# GENERALIZED DILATIONS AND HOMOGENEITY

**S. Emre Tuna** *,[1] **Andrew R. Teel** *

* *Department of Electrical and Computer Engineering*
*University of California, Santa Barbara, CA 93106, USA*

Abstract: The goal of this article is to show that the class of homogeneous systems can be made very general if one considers generalized dilations (which are a class of group actions) and defines homogeneity with respect to them. It turns out that uniqueness of solutions (in both directions of time) is indeed a sufficient condition for a system to be homogeneous with respect to some generalized dilation. The relation between homogeneity and monotonicity is also studied and it is shown that if a system is monotone with respect to some $V$ (a positive function) then there exists a generalized dilation with respect to which both the system and $V$ are homogeneous. Another result presented in the paper is the equivalence of local monotonicity and global monotonicity under homogeneity. 

Keywords: Generalized dilations, homogeneous systems, monotonicity

## 1. INTRODUCTION

An autonomous system defined by the differential equation $\dot{x} = g(x)$, with $g : \mathbb{R}^n \to \mathbb{R}^n$, is said to be homogeneous with respect to dilation $\Delta$ if for all $x \in \mathbb{R}^n$ and $\lambda > 0$ the righthand side satisfies $g(\Delta_\lambda x) = \lambda^d \Delta_\lambda g(x)$ for some fixed $d \in \mathbb{R}$ (so called the degree of homogeneity), where $\Delta_\lambda := \mathrm{diag}(\lambda^{r_1}, \lambda^{r_2}, \ldots, \lambda^{r_n})$ with $r_i > 0$ fixed. One can encounter the above definition in articles appeared as far back as (Kawski, 1988) or even earlier. This concept of dilations and homogeneity seems to have originated from the studies on nilpotent Lie groups. See, for instance, (Goodman, 1976). The concept was also extended to systems with a decision variable (i.e. $\dot{x} = g(x, u)$) and over the years homogeneity has proven itself useful in stability analysis and feedback design. In (Rosier, 1992) it was shown that local asymptotic stability (of the origin) is equivalent to global asymptotic stability for a homogeneous system and that for such a system a homogeneous Lyapunov function exists (a homogeneous Lyapunov function $V : \mathbb{R}^n \to [0, \infty)$ satisfies, aside from the generic properties that a Lyapunov function satisfies, that $V(\Delta_\lambda x) = \lambda^\kappa V(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$, where $\kappa > 0$ is fixed). Conditions that imply homogeneous feedback (a feedback law is homogeneous if the resulting closed-loop system preserves homogeneity) stabilizability were among the problems tackled. See, for instance, (Hermes, 1995) and (Grüne, 2000). Another interesting result appeared in (Bhat and Bernstein, 1997) where it was shown that if a system is homogeneous with a negative degree then the asymptotic stability of the origin implies that all solutions of the system converge to the origin in finite time. A brief summary of the results on homogeneous systems can be found in (Bacciotti and Rosier, 2001, §5.3).

In (Tuna and Teel, 2004*b*) an introduction of generalized dilations and a definition of homogeneity with respect to this broader class of operators than that of dilations were given for discrete-time systems. In a parallel work (Tuna and Teel, 2004*a*) authors studied homogeneous Lyapunov functions

[1] Corresponding author. Email: `emre@ece.ucsb.edu`

for difference inclusions that are homogeneous with respect to some generalized dilation. In this paper homogeneity of autonomous systems in continuous time is studied in this new sense. After generalized dilations are defined, homogeneity in terms of the solution to the system is introduced and a necessary and sufficient condition on the righthand side of the differential equation (describing the system) that implies homogeneity for a differentiable generalized dilation is given. In Theorem 9 it is shown that uniqueness of the solutions is enough in order for a system to be homogeneous with respect to some generalized dilation. Monotonicity of a system with respect to a positive function $V$ (which is sort of a Lyapunov function) is also defined and in Theorem 11 it is shown that for a system that is monotone with respect to some $V$ there exists a generalized dilation with respect to which both the system and $V$ are homogeneous. Finally, in Theorem 12 local monotonicity is shown to be equivalent to global monotonicity under homogeneity.

## 2. NOTATION AND DEFINITIONS

Consider the *system*

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathcal{D}$ is the *state* and $f$ is a map from $\mathcal{D}$ to $\mathbb{R}^n$. A *solution* of the system at time $t$, passing through (the initial condition) $x$ at $t = 0$ is denoted $\phi(t, x)$ for $t \in (\nu_x, \tau_x)$, where $-\infty \leq \nu_x < 0 < \tau_x \leq \infty$ for all $x \in \mathcal{D}$. Whenever $\tau_x$ is finite it is the smallest positive number beyond which the solution $\phi(\cdot, x)$ cannot be extended on $\mathcal{D}$. Likewise, whenever $\nu_x$ is finite it is the largest negative number beyond which an extension to the solution on $\mathcal{D}$ does not exist. Note that $\phi(0, x) = x$ by definition and the backward-in-time part of the solution is the forward-in-time solution to the differential equation $\dot{x} = -f(x)$.

The following assumption will prevail throughout the paper: for each initial condition $x \in \mathcal{D}$ the solution of the system (1) is unique and remains in $\mathcal{D}$ for all $t \in (\nu_x, \tau_x)$.

The notation $\mathbb{R}_{>0}$ stands for $\{z \in \mathbb{R} : z > 0\}$.

*Definition 1.* An operator $G : \mathbb{R}_{>0} \times \mathcal{D} \to \mathcal{D}$ is said to be a *generalized dilation* (or *gilation* in short) if the following conditions hold for all $x \in \mathcal{D}$.

(1) $G_1 x = x$.
(2) $G_{\lambda_1} G_{\lambda_2} x = G_{\lambda_1 \lambda_2} x$ for all $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$.

*Remark 2.* Note that a gilation $G$ is an *action* of the *group* $\mathbb{R}_{>0}$ on set $\mathcal{D}$.

Two obvious gilation examples in one dimension are $G_\lambda x = \lambda x$ on $\mathcal{D} = \mathbb{R}$ and $G_\lambda x = x^\lambda$ on $\mathcal{D} = \mathbb{R}_{>0}$. One not so obvious gilation example is

$$G_\lambda x = \operatorname{sgn}(x) \cdot \left(1 + \left(\frac{1 - x^2}{2\lambda x}\right)^2\right)^{0.5} - \frac{1 - x^2}{2\lambda x}$$

on $\mathcal{D} = (-1, 1) \setminus \{0\}$. Last example tells us that gilations are not always as easily recognizable as dilations (which are nothing but diagonal matrices) even in one dimension.

*Remark 3.* Let $H : \mathcal{D} \to \mathcal{D}^H$ be a transformation with an inverse $H^{-1} : \mathcal{D}^H \to \mathcal{D}$. Then $G^H$ is a gilation on $\mathcal{D}^H$ if it is defined as $G^H_\lambda z := H G_\lambda H^{-1} z$ provided that $G$ is a gilation on $\mathcal{D}$.

A gilation on $\mathcal{D}$ will be denoted *trivial* if $G_\lambda x = x$ for all $\lambda > 0$ and $x \in \mathcal{D}$. A gilation is *nontrivial* if it is not trivial.

*Definition 4.* Let $V : \mathcal{D} \to \mathbb{R}_{>0}$ be continuous. System (1) is said to be *monotone with respect to $V$ on $\mathcal{D}$* if for each $x \in \mathcal{D}$ and $t_1, t_2 \in (\nu_x, \tau_x)$

(1) $t_1 < t_2$ implies $V(\phi(t_2, x)) < V(\phi(t_1, x))$,
(2) $\lim_{t \to \tau_x} V(\phi(t, x)) = 0$, and
(3) $\lim_{t \to \nu_x} V(\phi(t, x)) = \infty$.

To give an example consider $\dot{x} = x$ with $\mathcal{D} = \mathbb{R} \setminus \{0\}$. This system is monotone with respect to $V(x) = |x|^{-1}$ on $\mathcal{D}$ since $\phi(t, x) = x \exp(t)$ for $t \in (-\infty, \infty)$.

*Definition 5.* Function $V : \mathcal{D} \to \mathbb{R}_{>0}$ is said to be *homogeneous with respect to gilation $G$ on $\mathcal{D}$* if

$$V(G_\lambda x) = \lambda^\kappa V(x) \tag{2}$$

with fixed $\kappa > 0$ for all $x \in \mathcal{D}$ and $\lambda > 0$.

*Definition 6.* System (1) is said to be *homogeneous with respect to gilation $G$ on $\mathcal{D}$* if for each $x \in \mathcal{D}$ the solution is unique and satisfies for all $\lambda > 0$ and $t \in (\nu_{G_\lambda x}, \tau_{G_\lambda x})$

$$\phi(t, G_\lambda x) = G_\lambda \phi(\theta_{\lambda,\, x}(t), x) \tag{3}$$

where $\theta_{\lambda,\, x} : (\nu_{G_\lambda x}, \tau_{G_\lambda x}) \to (\nu_x, \tau_x)$ is a zero at zero, continuous, and strictly increasing function for each fixed pair of $(\lambda, x)$ and will be called *correlator*.

Sometimes correlator $\theta_{\lambda,\, x}$ is written as $\theta_\lambda$ whenever it does not depend on $x$.

## 3. RESULTS

When one does not have to worry about differentiability (3) is equivalent to, thanks to chain rule and (1),

$$\partial G_\lambda \phi(\theta_{\lambda,\,x}(t),\,x)\cdot f(\phi(\theta_{\lambda,\,x}(t),\,x))\cdot\dot\theta_{\lambda,\,x}(t) = f(G_\lambda\phi(\theta_{\lambda,\,x}(t),\,x)) \qquad (4)$$

where $\partial G_\lambda\eta$ is shorthand for $\partial G_\lambda\eta/\partial\eta$. Moreover, if $\theta_{\lambda,\,x}$ is independent of $x$ then (4) boils down to

$$\partial G_\lambda x\cdot f(x)\cdot\dot\theta_\lambda(t) = f(G_\lambda x) \qquad (5)$$

which implies that $\dot\theta_\lambda(t)$ must be independent of time $t$.

For instance, consider the system $\dot x = x\ln(x)$ on $\mathcal{D} = \mathbb{R}_{>0}$. The solution to this system is $\phi(t,\,x) = x^{\exp(t)}$ which can be shown to be homogeneous with respect to $G_\lambda x = x^\lambda$ with $\theta_\lambda(t) = t$. Observe that

$$\begin{aligned}\partial G_\lambda x\cdot f(x)\cdot\dot\theta_\lambda(t) &= \lambda x^{\lambda-1}\cdot x\ln(x)\cdot 1\\ &= x^\lambda\ln(x^\lambda)\\ &= f(G_\lambda x)\ .\end{aligned}$$

That is one has (5).

*Remark 7.* Let system (1) be homogeneous with respect to some dilation $\Delta$ with degree $d$. Then the solutions satisfy (3) with $G_\lambda x = \Delta_\lambda x$ and $\theta_{\lambda,\,x}(t) = \lambda^d t$. Since then $\partial G_\lambda x = \Delta_\lambda$ and $\dot\theta_{\lambda,\,x}(t) = \lambda^d$, one has from (5) that $\lambda^d\Delta_\lambda f(x) = f(\Delta_\lambda x)$. Hence the standard definition of homogeneity with respect to a dilation is recovered.

*Lemma 8.* Given $x\in\mathcal{D}$ and $t\in(\nu_x,\,\tau_x)$ one has $\tau_{\phi(t,\,x)} = \tau_x - t$ and $\nu_{\phi(t,\,x)} = \nu_x - t$.

**PROOF.** Evident. ■

*Theorem 9.* Consider system (1). Suppose there exists $x\in\mathcal{D}$ and $t\in(\nu_x,\,\tau_x)$ such that $\phi(t,\,x)\neq x$. Then there exists a nontrivial gilation $G$ with respect to which the system is homogeneous on $\mathcal{D}$.

**PROOF.** One can reach the result by construction. Given some $x\in\mathcal{D}$ there are four possibilities regarding $\nu_x$ and $\tau_x$. Each case is studied below.

*Case 1, $\nu_x = -\infty$ and $\tau_x = \infty$:* Let $G_\lambda x := \phi(\ln(\lambda),\,x)$. Observe that $G_1 x = x$ and

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= G_{\lambda_1}\phi(\ln(\lambda_2),\,x)\\ &= \phi(\ln(\lambda_1),\,\phi(\ln(\lambda_2),\,x))\\ &= \phi(\ln(\lambda_1)+\ln(\lambda_2),\,x)\\ &= \phi(\ln(\lambda_1\lambda_2),\,x)\\ &= G_{\lambda_1\lambda_2}x\ .\end{aligned}$$

Therefore $G$ is a gilation. Also observe that

$$\begin{aligned}\phi(t,\,G_\lambda x) &= \phi(t,\,\phi(\ln(\lambda),\,x))\\ &= \phi(t+\ln(\lambda),\,x)\\ &= \phi(\ln(\lambda),\,\phi(t,\,x))\\ &= G_\lambda\phi(t,\,x)\ .\end{aligned}$$

Thus one has $\phi(t,\,G_\lambda x) = G_\lambda\phi(\theta_{\lambda,\,x}(t),\,x)$ with $\theta_{\lambda,\,x}(t) = t$. Note that $\theta_{\lambda,\,x}$ is a strictly increasing, continuous map from $(-\infty,\,\infty)$ to $(-\infty,\,\infty)$ and $\theta_{\lambda,\,x}(0) = 0$.

*Case 2, $\nu_x = -\infty$ and $\tau_x < \infty$:* Let $G_\lambda x := \phi(\tau_x(1-\lambda^{-1}),\,x)$. Observe that $G_1x = x$ and

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= G_{\lambda_1}\phi(\tau_x(1-\lambda_2^{-1}),\,x)\\ &= \phi(\tau_\eta(1-\lambda_1^{-1}),\,\eta)\end{aligned} \qquad (6)$$

where $\eta := \phi(\tau_x(1-\lambda_2^{-1}),\,x)$. Now since, by Lemma 8,

$$\begin{aligned}\tau_\eta &= \tau_x - \tau_x(1-\lambda_2^{-1})\\ &= \tau_x\lambda_2^{-1}\end{aligned}$$

one can proceed from (6) as

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= \phi(\tau_\eta(1-\lambda_1^{-1}),\,\eta)\\ &= \phi(\tau_x\lambda_2^{-1}(1-\lambda_1^{-1}),\,\eta)\\ &= \phi(\tau_x\lambda_2^{-1}(1-\lambda_1^{-1}),\,\phi(\tau_x(1-\lambda_2^{-1}),\,x))\\ &= \phi(\tau_x\lambda_2^{-1}(1-\lambda_1^{-1})+\tau_x(1-\lambda_2^{-1}),\,x)\\ &= \phi(\tau_x(1-(\lambda_1\lambda_2)^{-1}),\,x)\\ &= G_{\lambda_1\lambda_2}x\ .\end{aligned}$$

Therefore $G$ is a gilation. Also observe that

$$\begin{aligned}\phi(t,\,G_\lambda x) &= \phi(t,\,\phi(\tau_x(1-\lambda^{-1}),\,x))\\ &= \phi(t+\tau_x(1-\lambda^{-1}),\,x)\\ &= \phi((\tau_x-\lambda t)(1-\lambda^{-1})+\lambda t,\,x)\\ &= \phi((\tau_x-\lambda t)(1-\lambda^{-1}),\,\phi(\lambda t,\,x))\\ &= G_\lambda\phi(\lambda t,\,x)\end{aligned}$$

since $\tau_{\phi(\lambda t,\,x)} = \tau_x - \lambda t$ by Lemma 8. Hence $\phi(t,\,G_\lambda x) = G_\lambda\phi(\theta_{\lambda,\,x}(t),\,x)$ with $\theta_{\lambda,\,x}(t) = \lambda t$. Note that $\theta_{\lambda,\,x}$ is a strictly increasing, continuous map from $(-\infty,\,\tau_{G_\lambda x})$ to $(-\infty,\,\tau_x)$, and one has $\theta_{\lambda,\,x}(0) = 0$.

*Case 3, $\nu_x > -\infty$ and $\tau_x = \infty$:* Let $G_\lambda x := \phi(\nu_x(1-\lambda),\,x)$. Observe that $G_1x = x$ and

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= G_{\lambda_1}\phi(\nu_x(1-\lambda_2),\,x)\\ &= \phi(\nu_\eta(1-\lambda_1),\,\eta)\end{aligned} \qquad (7)$$

where $\eta := \phi(\nu_x(1-\lambda_2),\,x)$. Now since, by Lemma 8,

$$\begin{aligned}\nu_\eta &= \nu_x - \nu_x(1-\lambda_2)\\ &= \nu_x\lambda_2\end{aligned}$$

one can proceed from (7) as

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= \phi(\nu_\eta(1-\lambda_1),\, \eta)\\ &= \phi(\nu_x\lambda_2(1-\lambda_1),\, \eta)\\ &= \phi(\nu_x\lambda_2(1-\lambda_1),\, \phi(\nu_x(1-\lambda_2),\, x))\\ &= \phi(\nu_x\lambda_2(1-\lambda_1)+\nu_x(1-\lambda_2),\, x)\\ &= \phi(\nu_x(1-\lambda_1\lambda_2),\, x)\\ &= G_{\lambda_1\lambda_2}x\ .\end{aligned}$$

Therefore $G$ is a gilation. Also observe that

$$\begin{aligned}\phi(t,\, G_\lambda x) &= \phi(t,\, \phi(\nu_x(1-\lambda),\, x))\\ &= \phi(t+\nu_x(1-\lambda),\, x)\\ &= \phi((\nu_x-\lambda^{-1}t)(1-\lambda)+\lambda^{-1}t,\, x)\\ &= \phi((\nu_x-\lambda^{-1}t)(1-\lambda),\, \phi(\lambda^{-1}t,\, x))\\ &= G_\lambda\phi(\lambda^{-1}t,\, x)\end{aligned}$$

since $\nu_{\phi(\lambda^{-1}t,\, x)} = \nu_x - \lambda^{-1}t$ by Lemma 8. Hence $\phi(t,\, G_\lambda x) = G_\lambda\phi(\theta_{\lambda,\, x}(t),\, x)$ with $\theta_{\lambda,\, x}(t) = \lambda^{-1}t$. Note that $\theta_{\lambda,\, x}$ is a strictly increasing map from $(\nu_{G_\lambda x},\, \infty)$ to $(\nu_x,\, \infty)$, and $\theta_{\lambda,\, x}(0) = 0$.

*Case 4, $\nu_x > -\infty$ and $\tau_x < \infty$:* Let

$$G_\lambda x := \phi\left(\tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda\nu_x}\right),\, x\right)\ .$$

Observe that $G_1 x = x$ and

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= G_{\lambda_1}\phi\left(\tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_2\nu_x}\right),\, x\right)\\ &= \phi\left(\tau_\eta\left(1-\frac{\tau_\eta-\nu_\eta}{\tau_\eta-\lambda_1\nu_\eta}\right),\, \eta\right)\\ &= \phi\Bigg(\tau_\eta\left(1-\frac{\tau_\eta-\nu_\eta}{\tau_\eta-\lambda_1\nu_\eta}\right)\\ &\qquad +\tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_2\nu_x}\right),\, x\Bigg) \qquad (8)\end{aligned}$$

where $\eta := G_{\lambda_2}x$. Note that

$$\begin{aligned}\tau_\eta &= \tau_x - \tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_2\nu_x}\right)\\ &= \frac{\tau_x(\tau_x-\nu_x)}{\tau_x-\lambda_2\nu_x}\end{aligned}$$

and $\tau_\eta - \nu_\eta = \tau_x - \nu_x$, by Lemma 8. Let us define $T := \tau_x - \nu_x$. Then one can write

$$\begin{aligned}&\tau_\eta\left(1-\frac{\tau_\eta-\nu_\eta}{\tau_\eta-\lambda_1\nu_\eta}\right)\\ &= \frac{\tau_x T}{\tau_x-\lambda_2\nu_x}\left(1-\frac{T}{\frac{\tau_x T}{\tau_x-\lambda_2\nu_x}(1-\lambda_1)+\lambda_1 T}\right)\\ &= \frac{\tau_x T}{\tau_x-\lambda_2\nu_x} - \frac{\tau_x T^2}{\tau_x T(1-\lambda_1)+\lambda_1 T(\tau_x-\lambda_2\nu_x)}\\ &= \frac{\tau_x(\tau_x-\nu_x)}{\tau_x-\lambda_2\nu_x} - \frac{\tau_x(\tau_x-\nu_x)}{\tau_x-\lambda_1\lambda_2\nu_x}\ .\end{aligned}$$

Therefore one can write

$$\begin{aligned}\tau_\eta\left(1-\frac{\tau_\eta-\nu_\eta}{\tau_\eta-\lambda_1\nu_\eta}\right)+\tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_2\nu_x}\right)\\ = \tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_1\lambda_2\nu_x}\right) \quad (9)\end{aligned}$$

Combining (8) and (9) one obtains

$$\begin{aligned}G_{\lambda_1}G_{\lambda_2}x &= \phi\left(\tau_x\left(1-\frac{\tau_x-\nu_x}{\tau_x-\lambda_1\lambda_2\nu_x}\right),\, x\right)\\ &= G_{\lambda_1\lambda_2}x\ .\end{aligned}$$

Therefore $G$ is a gilation. Also, although the intermediate steps of the calculation are not given, it can be shown that

$$\phi(t,\, G_\lambda x) = G_\lambda\phi(\theta_{\lambda,\, x}(t),\, x)$$

with correlator

$$\theta_{\lambda,\, x}(t) = \frac{(\tau_x-\lambda\nu_x)^2 t}{\lambda(\tau_x-\nu_x)^2+(1-\lambda)(\tau_x-\lambda\nu_x)t}\ .$$

It is not hard to check that $\theta_{\lambda,\, x}(t)$ has a positive time derivative, hence strictly increasing, and it continuously maps $(\nu_{G_\lambda x},\, \tau_{G_\lambda x})$ to $(\nu_x,\, \tau_x)$. Also $\theta_{\lambda,\, x}(0) = 0$.

Since in each case the solution to the system is used to construct $G$, the assumption that there exists $x \in \mathcal{D}$ and $t \in (\nu_x,\, \tau_x)$ such that $\phi(t,\, x) \neq x$ rescues $G$ from being trivial. Hence the result. ■

*Remark 10.* Consider the proof of Theorem 9. Observe that

$$\lim_{\nu\to-\infty}\tau\left(1-\frac{\tau-\nu}{\tau-\lambda\nu}\right) = \tau(1-\lambda^{-1})$$

and

$$\lim_{\tau\to\infty}\tau\left(1-\frac{\tau-\nu}{\tau-\lambda\nu}\right) = \nu(1-\lambda)\ .$$

Moreover

$$\lim_{\nu\to-\infty}\frac{(\tau-\lambda\nu)^2 t}{\lambda(\tau-\nu)^2+(1-\lambda)(\tau-\lambda\nu)t} = \lambda t$$

and

$$\lim_{\tau\to\infty} \frac{(\tau-\lambda\nu)^2 t}{\lambda(\tau-\nu)^2 + (1-\lambda)(\tau-\lambda\nu)t} = \lambda^{-1} t \ .$$

That is, the constructed gilations and correlators in Case 2 and Case 3 are the limiting cases of their counterparts in Case 4.

Theorem 9 is an existence result. The construction method in the proof uses the solution which is almost never explicitly known for an arbitrary system. Although it seems a little technical at first sight the proof is simply based on the idea of finding a continuous map between the domain of gilation coefficient $\lambda$ which is $(0,\,\infty)$ and domain of time $t$ which is $(\nu_x,\,\tau_x)$ and using the solution itself as the gilation. Hence it is required that the solution is unique for each initial condition $x \in \mathcal{D}$ both in forward and backward time. This uniqueness assumption which also is embedded in Definition 1 might seem to degrade the generality of the result since even a very simple system such as $\dot{x} = |x|^{\frac{1}{2}}$ with $\mathcal{D} = \mathbb{R}$ does not satisfy the uniqueness assumption. The fix is simple though: just remove (disregard) the stationary point(s) for the analysis. Note that the solution to $\dot{x} = |x|^{\frac{1}{2}}$ *is* unique on $\mathcal{D} = \mathbb{R} \setminus \{0\}$.

*Theorem 11.* Given $V : \mathcal{D} \to \mathbb{R}_{>0}$, let system (1) be monotone with respect to $V$ on $\mathcal{D}$ and its solutions be continuous. Then there exists a continuous nontrivial gilation $G$ with respect to which both the system and $V$ are homogeneous on $\mathcal{D}$.

**PROOF.** Let $\mathcal{C} := \{z \in \mathcal{D} : V(z) = 1\}$. Then let function $\omega : \mathcal{C} \times \mathbb{R}_{>0} \to \mathbb{R}$ satisfy

$$V(\phi(\omega(z,\,\lambda),\,z)) = \lambda$$

and let $\varphi : \mathcal{D} \to \mathcal{C}$ be such that, for all $x \in \mathcal{D}$, $V(\varphi_x) = 1$ and $\varphi_x = \phi(t,\,x)$ for some $t \in (\nu_x,\,\tau_x)$. Both $\omega$ and $\varphi$ are well defined and continuous since the solution and $V$ are continuous and $V$ satisfies the conditions of Definition 4. Let us define gilation $G$ as follows

$$G_\lambda x := \phi(\omega(\varphi_x,\,\lambda V(x)),\,\varphi_x) \ .$$

Note that $V(G_\lambda x) = \lambda V(x)$ by definition. Due to monotonicity, $V(\phi(t_1,\,x)) = V(\phi(t_2,\,x))$ implies $t_1 = t_2$. Therefore

$$G_1 x = x$$

due to that $V(G_1 x) = V(x)$ and that $G_1 x = \phi(t,\,x)$ for some $t \in (\nu_x,\,\tau_x)$. Moreover,

$$\begin{aligned} G_{\lambda_1} G_{\lambda_2} x &= G_{\lambda_1} \phi(\omega(\varphi_x,\,\lambda_2 V(x)),\,\varphi_x) \\ &= \phi(\omega(\varphi_\eta,\,\lambda_1 V(\eta)),\,\varphi_\eta) \end{aligned}$$

where $\eta = G_{\lambda_2} x$. Note that $\varphi_\eta = \varphi_x$ and $V(\eta) = \lambda_2 V(x)$. Therefore one can continue as

$$\begin{aligned} G_{\lambda_1} G_{\lambda_2} x &= \phi(\omega(\varphi_x,\,\lambda_1\lambda_2 V(x)),\,\varphi_x) \\ &= G_{\lambda_1\lambda_2} x \ . \end{aligned}$$

Hence $G$ is a gilation. Gilation $G$ is continuous due to the continuity of its constituent functions. Given $x$ and $\lambda$ let $\theta_{\lambda,\,x} : (\nu_{G_\lambda x},\,\tau_{G_\lambda x}) \to (\nu_x,\,\tau_x)$ be a function satisfying

$$\begin{aligned} &\omega(\varphi_x,\,\lambda V(\phi(\theta_{\lambda,\,x}(t),\,x))) \\ &\qquad = t + \omega(\varphi_x,\,\lambda V(x)) \qquad (10) \end{aligned}$$

Note that (10) implies, by the way $\omega$ is defined,

$$\begin{aligned} &V(\phi(\theta_{\lambda,\,x}(t),\,x)) \\ &\qquad = \lambda^{-1} V(\phi(t + \omega(\varphi_x,\,\lambda V(x)),\,\varphi_x)) \\ &\qquad = \lambda^{-1} V(\phi(t,\,G_\lambda x)) \qquad (11) \end{aligned}$$

which in turn implies $\theta_{\lambda,\,x}$ is continuous since $V$ and $\phi$ are continuous. Equation (11) also implies that $\theta_{\lambda,\,x}$ is strictly increasing since $V(\phi(t,\,x))$ is strictly decreasing in $t$. Also $\theta_{\lambda,\,x}(0) = 0$. Using (10) and the definition of $G$ one can show that

$$\begin{aligned} \phi(t,\,G_\lambda x) &= \phi(t,\,\phi(\omega(\varphi_x,\,\lambda V(x)),\,\varphi_x)) \\ &= \phi(t + \omega(\varphi_x,\,\lambda V(x)),\,\varphi_x) \\ &= \phi(\omega(\varphi_x,\,\lambda V(\phi(\theta_{\lambda,\,x}(t),\,x))),\,\varphi_x) \\ &= G_\lambda \phi(\theta_{\lambda,\,x}(t),\,x) \end{aligned}$$

since $\varphi_x = \varphi_{\phi(\theta_{\lambda,\,x}(t),\,x)}$. Hence the result. ■

Theorem 11 points out that a system and a positive function $V$ with respect to which the system is monotone can be tied with a common gilation with respect to which both the system and $V$ are homogeneous. Function $V$ can be thought as a generalized Lyapunov function, except the fact that its range excludes zero. That exclusion is due to the fact that monotonicity and hence homogeneity break down when $V(x) = 0$, likewise when $V(x) = \infty$. Note that if $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for some system, then it is an immediate observation that the system is monotone with respect to $V$ on $\mathcal{D} \setminus \{z \in \mathcal{D} : V(z) = 0\}$.

*Theorem 12.* Let system (1) be homogeneous with respect to some gilation $G$ and $V : \mathcal{D} \to \mathbb{R}_{>0}$ be a continuous function that is homogeneous with respect to $G$ on $\mathcal{D}$. Suppose for each $x \in \mathcal{D}$ satisfying $V(x) = 1$ there exist $\tau \in \mathbb{R}_{>0}$ and $\nu \in \mathbb{R}_{<0}$ such that, for all $t_1,\,t_2 \in [\nu,\,\tau]$

- $t_1 < t_2$ implies $V(\phi(t_2,\,x)) < V(\phi(t_1,\,x))$,
- $V(\phi(\tau,\,x)) = 2^{-1}$, and
- $V(\phi(\nu,\,x)) = 2$.

Then the system is monotone with respect to $V$ on $\mathcal{D}$.

**PROOF.** Since $V$ is homogeneous it satisfies (2). Without loss of generality, let $\kappa$ in (2) be unity. Since system (1) is homogeneous its solutions satisfy (3). Let $x \in \mathcal{D}$ be given. Define $\lambda := V(x)^{-1}$. Observe that $V(G_\lambda x) = 1$. Let $\theta_{\lambda,\,x}$ be the function in (3) and let $\theta^{-1}_{\lambda,\,x}$ be its inverse which exists since $\theta_{\lambda,\,x}$ is continuous and strictly increasing. Hence there exist $\nu < 0$ and $\tau > 0$ such that $\nu \leq t_1 < t_2 \leq \tau$ implies $V(\phi(t_2,\, G_\lambda x)) < V(\phi(t_1,\, G_\lambda x))$. In addition $V(\phi(\tau,\, G_\lambda x)) = 2^{-1}$ and $V(\phi(\nu,\, G_\lambda x)) = 2$. Then for each $t \in [\theta_{\lambda,\,x}(\nu),\, \theta_{\lambda,\,x}(\tau)]$ one can write

$$\begin{aligned} V(\phi(t,\, x)) &= \lambda^{-1} V(G_\lambda \phi(t,\, x)) \\ &= V(x) V(\phi(\theta^{-1}_{\lambda,\,x}(t),\, G_\lambda x)) \ . \end{aligned}$$

Thence one can infer, since $\theta^{-1}_{\lambda,\,x}$ is a continuous and strictly increasing function, that $\nu_1 \leq t_1 < t_2 \leq \tau_1$ implies $V(\phi(t_2,\, x)) < V(\phi(t_1,\, x))$. Also, $V(\phi(\tau_1,\, x)) = 2^{-1}V(x)$ and $V(\phi(\nu_1,\, x)) = 2V(x)$, where $\nu_1 := \theta_{\lambda,\,x}(\nu)$ and $\tau_1 := \theta_{\lambda,\,x}(\tau)$. One can generalize this to, for $k \in \{1,\, 2,\, \ldots\}$ and $t_1,\, t_2 \in [\nu_k,\, \tau_k]$

- $t_1 < t_2$ implies $V(\phi(t_2,\, x)) < V(\phi(t_1,\, x))$,
- $V(\phi(\tau_k,\, x)) = 2^{-k}V(x)$, and
- $V(\phi(\nu_k,\, x)) = 2^k V(x)$,

where $\nu_x < \nu_{k+1} < \nu_k$ and $\tau_k < \tau_{k+1} < \tau_x$. The result hence follows as $k \to \infty$. ■

Theorem 12 is, in some sense, the generalization of the result that the local asymptotic stability (of the origin) for a homogeneous (with respect to a dilation) system is equivalent to the global asymptotic stability. The generality comes from that monotonicity is a more general concept than asymptotic stability is and that homogeneity with respect to a generalized dilation is considered.

## 4. CONCLUSION

Using generalized dilations, a more general definition for continuous-time homogeneous systems was introduced. It was shown that uniqueness of solutions is enough for a system to be homogeneous with respect to some generalized dilation. The relation between monotonicity and homogeneity was also studied and two basic results were presented. The first one is that if a system is monotone with respect to some positive function $V$ then there exists a generalized dilation with respect to which both the system and $V$ are homogeneous. The second result is that if a system is locally monotone with respect to some $V$ and there exists a generalized dilation with respect to which both the system and $V$ are homogeneous then the system is monotone with respect to $V$.

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