DISCRETE PORT HAMILTONIAN SYSTEMS

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Abstract: Either from a control theoretic viewpoint or from an analysis viewpoint it is necessary to convert smooth systems to discrete systems, which can then be implemented on computers for numerical simulations. Discrete models can be obtained either by discretizing a smooth model, or by directly modeling at the discrete level itself. One of the goals of this paper is to *model* port-Hamiltonian systems at the discrete level. We also show that the dynamics of the discrete models we obtain *exactly* correspond to the dynamics obtained via a usual discretization procedure. In this sense we offer an *alternative* to the usual procedure of modeling (at the smooth level) and discretization.

Keywords: Discrete geometry, Discrete port-Hamiltonian systems.

1. INTRODUCTION

In previous work, see e.g. (van der Schaft and Maschke, 1995; Maschke *et al.*, 1992; van der Schaft, 2000), it has been shown how port-based network modeling of complex lumped-parameter physical systems naturally leads to a generalized Hamiltonian formulation of the dynamics. In fact, the Hamiltonian is given by the total energy of the energy-storing elements in the system, while the geometric structure, defining together with the Hamiltonian the dynamics of the system, is given by the power-conserving interconnection structure of the system, and is called a Dirac structure. Furthermore, energy-dissipating elements may be added by terminating some of the system ports. The resulting class of open dynamical systems has been called "port-Hamiltonian systems" ((van der Schaft and Maschke, 1995; van der Schaft, 2000)). The port-Hamiltonian framework offers many fundamental benefits. Firstly, it is instrumental in finding the most convenient representation of the equations of motion of the system; in the format of purely differential equations or of mixed sets of differential and algebraic equations (DAEs). From an analysis point of view it allows to use powerful methods from the theory of Hamiltonian systems. Finally, the Hamiltonian structure may be fruitfully used in control design, e.g. by the explicit use of the energy function and conserved quantities for the construction of a Lyapunov function (possibly after the connection with another port-Hamiltonian controller system), or by directly modifying by feedback the interconnection and dissipation structure and shaping the internal energy. We refer to (Ortega *et al.*, April 2001; van der Schaft, 2000) for various work in this direction.

It is well known that for the study of complex physical systems, numerical simulation plays an important role. One of the most important areas of numerical analysis is in understanding the role that the structure (conservation laws, symmetries etc.) of the physical system plays in simulations, c.f. (Marsden and West, 2001). It has been well established that, for example, the exact conservation of a momentum integral is very important in attitude control in satellite dynamic simulations, or that energy conserving numerical algorithms has very good stability properties, or that preser-

vation of the symplectic form is important for long time runs in molecular dynamics simulations, and so on. For port-Hamiltonian systems, both from a numerical simulation and a control theoretic viewpoint, we would like to preserve the port-Hamiltonian structure at the discrete level also. For simulation we have just seen (with the examples of satellite dynamics, or for stability purposes etc.) why it is important to preserve certain structure at the discrete level. Next, for the purposes of control (digital control) we will need to set up a discrete model of the smooth port-Hamiltonian system (or controller) in the computer. One interesting application is in the area of haptics, where we are required to interconnect a smooth port-Hamiltonian system with a discrete system which should preferably be port-Hamiltonian . For this, we will need to understand what discrete port-Hamiltonian systems are. Their structure, conservation laws, symmetries etc will need to be formally studied.
So the *basic motivation of this paper* is to formalize the geometric/mathematical structure that port-Hamiltonian systems have at the discrete level. Discrete systems themselves can be derived in two ways. Either we can discretize smooth port-Hamiltonian systems (there exist a wide variety of techniques for doing so), or we can directly model at the discrete level itself. In this paper we proceed along the latter lines, and we show that the discrete models which we obtain as a result of our modeling process *exactly* coincide with discretized models!, thus offering an *alternative approach* towards the simulation of port-Hamiltonian systems.

The outline of the paper is as follows. We briefly recall discrete Hamiltonian mechanics and certain geometrical concepts in Section 2. Discrete Dirac structures, their representations and interconnection properties are introduced in Section 3. The interconnection properties of discrete Dirac structure are derived in Section 4, and discrete port-Hamiltonian systems are defined in Section 5.

2. GEOMETRY AND HAMILTONIAN MECHANICS ON DISCRETE SPACES

In this section we briefly recall certain concepts of discrete Hamiltonian mechanics, for more details c.f. (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*). The first requirement is to choose an appropriate discrete analogue for the reals $\mathbb{R}$. We can use discrete lattices (which have a ring structure), or the space of floating point numbers $\mathbb{F}$ which have a quasi-ring (c.f. (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*)) structure. Since computers use floating-point numbers, and since our main focus is numerical simulation, $\mathbb{F}$ will be our choice. A **discrete vector** at the point $p \in \mathbb{F}^n$ is a pair (p, q) where $q \in \mathbb{F}^n$. We will denote by $T_p\mathbb{F}^n$ the set defined as the union of all possible vectors defined at the point p, i.e. $T_p\mathbb{F}^n = \{(p, q) \in \mathbb{F}^n \times \mathbb{F}^n\} \sim \mathbb{F}^n$. Unlike in the smooth setting, there are several representations of discrete vectors. *Each representation corresponds to a certain numerical integration technique.* We recall two representations here, the Euler discrete vector and the Runge-Kutta 2 vector. These correspond to the Euler forward difference and the second order Runge-Kutta integration techniques. In (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*) we have defined others like Runge-Kutta vectors of any order, Leap-Frog vectors, central difference vectors etc. Euler vectors or Runge-Kutta 2 vectors are defined as: $v(f(p)) = \frac{f(p+\epsilon)-f(p)}{h}$. Where ϵ is the smallest possible distance from the point p to the next floating point number. The difference between Euler vectors and Runge-Kutta 2 vectors is of course in the actual value of $f(p+\epsilon)$. The point we are trying to make is that discrete vectors have the same finite-difference structure, they only differ in the values! A discrete vector [1] does not satisfy the usual Leibniz (or product) rule for derivations, rather it is a linear map $v_i : A_p(\mathbb{F}^n) \to \mathbb{F}$ which satisfies the **modified Leibniz rule**: $v(f \cdot g) = v(f) \cdot g(p) + Aut_v(f(p)) \cdot v(g), \quad \forall f, g \in A_p(\mathbb{F}^n)$, where Aut_v is an automorphism which is a linear map $Aut_v : A_p(\mathbb{F}^n) \to \mathbb{F}$, corresponding to the discrete vector v, defined as: $Aut_v(f(p)) := f(p+\epsilon), \quad p \in \mathbb{F}^n$ such that $Aut_v(f \cdot g) = Aut_v(f) \cdot Aut_v(g); \forall f, g \in A_p(\mathbb{F}^n)$
Discrete covectors are defined as mapping pairs of points (i.e. discrete vectors) to a floating point number, i.e. $v^* : (p, q) \to \mathbb{F}$. The set of discrete covectors forms the discrete cotangent space.

Then, we can define **discrete vector fields** as the mapping X which assigns to each point $p \in \mathbb{F}^n$ a discrete vector, i.e. $\forall p \in \mathbb{F}^n, X(p) = (p, q), q \in \mathbb{F}^n$. The flow of the discrete vector field X is defined as the sequence of points $p_o, p_1, p_2, \cdots$ in $\mathbb{F}^n$ such that $X(p_i) = (p_i, p_{i+1})$. Likewise we can define discrete one-forms as assigning a discrete covector to each point. A function $f : \mathbb{F}^n \to \mathbb{F}$ is said to be **discrete-differentiable** at $p \in \mathbb{F}^n$ iff there exists a mapping $G : A(\mathbb{F}^n) \to \mathbb{F}^n$ s.t. $\frac{f(p+\epsilon)-f(p)-G(f(p)\cdot\epsilon}{\epsilon} = 0$. Note that the above definition does classify discrete functions between those that are discrete differentiable and those which are not. This is easy to see, since we use floating point numbers, the computation - $\frac{f(p+\epsilon)-f(p)-G(f(p)\cdot\epsilon}{\epsilon}$ can easily result in a floating point overflow.
The *discrete exterior differential* is a mapping: $\Delta : \bigwedge^k(\mathbb{F}^n) \to \bigwedge^{k+1}(\mathbb{F}^n)$, defined in the following way. Consider, for instance, a function $f \in A(\mathbb{F}^n)$. The function corresponds to the assignment of an element of $\mathbb{F}$ at each point of the discrete space. The definition of a discrete one-form implies that we must construct a covector at each point. We can do that in many different ways, but if we want to preserve at the discrete level the smooth

[1] In (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*) we have shown that a collection of discrete vectors (Euler vectors, Runge-Kutta vectors etc.) form a 'discrete' tangent space.

property $X(f) = \langle X, \Delta f\rangle$, the definition of the exterior differential must take into account the type of action that vector fields have on functions. For the forward difference method, this leads us to a definition of the exterior differential such as to define the one-form $\Delta f \in \bigwedge^1(\mathbb{F}^n)$ which for every point $p \in \mathbb{F}^n$ assigns to the one-dimensional hypersurface (i.e. a link) connecting each pair of points (p, q), where the pair of points are defining a discrete vector, the value $f(q) - f(p)$ (note that this definition can be easily extended for higher-order forms). Hence, in the natural basis, we would obtain as a representation: $\Delta f(p) = \sum_i (f(p + h\epsilon_i) - f(p))dx^i$, where h is the smallest possible distance from the point p to the next floating point number in the i-th direction of the point p, and $\epsilon_i = [0, \cdots, 1, 0, \cdots]^T$. The concept of **discrete manifolds** has been introduced in (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*). Discrete manifolds are those that locally look like $\mathbb{F}^n$, on these we can define the discrete analogues of charts. atlases etc. Since $\mathbb{F}^n$ has a discrete-differentiable structure, this structure can be transferred onto discrete manifolds via chart mappings.

Let us conclude this section with discrete Hamiltonian mechanics. One way to do that would be by defining a discrete Poisson bracket as follows. Let $\mathcal{Z}$ be a discrete manifold and consider the algebra of discrete differentiable functions $A(\mathcal{Z})$ on $\mathcal{Z}$. This is endowed with a discrete Poisson structure if there exists a mapping from $A(\mathcal{Z})$ to the set of discrete vector fields $\mathfrak{X}(\mathcal{Z})$ which defines an intrinsic operation as: $\{f, g\} := X_f(g)$. This definition easily satisfies the required properties of skew-symmetricity, bilinearity and the modified Leibniz rule. And then we can define discrete Hamiltonian dynamics as follows. We have a canonical mapping from the algebra $A(\mathcal{Z})$ onto the space of discrete vector fields $\mathfrak{X}(A)$ of the algebra: $f \mapsto X_f = \{f, \cdot\}, \quad \forall f \in A(\mathcal{Z})$ The discrete Poisson dynamics are defined as follows. for any $f \in A(\mathcal{Z})$: $\frac{\Delta f(t)}{\Delta t} = \{f, H\} \Rightarrow f_{n+\delta} = f_n + \delta X_H(f_n)$. So in the limit as $\delta \to 0$ we recover the definition of dynamics in the smooth case using the smooth Poisson bracket: $\dot{f} = \{f, H\} = X_H(f)$.

3. DISCRETE DIRAC STRUCTURES

In this section we focus on the mathematical formalization of power-conserving interconnections in a discrete setting. The interconnection of discrete physical systems can be formalized by discrete-Dirac structures, first we consider the special case of constant discrete-Dirac structures. Consider a free quasi-module. $\mathbb{F}^n$ and its dual $\mathbb{F}^{n*}$. We call the product space $\mathbb{F}^n \times \mathbb{F}^{n*}$ as the space of *power variables* and on this product space we define the *power* as: $P = \langle e|f\rangle, \quad (f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*}$, with P taking values in $\mathbb{F}$. $\mathbb{F}^n$ is called the space of flows, and $\mathbb{F}^{n*}$ the space of efforts. On $\mathbb{F}^n \times \mathbb{F}^{n*}$ there exists a canonically defined bilinear form $\ll, \gg$ given by $\forall (f_1, e_1), (f_2, e_2) \in \mathbb{F}^n \times \mathbb{F}^{n*}$:

$$\ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1|f_2\rangle + \langle e_2|f_1\rangle,$$

Definition 1. A constant discrete Dirac structure on a finite-dimensional q-module $\mathbb{F}^n$ is a n-dimensional subspace $\mathcal{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ with the property that

$$\langle e_1|f_2\rangle + \langle e_2|f_1\rangle = 0, \qquad \forall (f_1, e_1), (f_2, e_2) \in \mathcal{D} \tag{1}$$

where $\langle|\rangle$ denotes the natural pairing between $\mathbb{F}^n$ and $\mathbb{F}^{n*}$.

Proposition 1. A constant Dirac structure on $\mathbb{F}^n$ is an n-dimensional subspace $\mathcal{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ with the property that: $\langle e|f\rangle = 0, \quad \forall (f, e) \in \mathcal{D}$.

Proof. Let $(f_1, e_1) = (f_2, e_2)$ then (1) gives $\langle e_1|f_1\rangle + \langle e_1|f_1\rangle = 0$ and hence $\langle e|f\rangle = 0$. Conversely, by linearity for all $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$ we have:

$$\begin{aligned} 0 =& \langle e_1|f_1\rangle + \langle e_2|f_1\rangle + \langle e_1|f_2\rangle + \langle e_2|f_2\rangle \\ =& \langle e_2|f_1\rangle + \langle e_1|f_2\rangle \end{aligned}$$

■

3.1 Representations of Dirac structures

The following representation will be used later on to prove that interconnection of Dirac structures results again in a Dirac structure. The setting is very similar to the smooth setting of (van der Schaft, 1999). Consider an n-dimensional q-module $\mathbb{F}^n$ and its dual n-dimensional q-module $\mathbb{F}^{n*}$. Also consider linear maps $F : \mathbb{F}^n \to W$, $E : \mathbb{F}^{n*} \to W$, with W an n-dimensional q-module. Then define $F + E : \mathbb{F}^n \times \mathbb{F}^{n*} \to W$ as: $(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*} \overset{F+E}{\longmapsto} F(f) + E(e) \in W$. Then we have:

Proposition 2.

- Every Dirac structure $\mathcal{D} \subset \mathbb{F}^n \times \mathbb{F}^{n*}$ can be written as $\mathcal{D} = ker(F + E)$ for linear maps as defined above. Furthermore any such E and F satisfy: $E\,F^* + F\,E^* = 0$.
- Every n-dimensional subspace $\mathcal{D} = \mathrm{ker(F + E)}$ defined by the above linear maps and satisfying $E\,F^T + F\,E^T = 0$, defines a Dirac structure.
- $\mathcal{D}$ can be written in an image representation as: $\mathcal{D} = \{(f, e) \in \mathbb{F}^n \times \mathbb{F}^{n*} | f = E^T\lambda, \; e = F^T\lambda, \; \lambda \in \mathbb{F}^n\}$

Proof. The proof is very similar to that in (van der Schaft, 1999), so we only present one technical detail important for our discrete setting. In the smooth setting, to simplify the proof, (van der Schaft, 1999) identify $\mathbb{F}^n$ with $\mathbb{R}^n$ and also $\mathbb{F}^{n*}$ with $\mathbb{R}^n$, and this was done using the Euclidean inner product on the reals. In our discrete setting we identity $\mathbb{F}^n$ with $\mathbb{F}^n$ and also $\mathbb{F}^{n*}$ with $\mathbb{F}^n$ and we can do this since there is a natural isomorphism between discrete vectors and discrete covectors, see (Talasila *et al.*, 2004*a*; Talasila *et* *al.*, July 5-9, 2004*b*). And then the rest of the proof is the same. ■

4. INTERCONNECTION OF DISCRETE DIRAC STRUCTURES

In this subsection we discuss the interconnection properties of discrete Dirac structures. In the smooth setting a fundamental result in the framework of port-Hamiltonian systems is that the interconnection of a number of Dirac structures results again in a Dirac structure. Physically it is clear that the composition of a number of power-conserving interconnections should result again in a power-conserving interconnection. In the smooth setting this has been formally proved, c.f. (van der Schaft, 1999; van der Schaft and Cervera, n.d.). The question now is if the same property would hold true for discrete models. We consider the composition of two discrete Dirac structures with partially shared variables.

We follow the same sign conventions as in (van der Schaft and Cervera, n.d.) for the power flow corresponding to the power variables $(f_2, e_2) \in \mathcal{D}_b$. Then the interconnection $\mathcal{D}_a || \mathcal{D}_b$ of the Dirac structures $\mathcal{D}_a$ and $\mathcal{D}_b$ is defined as:

$$\mathcal{D}_a||\mathcal{D}_b := \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* | \\ \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \quad \text{s.t.} (f_1, e_1, f_2, e_2) \in \mathcal{D}_a \\ \text{and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_b\}$$

First we will need the following result:

Lemma 1. Given $\lambda \in \mathbb{F}^n$:

$$\big(\exists \lambda \text{ s.t. } C\lambda = d\big) \Leftrightarrow \big(\forall \alpha \text{ s.t. } \alpha^T C = 0 \Rightarrow \alpha^T d = 0\big)$$

Proof. Let us prove first from left to right, this is simple.

$$C\lambda = d \Rightarrow \alpha^* C\lambda = \alpha^* d, \forall \alpha \\ \text{if } \forall \alpha, \ \alpha^* C = 0 \Rightarrow \alpha^* d = 0$$

Now the other way. Suppose $C\lambda \neq d$,

$$\Rightarrow d \notin span(C_i)$$

Define $\hat{C} := \{\alpha | \alpha^* C = 0\}$.

$$\Rightarrow \hat{C}^\perp = span(C_i) \\ \Rightarrow d \notin \hat{C}^\perp \\ \Rightarrow \exists \alpha^* \in \hat{C} \quad \text{s.t.} \quad \alpha^* d \neq 0$$

Hence proved. ■

With the above result we can now prove the following:

Theorem 1. Let $\mathcal{D}_a, \mathcal{D}_b$ be Dirac structures w.r.t. $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$. Then $\mathcal{D}_a||\mathcal{D}_b$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.

Proof. The proof presented here follows the same spirit as the proof for the smooth setting, c.f. (van der Schaft and Cervera, n.d.). Consider the Dirac structures $\mathcal{D}_a$ and $\mathcal{D}_b$ defined in matrix (more correctly - 'module') kernel representation by:

$$\mathcal{D}_a = \{(f_1, e_1, f_a, e_a) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* | \\ F_1 f_1 + E_1 e_1 + F_{2a} f_a + E_{2a} e_a = 0\} \\ \mathcal{D}_b = \{(f_b, e_b, f_3, e_3) \in \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^* | \\ F_{2b} f_b + E_{2b} e_b + F_3 f_3 + E_3 e_3 = 0\}$$

Using Proposition 2. we can easily see that $\mathcal{D}_a$ and $\mathcal{D}_b$ are alternatively given in the 'matrix' image representation as:

$$\mathcal{D}_a = [E_1^* \ F_1^* \ E_{2a}^* \ F_{2a}^* \ 0 \ 0]^*, \quad \mathcal{D}_b = [0 \ 0 \ E_{2b}^* \ F_{2b}^* \ E_3^* \ F_3^*]^*$$

Hence,

$$\Leftrightarrow (f_1, e_1, f_3, e_3) \in \mathcal{D}_a||\mathcal{D}_b \Leftrightarrow \exists \lambda_a, \lambda_b \text{ s.t. } [f_1 \ e_1 \ 0 \ 0 \ f_3 \ e_3]^* \\ = \begin{bmatrix} E_1^* & F_1^* & E_{2a}^* & F_{2a}^* & 0 & 0 \\ 0 & 0 & E_{2b}^* & -F_{2b}^* & E_3^* & F_3^* \end{bmatrix}^* \begin{bmatrix} \lambda_a \\ \lambda_b \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \forall (\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3) \text{ s.t.} \\ (\beta_1^* \alpha_1^* \beta_2^* \alpha_2^* \beta_3^* \alpha_3^*) \begin{bmatrix} E_1^* & F_1^* & E_{2a}^* & F_{2a}^* & 0 & 0 \\ 0 & 0 & E_{2b}^* & -F_{2b}^* & E_3^* & F_3^* \end{bmatrix}^* = 0 \\ \beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0 \Leftrightarrow \\ \Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \text{ s.t.}$$

$$\begin{bmatrix} F_1 & E_1 & F_{2a} & E_{2a} & 0 & 0 \\ 0 & 0 & -F_{2b} & E_{2b} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \alpha_3 \\ \beta_3 \end{bmatrix} = 0$$

$$\beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0 \Leftrightarrow \\ \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \in \mathcal{D}_a||\mathcal{D}_b, \beta_1^* f_1 + \alpha_1^* e_1 + \beta_3^* f_3 + \alpha_3^* e_3 = 0 \\ \Leftrightarrow (f_1, e_1, f_3, e_3) \in (\mathcal{D}_a||\mathcal{D}_b)^\perp$$

Thus $\mathcal{D}_a||\mathcal{D}_b = (\mathcal{D}_a||\mathcal{D}_b)^\perp$, and hence it is a Dirac structure. ■

5. DISCRETE PORT-HAMILTONIAN SYSTEMS

From a network modeling perspective a finite-dimensional physical system is naturally described by a set of energy-storing elements, a set of energy-dissipating elements and a set of external ports (via which the interaction with the environment can take place) - interconnected to each other by a *power-conserving interconnection.* Associated with the energy storing elements are energy variables $z_1, \cdots, z_n$ being coordinates for some n-dimensional state discrete-manifold $\mathcal{Z}$, and a total energy $H : \mathcal{Z} \to \mathbb{F}$. First we formalize the power-conserving interconnection by a constant Dirac structure $\mathcal{D}$ on the finite-dimensional space $\mathcal{F} := \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$, with $\mathcal{F}_S$ denoting the space of flows f_S connected to the energy-storing elements, $\mathcal{F}_R$ denoting the space of flows f_R connected to the energy dissipating elements, and $\mathcal{F}_P$ denoting the space of external flows f_P which can be connected to the environment. Dually we write $\mathcal{E} := \mathcal{E}_S \times \mathcal{E}_R \times \mathcal{E}_P$ with the efforts $e_S \in \mathcal{E}_S, e_R \in \mathcal{E}_R, e_P \in \mathcal{E}_P$ being the corresponding dual variables of $f_S \in \mathcal{F}_S, f_R \in \mathcal{F}_R, f_P \in \mathcal{F}_P$, i.e. with $\mathcal{E}_S = \mathcal{F}_S^*, \mathcal{E}_R = \mathcal{F}_R^*, \mathcal{E}_P = \mathcal{F}_P^*$

Definition 2. Let $\mathcal{Z}$ be a discrete n-dimensional manifold of energy variables, and let $H : \mathcal{Z} \to \mathbb{F}$ be a discrete Hamiltonian. Furthermore, let $\mathcal{F}_P$ be the space of external flows f, with $\mathcal{E}_P$ the dual space of external effort e. Consider a Dirac structure on the product space $\mathcal{Z} \times \mathcal{F}_P$, only depending on z. The implicit discrete port-Hamiltonian system corresponding to $\mathcal{Z}$, $\mathcal{D}$, H and $\mathcal{F}_P$ is defined by the specification:

$$\left(-\frac{\Delta z}{\Delta t},\ f,\ \partial_z H(z),\ e \right) \in \mathcal{D}(z)$$

Note that the minus sign in front of the flow $\frac{\Delta z}{\Delta t}$ physically means that the ingoing power is positive. The efforts and flows corresponding to the energy-storing elements are given as $f_S = \frac{\Delta z}{\Delta t},\quad e(z) = \partial_z H(z)$, and then it follows that the physical system is described by the set of *Difference Algebraic Equations*:

$$F_z \frac{\Delta z(t)}{\Delta t} + E_z \partial_z H(z) + F f(t) + E e(t) = 0$$

In the smooth setting the next step would be to define the energy balance as follows. For all $(X, f, \alpha, -e) \in \mathcal{D}$ we have: $\langle \alpha | X \rangle - \langle e | f \rangle = 0$, due to which it follows that an implicit smooth port-Hamiltonian system satisfies the energy balance, c.f. (Dalsmo and van der Schaft, 1998; van der Schaft, 1999; van der Schaft and Maschke, 1995), $\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} = e^T f$. In the above computation one uses the chain rule for differentiation. The chain rule however does not work in the discrete setting. Let us see this with a very simple example: For instance, consider the action of a derivation on the function $f(x) = x^2$. From the definition of twisted derivation we have: $X(f)(x) = X(x^2) = X(x) \cdot x + Aut_X(x) \cdot X(x)$. Only if $Aut_X(x) = x$ for any vector field X, the chain rule is satisfied. So in general we do not have a discrete version of the chain rule. What does this imply?

First of all note that in the discrete case we do have the following: For all $(X, f, \alpha, -e) \in \mathcal{D}$ we have:

$$\langle \alpha | X \rangle - \langle e | f \rangle = 0 \Rightarrow \partial_z H(z) \cdot \frac{\Delta z}{\Delta t} - e^T f = 0$$

However $\frac{\Delta H}{\Delta t} \neq \partial_z H(z) \frac{\Delta z}{\Delta t}$, since the chain rule is not valid in the discrete setting. And indeed, it is well known that there exist no basic integration techniques (like Euler integration, Runge-Kutta etc.) that preserve the energy balance relation. There exist many special integration techniques that do preserve the energy balance, but these techniques dramatically alter the geometric structure of the Dirac framework. We have discussed these aspects in (Talasila *et al.*, 2004*a*; Talasila *et al.*, July 5-9, 2004*b*) on discrete Hamiltonian systems where we showed how the Poisson structure can get dramatically modified under structure preserving algorithms. Similar analysis also holds for Dirac structures, this will be the subject of future work. In any case, in general we would have an energy relation of the following type:

$$\frac{\Delta H}{\Delta t} = \partial_z H(z) \cdot \frac{\Delta z}{\Delta t} - e^T f - \tilde{H} = 0$$

where $\tilde{H}$ is the extra energy that is *created* in the system as a result of the discrete process. In the continuum limit $\tilde{H} \to 0$.

6. EXAMPLES

In this section we present two examples of the modeling and simulation of port-Hamiltonian systems in the discrete setting. We will show that the simulations from our discrete model *exactly* coincide with the simulations that we get via a corresponding discretization technique.

Example 1. Consider the electrical circuit as shown in Figure 1.

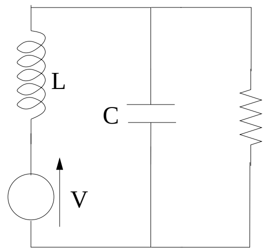

Fig. 1. A driven RLC circuit

For notational simplicity we assume $L = C = 1$. The Hamiltonian function is given by: $H(q, \phi) = \frac{1}{2}(q(t)^2 + \phi(t)^2)$. Then the discrete dynamics are, using the Dirac structure, given by:

$$\frac{\Delta q}{\Delta t} = \partial_\phi H - \frac{1}{R}\partial_\phi H, \qquad \frac{\Delta \phi}{\Delta t} = -\partial_q H + V$$

Note that $\partial_q H = \partial_q \frac{q^2}{2} = \frac{(q+\epsilon)^2 - q^2}{\epsilon} = q + \frac{\epsilon}{2}$. However since ϵ is extremely small (on the order of 10e-16) so for the examples considered here it does not affect the simulation results, and hence we can safely ignore the ϵ terms. So $\partial_q H = q$ and $\partial_\phi H = \phi$. Let us use a Runge-Kutta 2 discrete vector, and let us compare the simulation results with the usual second order Runge-Kutta technique. The simulation results in Figure 2 show an exact matching between the two approaches.

Example 2. Now we model the Van der Pol circuit in our discrete setting. The Hamiltonian is: $H(q, \phi) = \frac{1}{2}(q(t)^2 + \phi(t)^2)$. The discrete dynamics are defined as follows:

$$\frac{\Delta q}{\Delta t} = \partial_\phi H + Ep \cdot q(t) \cdot (1 - \phi(t)^2), \qquad \frac{\Delta \phi}{\Delta t} = \partial_q H$$

The simulation results are shown in Figure 3 and we have used the Runge-Kutta 2 discrete vector, again the comparison with a second order Runge Kutta technique shows an exact matching.

7. CONCLUSIONS AND FUTURE WORK

In this paper we provided an alternative to the usual two stage process of modeling and dis-

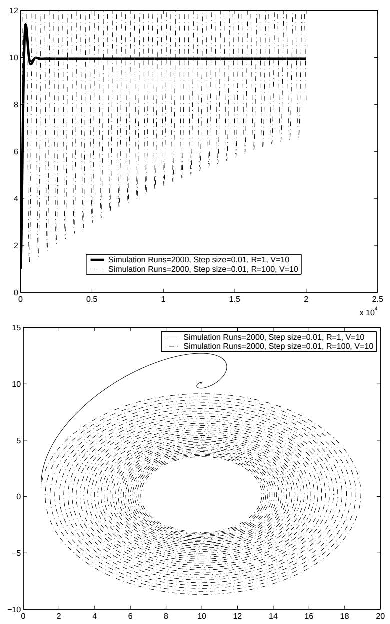

Fig. 2. Discrete dynamics of a driven RLC circuit.

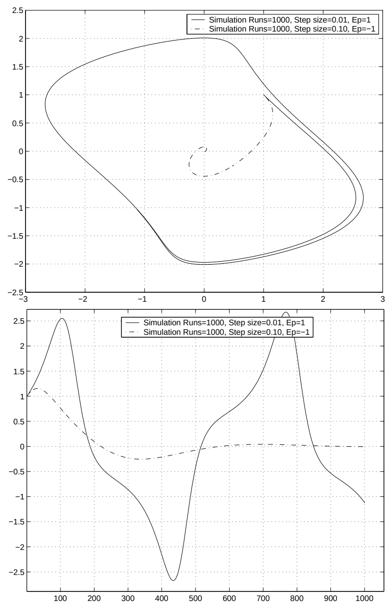

Fig. 3. Hopf bifurcation in the discrete dynamics of the Van der Pol circuit.

cretization of port-Hamiltonian systems - we defined a framework for the discrete modeling of such systems, so as to provide models that are trivially implementable on computers for either numerical simulation or digital control. Moreover all the geometric/mathematical structure, and the corresponding analysis, presented in this paper is also perfectly valid for discretized port-Hamiltonian systems.

This paper is the first stage of the process of formalizing the discrete structure of port-Hamiltonian systems which we would later like to use for *modular simulation*. The port-Hamiltonian framework is suitable for the modular approach to modeling complex physical systems. Regarding simulation, it is well known that basic integration algorithms do not preserve important structure. It is much harder to design structure preserving algorithms for an entire discretized system, than designing structure preserving algorithms for each individual discretized submodel and then interconnecting all these discrete submodels. Of course, to do so we need a formal interconnection theory at the discrete level, which we have provided in this paper. Our future work will concern developing the concept of modular simulations.

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