$ of the observation signal Y at the instant k. In other words, $\left(\mathcal{F}_k^Y\right)_{-\infty<k<\infty}$ is the flow of σ-algebras in $\mathcal{F}$ generated by the sequence Y.

Admissible controller (3.10) is called state-estimating, if its n-dimensional internal state Ξ coincides with the sequence of one-step predictors for the internal state X of the system F via the observation signal Y under the worst-case input disturbance W, i.e. if

$$\xi_k = \mathbf{E}\,\left(x_k \mid \mathcal{F}_{k-1}^Y\right), \quad -\infty < k < +\infty,$$

where $W = G \otimes V$ with the worst-case generating filter G (here, $\mathbf{E}\,(\cdot \mid \cdot)$ stands for the conditional expectation).

The system with the worst BP- and BS-inputs has the following state-space realization:

$$\mathcal{L}(\tilde{F}, K)\begin{bmatrix} I & 0 \\ 0 & \widetilde{G}_1 \end{bmatrix}\widetilde{G}_0 = \left[\begin{array}{cc|c} \widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{B}_1 \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{B}_2 \\ \hline * & * & * \\ \widetilde{C}_{21} & \widetilde{C}_{22} & \widetilde{D} \end{array}\right],$$

where $\tilde{A}$ is the matrix divided into uniform blocks corresponding with x and ξ.

Prior to formulate a criterion for the state-estimating property, we consider the Riccati equation

$$S = \widetilde{A}_{11} S \widetilde{A}_{11}^T + \widetilde{B}_1 \widetilde{B}_1^T - \Lambda\Theta\Lambda^T, \quad (7.29)$$

$$\Theta = \widetilde{C}_{21} S \widetilde{C}_{21}^T + \widetilde{D}\widetilde{D}^T, \quad (7.30)$$

$$\Lambda = (\widetilde{A}_{11}\widetilde{C}_{21}^T + \widetilde{B}_1\widetilde{D})\Theta^{-1}, \quad (7.31)$$

where the matrices Σ and L are defined in Theorem 4.

A solution $S = S^T \in I\!R^{n\times n}$ of equation (7.29)–(7.31) is called admissible if the matrix S is positive semidefinite and $\widetilde{A}_{11} - \Lambda\widetilde{C}_{21}$ is asymptotically stable.

Theorem 5. Let system (3.6) satisfy Assumption 1, and let the state-space realization matrices of admissible controller (3.10) obey the relations

$$\begin{aligned} \widehat{A} &= \widetilde{A}_{11} + \widetilde{A}_{12} - \Lambda(\widetilde{C}_{21} + \widetilde{C}_{22}), \\ \widehat{B} &= \Lambda, \end{aligned} \quad (7.32)$$

where the matrix Λ is expressed through the admissible solution of Riccati equation (7.29)–(7.31), where, in turn, the matrices Σ and L determine the worst-case generating filter as described in Theorem 4. Then controller (3.10) is state-estimating.

8. STATE-SPACE FORMULAS FOR THE OPTIMAL CONTROLLER

In order to formulate the final result, we consider the following Riccati equation

$$T = A_u^T T A_u + C_u^T C_u - N^T \Upsilon N, \quad (8.33)$$

$$\Upsilon = B_u^T T B_u + D_{12}^T D_{12}, \quad (8.34)$$

$$\begin{aligned} N &\equiv \begin{bmatrix} N_1 & N_2 \end{bmatrix} \\ &= -\Upsilon^{-1}(B_u^T T A_u + D_{12}^T C_u), \end{aligned} \quad (8.35)$$

where the matrix N is partitioned into two blocks $N_1, N_2 \in I\!R^{m_2 \times n}$, and the matrices $A_u \in I\!R^{2n\times 2n}$ $B_u \in I\!R^{2n\times m_2}$ and $C_u \in I\!R^{p_1\times 2n}$ are given by

$$A_u = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} - B_2\widehat{C} \\ 0 & \widetilde{A}_{11} + \widetilde{A}_{12} - \Lambda(\widetilde{C}_{21} + \widetilde{C}_{22}) \end{bmatrix},$$

$$B_u = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \; C_u = \begin{bmatrix} C_1 & 0 \end{bmatrix}.$$

A solution $T = T^T \in I\!R^{2n\times 2n}$ of equation (8.33)–(8.34) is called admissible if the matrix T is non-negative definite and $A_u + B_u N$ is asymptotically stable.

Theorem 6. Let system (2.1) satisfy assumptions (A)-(E) and let the state-space realization matrices of state-estimating controller (3.10) obey (7.32) in combination with the following equation:

$$\widehat{C} = N_1 + N_2\,, \quad (8.36)$$

where the matrices N_1, N_2 are expressed via the admissible solution of the Riccati equation (8.33)–(8.34). Then the controller is a solution to Problem 3.

9. EXPLICIT FORMULAS FOR STATE ESTIMATING CONTROLLER

Now we can collect the results derived above. The solution of the RSAHO problem can be divided into several steps. First, we fix values $\gamma_i \neq 0$. Then, cross-coupled Riccati equations (5.17), (6.25)-(6.26), (7.29)-(7.31), (8.33)-(8.35), the Lyapunov equation (6.24) and the equation of special type (6.28) should be solved. These equations can be solved numerically using homotopic methods (Diamond *et al.*, 1997). A solution of these equations gives the controller $K_\gamma(s)$, which is suboptimal solution for the original problem. To obtain the optimal solution, it is necessary to find the minimal value γ_{min} such that the Riccati equations above have admissible solutions. This can be accomplished by gradually decreasing the parameters γ_i. The controller $K_{\gamma_{min}}(s)$ tends to be the optimal one in the sense (2.4).

10. CONCLUSION

In this paper a state-space solution to the robust anisotropy-based stochastic H_∞ optimization problem for discrete finite-dimensional LTI systems was proposed. It is shown that solving the problem for uncertain system can be replaced by solving mixed H_2/H_∞-problem. The solution of the last problem is reduced to solving four cross-coupled algebraic Riccati equations, Lyapunov equation and one equation of a special type.

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