MULTIRATE SAMPLING AND DELAYS IN RECEDING HORIZON STABILIZATION OF NONLINEAR SYSTEMS

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Abstract: In this paper, a multirate version of the receding horizon algorithm for the stabilization of sampled-data nonlinear systems is presented. The computations are based on discrete-time approximate models. "Low measurement rate" is assumed, and the presence of measurement and computational delays are taken into account. It is shown that, under reasonable assumptions, the proposed algorithm gives a closed-loop system which is stable in appropriate sense.

Keywords: Nonlinear predictive control, Multirate sampled-data systems, Time delays, Optimal control, Stability.

1. INTRODUCTION

During last decade, considerable attention has been paid to the stabilization problem of nonlinear systems. Among the solution methods for this problem, receding horizon control strategies, also known as model predictive control (MPC), have become quite popular (see for example (Mayne, *et al.*, 2000; Findeisen, *et al.*, 2003) for an overview). Owing to the use of computers in the implementation of the controllers, the investigation of sampled-data control systems has become an important area of control science. An overview and analysis of existing approaches for the stabilization of sampled-data systems can be found in (Nešić, *et al.*, 1999; Nešić and Teel, 2004) (see also Gyurkovics and Elaiw, 2004; Polushin and Marquez, 2004) and the references of these papers). Two main approaches of sampled-data control design can be distinguished: the first one consists in the implementation of a continuous-time stabilizing control law at a sufficiently high sampling rate, while the second way is to discretize the continuous-time model and design a stabilizing controller on the basis of the approximate discrete-time model. In recent papers (Nešić, *et al.*, 1999; Nešić and Teel, 2004), sufficient conditions are presented which guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant. These results yield a general framework for the control design. Significant amount of work is also devoted to the study of control design methods which results in controllers satisfying the above mentioned sufficient conditions (see e.g. Grüne and Nešić 2003; Gyurkovics and Elaiw, 2004). All of this investigations deal with the case when the sampling rates of the control function and the state measurement coincide i.e. a single-rate approach is presented. Moreover, the measurement result and the corresponding controller are as-

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sumed to be available instantaneously. The latter assumption is of course unrealistic and may be considered as one of the reasons why different rates of control and measurement samplings have to be taken into account: it is meaningless or impossible to perform a new measurement until the results of the previous one becomes available and worked up. Besides the measurement and computational delay, the nature of the problem may involve different measurement and control sampling rate (see e.g. Elaiw and Gyurkovics, 2004). The notion of multirate sampled-data feedback (which was introduced to the best of our knowledge by Polushin and Marquez, 2004) is used in the present paper in this sense. Polushin and Marquez addresses the design of multirate controllers based on the knowledge of a continuous-time stabilizing feedback for the exact model as well as on that of a discrete-time stabilizing feedback for the approximate model under the assumption of "low measurement rate" and in the presence of measurement delay. The computational delays in MPC is the main issue in (Chen, *et al.*, 2000; Findeisen and Allgöwer, 2004), where the proposed approaches are based on the exact continuous time model.

The aim of the present paper is to drive a multirate version of the receding horizon algorithm based on discrete-time approximate models of the plant, and establish sufficient conditions which guarantee that the proposed control stabilizes the original exact model in the presence of measurement and computational delays. The basic idea of handling the delays is very similar to that of (Chen, et al., 2000; Findeisen and Allgöwer, 2004), but, in contrast to these works, the design of the controller is based on the approximate model in the present paper. The importance of taking into account this fact is supported by a series of counter-examples (see e.g. Nešić, *et al.*, 1999; Nešić and Teel, 2004; Gyurkovics and Elaiw, 2004), which show that it is not sufficient to require small computational errors only.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1 The models

Consider the nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbf{R^n}$, $u(t) \in U \subset \mathbf{R^m}$, $f : \mathbf{R^n} \times \mathbf{U} \to \mathbf{R^n}$, with $f(0,0) = 0$, U is closed and $0 \in U$. We shall assume that f is continuous and for any pair of positive numbers (Δ', Δ'') there exists an $L_f = L_f(\Delta', \Delta'')$ such that

$$\|f(x,u) - f(y,u)\| \le L_f \|x - y\|,$$

for all $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$. Let $\Gamma \subset \mathbf{R^n}$ be a given compact set containing the origin and consisting of all initial states to be taken into account. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(iT) =: u_i$, if $t \in [iT, (i+1)T)$, $i \in \mathbf{N}$, where $T > 0$ is the control sampling period which is assumed to be fixed. Under the conditions on f, for any $\overline{x} \in \mathcal{B}_{\Delta'}$ and $\overline{u} \in \mathcal{B}_{\Delta''}$ there exists an $\omega = \omega(\overline{x}, \overline{u}) > 0$ such that equation (1) with $u(t) \equiv \overline{u}$, $(t \in [0, \omega))$ and initial condition $x(0) = \overline{x}$ has a unique solution on $[0, \omega)$ denoted by $\phi^E(., \overline{x}, \overline{u})$. Then, the *exact discrete-time model* of the system can be defined as

$$x_{i+1}^E = F_T^E(x_i^E, u_i), \quad i = 0, 1, ... \tag{2}$$

where $F_T^E(x,u) := \phi^E(T; x, u)$, if $T < \omega(x,u)$ otherwise $F_T^E(x,u)$ is defined to be an arbitrary element of $\mathbf{R^n}$ with sufficiently large norm. (A discussion about the case of finite escapes can be found e.g. in (Nešić, *et al.*, 1999).) We emphasize that F_T^E in (2) is not known in most cases, therefore the controller design can be carried out by means of an *approximate discrete-time model*

$$x_{i+1}^A = F_{T,h}^A\left(x_i^A, u_i\right), \quad i = 0, 1, ..., \tag{3}$$

where T is again the control sampling period, while parameter h refers to certain modeling parameters connected typically with the underlying numerical method: $F_{T,h}^A$ is derived by the multiple application of some numerical scheme (e.g. a Runge-Kutta formula) with step sizes $h_0^i, ..., h_{m_i}^i$, where $0 < h_k^i \le h$ and $h_0^i + ... + h_{m_i}^i = T$. In what follows, we shall refer to such a subdivision by h, for simplicity. For the solutions of (2) and (3) with the control sequence $\mathbf{u} = \{u_0, u_1, ...\}$ satisfying the initial conditions $x_0^E = x'$ and $x_0^A = x''$ we shall use the notation $\phi_i^E(x', \mathbf{u})$ and $\phi_i^A(x'', \mathbf{u})$, respectively.

Assumption A1 $F_{T,h}^A(0,0) = 0$, $F_{T,h}^A$ is continuous in both variables uniformly in small h, and it satisfies a local Lipschitz condition: there is a $h^* > 0$ such that for any pair of positive numbers (Δ', Δ'') there exists $L_{F^A} > 0$ such that

$$\|F_{T,h}^A(x,u) - F_{T,h}^A(y,u)\| \le e^{L_{F^A} T} \|x - y\|,$$

holds for all $u \in \mathcal{B}_{\Delta''}$, $x, y \in \mathcal{B}_{\Delta'}$ and $h \in (0, h^*]$.

The use of (3) in control design will result in controllers, which depend on the parameters characterized by h. Let $\mathcal{U}^h$ denote a family of control sequence parameterized by h: $\mathbf{u}^h \in \mathcal{U}^h$ if $\mathbf{u}^h = \{u_0^h, u_1^h, ...\}$ and $u_i^h \in U$, $i = 0, 1, ...$.

Definition 1 System (2) is *practically asymptotically controllable* (PAC) from $\Omega \subset \mathbf{R^n}$ to the origin with the parametrized family $\mathcal{U}^h$, if there exist a $\beta(.,.) \in \mathcal{KL}$ and a continuous, positive and nondecreasing function $\sigma(.)$ which are independent of h, and such that for any $r > 0$ there exists a $h^* > 0$ so that for all $x \in \Omega$ and for all $h \in (0, h^*]$ there exists a control sequence $\mathbf{u}^h(x) \in \mathcal{U}^h$, such that $\left\|u_i^h(x)\right\| \leq \sigma(\|x\|)$, and the corresponding solution ϕ^E of (2) satisfies the inequality

$$\left\|\phi_i^E(x, \mathbf{u}^h(x))\right\| \leq \max\left\{\beta(\|x\|, iT), r\right\}, \quad i \in \mathbf{N}.$$

Assumption A2 There exists $h^* > 0$ such that the exact discrete-time model (2) is PAC from a set $\Omega \supset \Gamma$ to the origin with $\mathcal{U}^h$ for all $h \in (0, h^*]$.

Remark 1 Observe that Assumption A2 implies that for any $x \in \Omega$ there exists a control function $\mathbf{u}^h(x) \in \mathcal{U}^h$ for which no finite escape time occurs.

To investigate the stability behavior of the exact model with a controller designed to stabilize the approximate model we need an assumption describing the "closeness" of these two models.

Assumption A3 Let T be given. For any $\Delta' > 0$ and $\Delta'' > 0$ there exist $\gamma \in \mathcal{K}$ and $h^* > 0$ such that

$$\left\|F_{T,h}^A(x,u) - F_T^E(x,u)\right\| \leq T\gamma(h), \tag{4}$$

for all $(x,u) \in \mathcal{B}_{\Delta'} \times \mathcal{B}_{\Delta''}$, and $h \in (0, h^*]$.

In this paper we address the problem of state feedback stabilization of (2) under "low measurement rate" in the presence of measurement and computational delays. More precisely, we shall assume that state measurements can be performed at the time instants jT^m, $j = 0, 1, ...$:

$$y_j := x(jT^m), \;\; j = 0, 1, ...$$

The result of the measurement y_j becomes available for the computation of the controller at $jT^m + \tau_1 (> \; jT^m)$, while the computation requires $\tau_2 > 0$ length of time i.e. the (re)computed controller is available at $T_j^* := jT^m + \tau_1 + \tau_2$, $j = 0, 1, ...$. We assume that $\tau_1 = \ell_1 T$, $\tau_2 = \ell_2 T$ and $T^m = \ell T$ for some integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ and $\ell \geq \ell_1 + \ell_2 =: \overline{\ell}$.

Because of the measurement and computational delays, on the time interval $[0, \tau_1 + \tau_2)$ a precomputed control function $\mathbf{u}^c$ can only be used. It is reasonable or assume that initial states can be kept within the PAC domain of the exact system with such a precomputed controller. More precisely:

Assumption A4 There exists a $\Delta_0 > 0$, and a control sequence $\mathbf{u}^c = \{u_0^c, ..., u_{\overline{\ell}-1}^c\}$ with $u_i^c \in U$ can be given so that $\phi_k^E(x, \mathbf{u}^c) \in \Omega \cap \mathcal{B}_{\Delta_0}$, $\phi_k^A(x, \mathbf{u}^c) \in \Omega \cap \mathcal{B}_{\Delta_0}$, $k = 0, 1, ..., \overline{\ell}$ for all $x \in \Gamma$.

Furthermore, a "new" controller computed according to the measurement $y_j = x(jT^m)$ will only be available from $jT^m + \overline{\ell}T$, in the time interval $[jT^m, jT^m + \overline{\ell}T)$, the "old" controller has to be applied. Since the corresponding exact trajectory is unknown, an approximation ζ_j^A to the exact state $x\left(jT^m + \overline{\ell}T\right)$ can only be used which can be defined as follows. Assume that a control sequence $\left\{u_0\left(\zeta_{j-1}^A\right), ..., u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\}$ has been defined for $j \geq 1$. Let

$$\mathbf{v}^p\left(\zeta_{j-1}^A\right) = \left\{u_{\ell-\overline{\ell}}\left(\zeta_{j-1}^A\right), ..., u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\}$$

and define ζ_j^A by

$$\zeta_j^A = \mathcal{F}_{\overline{\ell}}^A\left(y_j, \mathbf{v}^p\left(\zeta_{j-1}^A\right)\right), \; \zeta_0^A = \phi_{\overline{\ell}}^A(x, \mathbf{u}^c),$$

where $\mathcal{F}_{\overline{\ell}}^A\left(y, \{u_0, ..., u_{\overline{\ell}-1}\}\right) =$

$F_{T,h}^A\left(...F_{T,h}^A\left(F_{T,h}^A(y, u_0), u_1\right) ..., u_{\overline{\ell}-1}\right)$.

In the stability analysis of the exact discrete-time model in the case of multirate sampling with delays outlined above, the following ℓ-step exact discrete-time model plays an important role: let $\mathbf{v}^{(j)} = \left\{u_0^{(j)}, ..., u_{\ell-1}^{(j)}\right\}$ and let

$$\xi_{j+1}^E = \mathcal{F}_\ell^E(\xi_j^E, \mathbf{v}^{(j)}), \qquad \xi_0^E = \phi_{\overline{\ell}}^E(x, \mathbf{u}^c), \tag{5}$$

where $\mathcal{F}_\ell^E(\xi_j^E, \mathbf{v}) = \phi_\ell^E(\xi_j^E, \mathbf{v})$.

Our aim is to solve the following problem: for given T, T^m, τ_1 and τ_2 find a control strategy

$$\mathbf{v}_{\ell,h} : \widetilde{\Gamma} \to \underbrace{U \times U \times ... \times U}_{\ell \text{ times}}$$

$$\mathbf{v}_{\ell,h}(x) = \left\{u_0(x), ..., u_{\ell-1}(x)\right\},$$

using the approximate model (3) which stabilizes the origin for the exact system (2) in an appropriate sense, where $\widetilde{\Gamma}$ is a suitable set containing at least $\Omega \cap \mathcal{B}_{\Delta_0}$.

2.2 An optimization problem

In order to find a suitable controller $\mathbf{v}$, we shall apply a multistep version of the receding horizon method. To do so, we shall consider the following optimal control problem.

Let $0 < N \in \mathbf{N}$ be given. Let (3) be subject to the cost function

$$J_{T,h}(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} T l_h(x_k^A, u_k) + g(x_N^A),$$

where $\mathbf{u} = \{u_0, u_1, ..., u_{N-1}\}$, $x_k^A = \phi_k^A(x, \mathbf{u})$, $k = 0, 1, ..., N$, denote the solution of (3), l_h and g are given functions, satisfying assumptions to be formulated later.

Consider the optimization problem

$$P_{T,h}^{A}(N,x): \quad \min\left\{J_{T,h}(N,x,\mathbf{u}) : u_k \in U\right\}.$$

If this optimization problem has a solution denoted by $\mathbf{u}^* = \left\{u_0^*, ..., u_{N-1}^*\right\}$, then the first ℓ elements of $\mathbf{u}^*$ are applied at the state x i.e.,

$$\mathbf{v}_{\ell,h}(x) = \left\{u_0^*(x), ..., u_{\ell-1}^*(x)\right\}$$

The optimization problem $P_{T,h}^{A}$ will be investigated under the following conditions imposed on the choice of the stage and the terminal cost functions.

Assumption A5 (i) g is continuous, positive definite, and for any Δ' there exists a constant $L_g = L_g(\Delta') > 0$ such that $|g(x)-g(y)| \leq L_g\|x-y\|$ for all $x \in \mathcal{B}_{\Delta'}$.

(ii) l_h is continuous with respect to x and u, uniformly in small h, and for any $\Delta' > 0$, $\Delta'' > 0$ there exist $h^* > 0$ and $L_l = L_l(\Delta', \Delta'') > 0$ such that $|l_h(x,u) - l_h(y,u)| \leq L_l\|x-y\|$ for all $h \in (0, h^*]$, $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$.

(iii) There exist a $h^* > 0$ and two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that the inequality

$$\varphi_1(\|x\|) + \varphi_1(\|u\|) \leq l_h(x,u) \leq \varphi_2(\|x\|) + \varphi_2(\|u\|),$$

holds for all $x \in \mathbf{R^n}$, $\mathbf{u} \in \mathbf{U}$ and $h \in (0, h^*]$.

Let $\Delta_0 > 0$ be given in Assumption A4, let $\beta(.,.)$ and $\sigma(.)$ be functions generated by Assumption A2 and let Δ_1 be such that $\Delta_1 \geq 1 + \beta(\Delta_0, 0)$. Moreover, for $0 < \rho \leq \Delta_1$, we introduce the notation $U_\rho = U \cap \mathcal{B}_{\sigma(\rho)}$.

The terminal cost function g and/or a terminal constraint set given explicitly or implicitly play crucial role in establishing the desired stabilizing property. We shall assume that g is chosen according to the following assumption.

Assumption A6 There exist $h^* > 0$ and $\eta > 0$ such that for all $x \in \mathcal{G}_\eta = \{x : g(x) \leq \eta\}$ there exists a $\kappa(x) \in U_{\rho_0}$ (which may depend on parameter h) such that inequality

$$Tl_h(x,\kappa(x)) + g\left(F_{T,h}^{A}(x,\kappa(x))\right) \leq g(x) \qquad (6)$$

holds true for all $h \in (0, h^*]$, where ρ_0 is such that $\mathcal{G}_\eta \subset \mathcal{B}_{\rho_0}$.

Without loss of generality, we may assume that $\mathcal{G}_\eta \subset \Omega \cap \mathcal{B}_{\Delta_1}$. For any $N > 0$ and $x \in \mathbf{R^n}$, let

$$V_N(x) = \inf\left\{J_{T,h}(N,x,\mathbf{u}) : u_k \in U\right\},$$

if the right hand side is finite, and let $V_N(x) = \infty$ otherwise. (Evidently, function V_N depends also on the on the parameter h, but, for simplicity, this dependence is not reflected in the notation.) Let h_0^* denote the minimum of the values h^* generated by Assumptions A1-A3 and A5-A6 with $\Delta' = \Delta_1$ and $\Delta'' = \sigma(\Delta_1)$. Then, from Assumptions A1-A6, it follows immediately that for any $x \in \Omega \cap \mathcal{B}_{\Delta_1}$and $h \in (0, h_0^*]$, $P_{T,h}^{A}(N,x)$ has a solution $\mathbf{u}^*(x)$, function $V_N(.)$ is positive definite and continuous uniformly in small h.

With argumentations standard in receding horizon literature one can prove the following lemma.

Lemma 1 Suppose that Assumptions A1, A5 and A6 hold true. Then for any $N \geq 1$ the following statements are valid:

(i) For any $x_0 \in \mathcal{G}_\eta$, $V_N(x_0) \leq g(x_0)$ and $\phi_N^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$.

(ii) If $\phi_N^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$ for some $x_0 \in \mathbf{R^n}$, then $V_{\overline{N}}(x_0) \leq V_N(x_0)$ for all $\overline{N} \geq N$, and

$$V_N(F_{T,h}^{A}(x_0, u_0^*(x_0))) - V_N(x_0) \leq -Tl_h(x_0, u_0^*(x_0)).$$

(iii) If for some $x_0 \in \mathbf{R^n}$ and for some $k \in \mathbf{N}$, $0 \leq k < N$, $\phi_k^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$, then $\phi_N^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$.

Lemma 2 If Assumptions A2, A3, A5 and A6 hold true, then there exist a h_1^* with $0 < h_1^* \leq h_0^*$, and a $\mathcal{T}_1^* > 0$ and for all $x \in \Omega \cap \mathcal{B}_{\Delta_1}$ and for all $h \in (0, h_1^*]$ there exists a $\widetilde{\mathbf{u}}(x) \in \mathcal{U}^h$ with $\|\widetilde{u}_i(x)\| \leq \sigma(\|x\|)$ such that

$$\left\|\phi_k^A(x, \widetilde{\mathbf{u}}(x))\right\| \leq \beta(\Delta_0, 0) + 1,$$

and $\phi_k^A(x, \widetilde{\mathbf{u}}(x)) \in \mathcal{G}_\eta$ if $k \geq N_1$ where N_1 is such that $\mathcal{T}_1^* \leq N_1 T \leq \mathcal{T}_1^* + T$. Moreover there exists a constant $V_{\max}^A > 0$ such that $V_N(x) \leq V_{\max}^A$ for all $x \in \Omega \cap \mathcal{B}_{\Delta_0}$ and for all $N \geq 1$.

The proof is omitted because of the lack of space.

Let $\Gamma_{\max} = \left\{x \in \mathbf{R^n} : \mathbf{V_N(x)} \leq \mathbf{V_{\max}^A}\right\}$. Obviously, $\Gamma \subset \Omega \cap \mathcal{B}_{\Delta_0} \subset \Gamma_{\max}$.

Lemma 3 If Assumptions A1-A6 hold true, then there exist two class-$\mathcal{K}_\infty$ functions σ_1 and σ_2, and constants $M > 0$ and $\widetilde{\Delta} > 0$ which are independent of N and h such that

$$\sigma_1(\|x\|) \leq V_N(x) \leq \sigma_2(\|x\|), \qquad (7)$$

$$\|u_k^*(x)\| \leq M, \quad \left\|\phi_k^A(x, \mathbf{u}^*(x))\right\| \leq \widetilde{\Delta}$$

$k = 0, 1, ..., N-1$, for all $x \in \Gamma_{\max}$ and $h \in (0, h_1^*]$.

The proof is omitted because of the lack of space.

Lemma 4 Suppose that Assumptions A1-A6 hold true. Let $x \in \Gamma_{\max}$ be arbitrary and let $\rho_1 > 0$ be such that $\mathcal{B}_{\rho_1} \subset \mathcal{G}_\eta$. If $h \in (0, h_1^*]$, and $N \in \mathbf{N}$ is chosen so that

$$TN > \mathcal{T}_2^* := \frac{V_{\max}^A - \eta}{\varphi_1(\rho_1)}, \qquad (8)$$

then $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$, and for all $k = 1, ..., \ell$

$$V_N\left(\phi_k^A(x, \mathbf{u}^*(x))\right) - V_N(x) \le -T\varphi_1(\|x\|). \quad (9)$$

The proof is omitted because of the lack of space.

Lemma 5 (Gyurkovics & Elaiw 2004). Suppose that Assumptions A1-A6 hold true. Then for any given $N \in \mathbf{N}$ there exist h_2^* $(0 < h_2^* \le h_1^*)$, $L_V > 0$ and $\delta_V > 0$ such that for all $h \in (0, h_2^*]$, inequality $|V_N(x) - V_N(y)| \le L_V \|x - y\|$ holds true for all $x, y \in \Gamma_{\max}$ with $\|x - y\| \le \delta_V$.

3. MULTISTEP RECEDING HORIZON CONTROL IN THE PRESENCE OF DELAYS

In this section we outline an approach to the problem how the occurring measurement and computational delays can be taken into account in the stabilization of multi-rate sampled-data system by receding horizon controller.

Suppose that a precomputed control sequence $\mathbf{u}^c$ satisfying Assumption A4 is given. Then the following Algorithm can be proposed.

Algorithm Let N be chosen according to (8), let $j = 0$, $T_{-1}^* = 0$ and let $\mathbf{u}^{(0)} = \mathbf{u}^{(p,0)} = \mathbf{u}^c = \{u_0^c, ..., u_{\ell-1}^c\}$. Measure the initial state $y(0) = x_0$.

Step j. (i) Apply the controller $\mathbf{u}^{(j)}$ to the exact system over the time interval $[T_{j-1}^*, T_j^*]$.

ii) Predict the state of the system by using the approximate model and let $\zeta_j^A = \phi_{\overline{\ell}}^A(y(j), \mathbf{u}^{(p,j)})$.

iii) Find the solution $\mathbf{u}^* = \{u_0^*, ..., u_{N-1}^*\}$ to the problem $P_{T,h}^A(N, \zeta_j^A)$, let $\mathbf{u}^{(j+1)} = \{u_0^*, ..., u_{\ell-1}^*\}$ and $\mathbf{u}^{(p,j+1)} = \{u_{\ell-\overline{\ell}}^*, ..., u_{\ell-1}^*\}$.

(iv) $j = j + 1$.

A schematic illustration of the Algorithm is sketched in Figure 1.

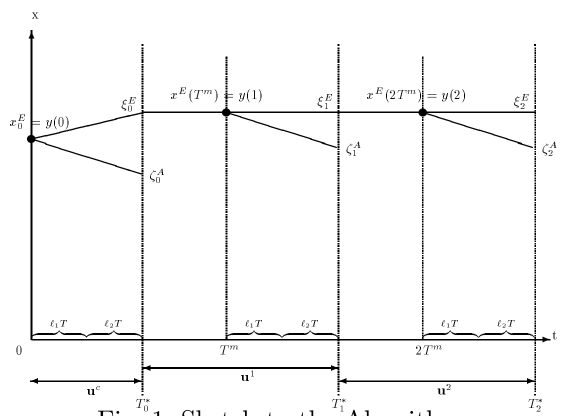

Fig. 1. Sketch to the Algorithm.

Lemma 6 Let $d > 0$ and $k \in \{1, 2, ..., \ell\}$ be arbitrary. Suppose that Assumptions A1-A6 are valid, N is chosen according to (8), and the following condition is satisfied:

(C) $\xi_{j-1}^E \in \Gamma_{\max}$, $\zeta_{j-1}^A \in \Gamma_{\max}$, and there exists a $\varepsilon_1(h) \in \mathcal{K}$ such that $\left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \le \varepsilon_1(h)$, if $h \in (0, h_2^*]$ $(j \in \mathbf{N}, j \ge 1)$ with some $0 < h_2^* \le h_1^*$.

Then there exists a $h_3^* > 0$ (independent of k) such that for any $h \in (0, h_3^*]$, inequality

$$\max\left\{V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right), V_N(\xi_{j-1}^E)\right\} \ge d \quad (10)$$

implies that

$$V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) - V_N(\xi_{j-1}^E) \le -\frac{T}{2}\varphi_1(\frac{1}{2}\left\|\xi_{j-1}^E\right\|),$$

where $\mathbf{u}^{(j)}$ is the optimal solution of problem $P_{T,h}^A(N, \zeta_{j-1}^A)$.

The proof is omitted because of the lack of space.

Corollary Under the conditions of Lemma 6 inequality $\max\left\{V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right), V_N(\xi_{j-1}^E)\right\} \ge d$ implies that $\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)}) \in \Gamma_{\max}$.

Theorem 1 Suppose that Assumptions A1-A6 hold true. Then there exist a $\mathcal{T}^* > 0$ and a $\beta \in \mathcal{KL}$, and for any $r_0 > 0$ there exists a $h^* > 0$ such that for any fixed $N \in \mathbf{N}$ with $NT \ge \mathcal{T}^*$, $h \in (0, h^*]$ and $x_0 \in \Gamma$, the trajectory of the ℓ-step exact discrete-time system

$$\xi_{k+1}^E = \mathcal{F}_\ell^E(\xi_k^E, \mathbf{v}_{\ell,h}(\zeta_k^A)),\ \xi_0^E = \phi_{\overline{\ell}}^E(x_0, \mathbf{u}^c) \quad (11)$$

with the ℓ-step receding horizon controller $\mathbf{v}_{\ell,h}$ obtained by the prediction

$$\zeta_{k+1}^A = \mathcal{F}_{\overline{\ell}}^A\left(y_{k+1}, \mathbf{v}^p\left(\zeta_k^A\right)\right),\ \zeta_0^A = \phi_{\overline{\ell}}^A(x_0, \mathbf{u}^c) \quad (12)$$

satisfies that $\xi_k^E \in \Gamma_{\max}$ and

$$\left\|\xi_k^E\right\| \le \max\left\{\beta\left(\left\|\xi_0^E\right\|, kT^m\right), r_0\right\}$$

for all $k \ge 0$. Moreover, $\zeta_k^A \in \Gamma_{\max}$, as well, and

$$\left\|\zeta_k^A\right\| \le \max\left\{\beta\left(\left\|\zeta_0^A\right\|, kT^m\right) + \delta_1, r_0\right\}$$

where δ_1 can be made arbitrarily small by suitable choice of h.

The proof is omitted because of the lack of space.

Remark 2 From Theorem 1 and Lemma 6 it follows that $\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})$ converges to the ball $\mathcal{B}_{r_0}$ as $j \to \infty$ for all k.

Remark 3 We note that the statement of Theorem 1 is similar to the practical asymptotic stability of the closed-loop system (11)-(12) with respect to the initial state ξ_0^E, ζ_0^A. This is not true for the original initial state x_0, because - due to the initial phase - the ball $\mathcal{B}_{r_0}$ is not invariant over the time interval $[0, \overline{\ell}T)$.

4. ILLUSTRATIVE EXAMPLE

Consider the continuous-time system (this example is taken from Chen and Allgöwer (1998))

$$\dot{x}_1 = x_2 + 0.5(1 + x_1)u,$$
$$\dot{x}_2 = x_1 + 0.5(1 - 4x_2)u.$$

Let the approximate discrete-time model be defined by Euler method as follows: let $z_0 = x_k^A$, $\overline{u} = u_k$, $h = T/m$ and let

$$z_{1,i+1} = z_{1,i} + h\left[z_{2,i} + 0.5(1 + z_{1,i})\overline{u}\right],$$
$$z_{2,i+1} = z_{2,i} + h\left[z_{1,i} + 0.5(1 - 4z_{2,i})\overline{u}\right].$$

$i = 0, 1, ..., m-1$. Take $x_{k+1}^A = z_m$. The running and the terminal costs are given by $l_h(x,u) = \frac{1}{2}\|x\|^4 + u^2$, $g(x) = 2.7778x_1^2 + 2.2223x_2^2$. All computations were carried out by MATLAB. Especially, the optimal control sequence was computed by the `constr` code of the Optimization toolbox. Simulations for the continuous-time system were carried out using `ode45` program in MATLAB when $T = 0.05$, $m = 10$, $\ell_1 = \ell_2 = 1$, $\ell = 3$.

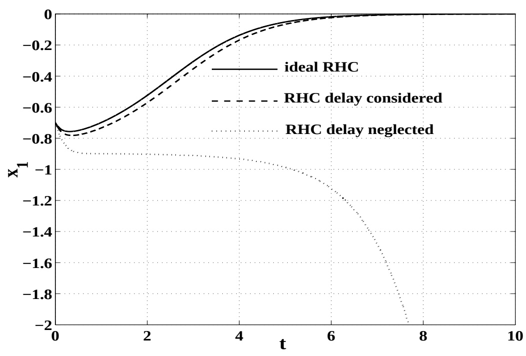

Fig. 2. The evolution of x_1 with different controllers.

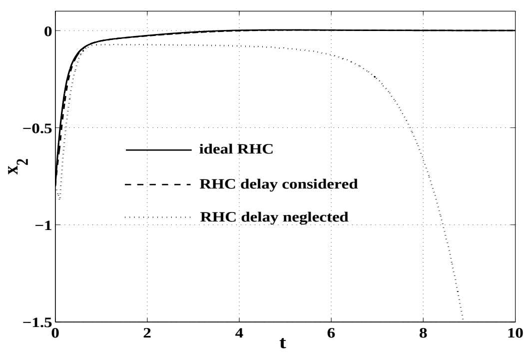

Fig. 3. The evolution of x_2 with different controllers.

The trajectories of the continuous-time system are shown in Figures 2-3. In these figures, three cases are shown; 1) The ideal instantaneous ℓ-step receding horizon controller is applied under the condition that no delays are presented (ideal RHC); 2) The ideal instantaneous receding horizon controller is applied without taking into account the occurring delays (RHC delay neglected); 3) The receding horizon controller obtained by the proposed Algorithm applied to the system when the delays are present (RHC delay considered).

REFERENCES

Chen, H., and Allgöwer, F. (1998). A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability, *Automatica*, 34, 1205-1217.

Chen, W.H., Ballance, D. J. and O'Reilly, J. (2000). Model predictive control of nonlinear systems: Computational burden and stability, *IEE Proc.-Control Theory Appl.,* 147, 387-394.

Elaiw, A.M. and Gyurkovics, E. (2004). Stabilization of HIV/AIDS model using an ℓ-step receding horizon control based treatment scheduling, *Proceedings of the IFAC Symposium on Nonlinear Control Systems, Stuttgart, Germany.*

Findeisen, R., Imsland, L., Allgöwer, F., and Foss, B. A., (2003). State and output feedback nonlinear model predictive control: An overview, *European J. Control*, 9, 190-206.

Findeisen, R. and Allgöwer, F. (2004). Computational delay in nonlinear model predictive control, *In Proceedings of Int. Symp. Adv. Control of Chemical Processes.*

Grüne, L. and Nešić, D. (2003). Optimization based stabilization of sampled-data nonlinear systems via their approximate discrete-time models, *SIAM J. of Control Optim.* 42, 98-122.

Gyurkovics, É. and Elaiw, A.M. (2004). Stabilization of sampled-data nonlinear systems by receding horizon control via discrete-time approximations, *Automatica*, 40, 2017-2028.

Mayne, D.Q., Rawlings, J.B., Rao, C.V. and Scokaert, P.O.M. (2000). Constrained model predictive control: Stability and optimality, *Automatica*, 36, 789-814.

Nešić, D. and Teel, A.R. (2004). A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models, *IEEE Trans. Autom. Control,* 49, 1103-1122.

Nešić, D., Teel, A.R. and Kokotović, P.V. (1999). Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximation, *System and Control Letters*, 38, 259-270.

Polushin, I. G. and Marquez, H. J. (2004). Multirate versions of sampled-data stabilization of nonlinear systems, *Automatica*, 40, 1035-1041.