FAULT TOLERANT CONTROL : THE PSEUDO-INVERSE METHOD REVISITED

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Abstract: In the pseudo-inverse method (PIM) the Frobenius norm based distance between the closed loop model of the faulty system and some reference model is minimized. Stability issues are considered in the Modified PIM (MPIM). This paper proposes to use a set of admissible models, rather than searching for an optimal one which does not provide any stability / adequation guarantee. The approach allows to characterize the set of accommodable faults, and to quantify the robustness of the fault adaptation scheme.

Keywords: Fault Tolerant Control, Diagnosis, Control Systems, Model Matching

1. INTRODUCTION

The Fault Tolerant Control problem (Blanke *et al.*, 2003), (Patton, 1997) has been addressed considering many different objectives : stability, disturbance attenuation (Niemann and Stoustrup, 2002), model matching (Gao and Antsaklis, 1991), (Huang and Strangel, 1990), predictive control (Maciejowski, 1997), optimal control (Staroswiecki, 2003*a*), (Staroswiecki, 2003*b*). Model matching and the pseudo-inverse method (PIM) have been first introduced in flight control systems, see e.g. (Caglayan *et al.*, 1988), (Gao and Antsaklis, 1991), (Ostroff, 1985), to deal with situations where pilots must keep faulty systems at hand. However, the PIM does not guarantee the stability of the obtained solution, a problem which has been later addressed by (Gao and Antsaklis, 1991).

In this paper, the model matching problem is revisited by searching for the solution within an admissible set of reference models, instead of finding the best approximation of an ideal one. It is organised as follows. Section 2 presents the nominal control problem. Section 3 addresses both the fault accommodation and the system reconfiguration strategies to solve the fault tolerant control problem. Section 4 first presents the classical PIM method, which is used when the problem has no exact solution, and then proposes a more convenient point of view leading to the definition of the admissible model matching approach. The example used in (Gao and Antsaklis, 1991) is revisited in Section 5 using this new point of view. Section 6 concludes the paper.

2. NOMINAL PROBLEM

Let subscript n stand for *nominal*. In the LTI model matching problem, the objective O_n is to design a control law for the system submitted to the constraints

$$C_n : \dot{x}(t) = A_n x(t) + B_n u(t) \tag{1}$$

such that the closed loop behavior follows the reference model

$$\dot{x}(t) = M^* x(t) + N^* e(t) \tag{2}$$

where $x(t) \in R^n$ is the system state, the pair M^*, N^* is given, and $e(t) \in R^q$ is an arbitrary input vector.

Considering state feedback, the set of admissible controls is

$$U_n : \left\{ \begin{array}{c} R^n \times R^q \longrightarrow R^m \\ (x(t), e(t)) \longmapsto u(t) = G_n e(t) - K_n x(t) \end{array} \right.$$

where G_n and K_n are matrices to be determined. Therefore, the nominal solution of the model matching problem is obtained by solving the system

$$\begin{aligned} A_n - B_n K_n &= M^* \\ B_n G_n &= N^* \end{aligned} \tag{3}$$

for K_n and G_n, which can always be done provided that

$$\begin{aligned} Im(A_n - M^*) &\subseteq Im(B_n) \\ Im(N^*) &\subseteq Im(B_n) \end{aligned} \tag{4}$$

Assume $rank(B_n) = m$, the unique solution is

$$\begin{aligned} K_n &= B_n^+ (A_n - M^*) \\ G_n &= B_n^+ N^* \end{aligned} \tag{5}$$

where B_n^+ is the left pseudo-inverse of B_n, i.e. a matrix such that $B_n^+ B_n = I$.

3. FAULT TOLERANT CONTROL

The fault tolerant control problem is defined by the triple $< O_n, C_f, U_f >$, where O_n is the (unchanged) objective, and C_f (resp. U_f) is the set of constraints (resp. of admissible controls) associated with the post-fault system. According to the performance of the FDI algorithm, fault accommodation or system reconfiguration have to be used.

3.1 *Fault accommodation*

Assume that the FDI algorithm does not only detect and isolate faults, but also identifies the faulty system model (which is assumed to be still LTI). The triple $< O_n, C_f, U_f >$ is

$$\begin{array}{l} O_n : \dot{x}(t) = M^* x(t) + N^* e(t) \\ C_f : \ \dot{x}(t) = A_f x(t) + B_f u(t) \\ U_f : \ u(t) = G_f e(t) - K_f x(t) \end{array}$$

and the new control law (K_f, G_f) can obviously be found as long as the consistency conditions (4) hold true for the faulty system. Therefore, the set of faults that can be accommodated is defined by the pairs (A_f, B_f) such that

$$\begin{aligned} Im(A_f - M^*) &\subseteq Im(B_f) \\ Im(N^*) &\subseteq Im(B_f) \end{aligned} \tag{6}$$

and (K_f, G_f) can be found by applying (5) with entries (A_f, B_f) instead of (A_n, B_n).

3.2 *System reconfiguration*

Assume now that only fault detection and isolation is available. From detection, it is known that the system is no longer described by (A_n, B_n), but since there is no estimation, the model (A_f, B_f) of the faulty system is not known. However, from fault isolation, the faulty components are isolated, and the model of the system in which they are switched off - say (A_r, B_r) - can be determined (it is assumed that switching off the faulty components is possible). Therefore, the conditions under which objective O_n can be achieved are the same as above, just replacing the pair (A_f, B_f) associated with the faulty system by the pair (A_r, B_r) associated with the reconfigured one. Note that when e.g. actuator faults are addressed, matrix B_r is quite simple to determine, since it is obtained by zeroing the columns of B_n associated with the faulty actuators.

4. OBJECTIVE RECONFIGURATION

When the pair (A_f, B_f) - resp. (A_r, B_r) - is such that the consistency conditions do not hold, neither accommodation nor reconfiguration can provide a solution. Strictly speaking, the objective is not tolerant to such faults. However, the exact requirement may seem too demanding, and approximate rather than exact solutions may be of interest. The PIM uses such an objective reconfiguration approach, since it sets a "best matching" objective instead of an "exact matching" one. This method is recalled and commented in this section, and the idea of an "admissible matching" objective is then proposed and developed. In the sequel, the same notation (A_f, B_f) is used for the two situations associated with fault accommodation and for system reconfiguration.

4.1 *Approximate model matching*

The approximate model matching problem was first stated in (Huang and Strangel, 1990). Since the closed loop matrices $A_f - B_f K_f$ and $B_f G_f$ cannot be made equal to M^* and N^*, approximate solutions are computed, which minimize the two criteria $J_1 = \|A_f - B_f K_f - M^*\|_F^2$ and $J_2 = \|B_f G_f - N^*\|_F^2$ where $\|P\|_F$ is the Frobenius norm of matrix P,

$$\|P\|_F^2 = \sum_{i,j} p_{ij}^2$$

Simple calculations show that the control law (K_f, G_f) which minimizes both J_1 and J_2 is still given by (5), hence the name Pseudo Inverse Method. Replacing K_f and G_f by their optimal value, the result is

$$A_f - B_f K_f - M^* = (I - B_f B_f^+)(A_f - M^*)$$
$$B_f G_f - N^* = (I - B_f B_f^+) N^*$$

which leads to the values

$$J_1 = \sum_{i=1}^{n} (a_f^i - m^{*i})^\tau (I - B_f B_f^+)(a_f^i - m^{*i})$$
$$J_2 = \sum_{i=1}^{n} (n^{*i})^\tau (I - B_f B_f^+)(n^{*i})$$

where a_f^i, m^{*i} and n^{*i} are respectively the i^{th} columns of A_f, $M^*$and N^*. Note that the values of J_1^* and J_2^* are zero when the compatibility conditions hold, but otherwise at least one is non-zero.

There are three major drawbacks to the standard PIM.

- First, exhibiting the closed loop behaviour nearest to the reference one, does not guarantee the accommodated system to be stable. Indeed, as pointed out in (Gao and Antsaklis, 1991), assume M^* is non-defective, and let V be its eigenvector matrix, then by the Bauer-Fike theorem (Steward, 1973) one has
$$|\lambda_f - \lambda^*| \leq \|V\|_2 \left\|V^{-1}\right\|_2 J_1 \quad (7)$$
for any $\lambda^* \in Sp[M^*]$ and $\lambda_f \in Sp[A_f - B_f K_f]$. Therefore, it may happen that $A_f - B_f K_f$ is unstable, although it is the best Frobenius norm approximation of the stable matrix M^*. Extensions have been proposed in the literature, using constrained optimisation, namely the criterion J_1 is minimised under the constraint that $A_f - B_f K_f$ is stable (Gao and Antsaklis, 1991). However, the constrained optimization problem is based on sufficient stability conditions (it may therefore provide very conservative solutions), and it may be complex to solve in real time, when computing power is small (as in embedded systems for example), and solutions are urgently needed.
- Second, finding the accommodated / reconfigured system closest to the reference one, does not guarantee that it will be close enough to exhibit a satisfactory dynamic behaviour (even when it is stable). Moreover, the very meaning of "closest" is questionable, since the choice of the Frobenius norm based distance is arbitrary.
- Finally, it follows from the problem setting that any fault (A_f, B_f) can be accommodated, since there is always a solution to the minimal distance problem. This is indeed a point that contradicts our feeling that in some fault situations, there is no accommodated / reconfigured control which achieves satisfactory approximation of (M^*, N^*).

In order to define solutions which overcome these drawbacks, another objective reconfiguration problem, namely the admissible model matching problem, is proposed.

4.2 *Admissible model matching*

4.2.1. *Problem definition* Let us define two sets, $\mathcal{M}$ and $\mathcal{N}$, such that any solution of

$$\dot{x}(t) = Mx(t) + Ne(t) \quad (8)$$
$$(M, N) \in \mathcal{M} \times \mathcal{N}$$

is admissible, i.e. it has acceptable dynamic behaviour. The admissible model matching problem is defined by the triple

$$\begin{array}{l} O_f : \begin{cases} A_f - B_f K_f \in \mathcal{M} \\ B_f G_f \in \mathcal{N} \end{cases} \\ C_f : \dot{x}(t) = A_f x(t) + B_f u(t) \\ U_f : u(t) = G_f e(t) - K_f x(t) \end{array} \quad (9)$$

In other words, the problem is to find a state feedback control law $u(t) = K_f x(t) - G_f e(t)$ such that the closed loop system

$$\dot{x}(t) = (A_f - B_f K_f)x(t) + B_f G_f e(t)$$

satisfies $A_f - B_f K_f = M$ and $B_f G_f = N$ **for some** $(M, N) \in \mathcal{M} \times \mathcal{N}$. The sets $\mathcal{M}$ and $\mathcal{N}$ being given, the computations to be done on-line, when a fault occurs, only consist in finding a pair $(M, N) \in \mathcal{M} \times \mathcal{N}$ such that consistency conditions (4) hold. A consequence is that the set of faults (A_f, B_f) which are accommodable can be characterized, since there exists a fault accommodating control that achieves objective (9) if and only if the faulty system (A_f, B_f) is such that consistency conditions (4) hold for some $(M, N) \in \mathcal{M} \times \mathcal{N}$.

4.2.2. *Admissible reference models* The sets of admissible reference models $\mathcal{M}$ and $\mathcal{N}$ are defined off-line, and specified such that any trajectory which results from (8) is admissible. In particular, any matrix $M \in \mathcal{M}$ is stable.

Let us first assume that the set of admissible reference models is defined by scalar constraints

$$\mathcal{M} = \{M \text{ s.t. } \Phi(M) \leq 0\} \quad (10)$$
$$\mathcal{N} = \{N \text{ s.t. } \Psi(N) \leq 0\}$$

where functions $\Phi : R^{n \times n} \to R$ and $\Psi : R^{n \times q} \to R$ are given.

4.2.3. *Problem solving* When a fault (A_f, B_f) occurs, the problem becomes that of finding matrices K_f and G_f such that

$$\begin{aligned} \Phi(A_f - B_f K_f) &\leq 0 \quad (11) \\ \Psi(B_f G_f) &\leq 0 \end{aligned}$$

Therefore, it is easily seen that a necessary and sufficient condition for the admissible model matching problem to have a solution is that the faults satisfy

$$\begin{aligned} \min_{K_f} \Phi(A_f - B_f K_f) &\leq 0 \quad (12) \\ \min_{G_f} \Psi(B_f G_f) &\leq 0 \end{aligned}$$

There are several good features characterizing this approach:

- (12) defines the set of faults that are accommodable;
- solving the minimization problem for

$$\begin{aligned} K_f^* &\triangleq \arg\min \Phi(A_f - B_f K_f) \\ G_f^* &\triangleq \arg\min \Psi(B_f G_f) \end{aligned}$$

  is possible but it is not compulsory, since it is enough to find two matrices such that (11) holds;
- finding $\left(K_f^*, G_f^*\right)$ calls for a **non-constrained optimization** procedure, and therefore the solution satisfies classical stationarity conditions

$$\begin{aligned} B_f^\tau \frac{\partial \Phi}{\partial M}(A_f - B_f K_f^*) &= 0 \quad (13) \\ B_f^\tau \frac{\partial \Psi}{\partial N}(B_f G_f^*) &= 0 \end{aligned}$$

(assuming that Φ and Ψ are differentiable);

- uniqueness properties and efficient algorithms are available when functions Φ and Ψ are convex;
- the approach provides some kind of robustness property. Indeed, assume that the nominal system (A_n, B_n), with nominal controller (K_n, G_n) runs on the time interval $[t_0, t_f[$, and that the fault (A_f, B_f) occurs at time t_f. It is not necessary to accommodate the fault as long as

$$\begin{aligned} \Phi(A_f - B_f K_n) &\leq 0 \\ \Psi(B_f G_n) &\leq 0 \end{aligned}$$

  holds, since the faulty system is controlled by the nominal control in such a way that its behavior is still admissible;
- let the system (A_f, B_f) be controlled by a pair (K_f, G_f) such that the closed loop behaviour is admissible. The differences

$$\begin{aligned} \Delta\Phi &\triangleq \Phi(A_f - B_f K_f) - \Phi(A_f - B_f K_f^*) \\ \Delta\Psi &\triangleq \Psi(B_f G_f) - \Psi(B_f G_f^*) \end{aligned}$$

  can be interpreted as the system robustness margins in this situation;
- let (K_f, G_f) be the actually applied admissible control, then it is possible to improve it (e.g. through a steepest descent algorithm) so as to increase the robustness margins (since this is not compulsory, it can be done only at those moments when the available computation power makes it possible);
- finally, the approach is very easily extended when the sets $\mathcal{M}$ and $\mathcal{N}$ are defined not only using one, but several scalar inequalities. Let now Φ and Ψ in (11) be vector functions. Then, condition (12) readily extends to

$$\begin{aligned} \mathcal{P}_K \left[\Phi(A_f - B_f K_f)\right] \cap \Phi_S^- &\neq \emptyset \\ \mathcal{P}_G \left[\Psi(B_f G_f)\right] \cap \Psi_S^- &\neq \emptyset \end{aligned}$$

  where $\mathcal{P}_X \left[\Lambda\left(X\right)\right]$ is the set of Pareto-optimal solutions associated with the minimization of the vector function Λ with respect to the decision variables X, and Λ_S^- is the negative cone in the Λ space.

4.2.4. *The PIM solution* Consider the particular case where admissible sets $\mathcal{M}$ and $\mathcal{N}$ are defined by

$$\begin{aligned} \Phi(M) &= \|M - M^*\|_F^2 - c_M \quad (14) \\ \Psi(N) &= \|N - N^*\|_F^2 - c_N \end{aligned}$$

where (M^*, N^*) is some ideal closed loop system, and the positive scalars (c_M, c_N) characterize admissible neighborhoods of the ideal system - note that, from (7), a choice of c_M which guarantees the system stability for any $M \in \mathcal{M}$ is

$$c_M < \frac{1}{\|V\|_2 \|V^{-1}\|_2} max_{Sp[M^*]} \left\{\lambda^*\right\}$$

Proposition : Let the admissible sets $\mathcal{M}$ and $\mathcal{N}$ be defined by (14), then, a necessary and sufficient condition for the fault (A_f, B_f) to be accommodable is that the faulty system satisfies

$$\begin{aligned} \sum_{i=1}^{n} (a_f^i - m^{*i})^\tau [I - B_f B_f^+](a_f^i - m^{*i}) &\leq c_M \\ \sum_{i=1}^{n} (n^{*i})^\tau [I - B_f B_f^+](n^{*i}) &\leq c_N \end{aligned}$$

Moreover, the pair (K_f, G_f) which satisfies the stationarity conditions (13) is the classical solution of the PIM :

$$\begin{aligned} K_f &= B_f^+(A_f - M^*) \quad (15) \\ G_f &= B_f^+ N^* \end{aligned}$$

The proof is straighforward from (12), (13) and the definitions in (14).

5. EXAMPLE

The LTI system

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \; B = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

was used as an illustration in (Gao and Antsaklis, 1991), where the reference model was

$$\dot{x}(t) = M^* x(t) \text{ with } M^* = \begin{pmatrix} -2 & 0 \\ -5 & -1 \end{pmatrix}$$

The closed loop control with

$$k_1 = 1, \; k_2 = 2$$

$$u = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

achieves perfect model matching, with closed loop system poles $\lambda_1 = -1$ and $\lambda_2 = -2$. In (Gao and Antsaklis, 1991) the fault $A_f = A$ and $B_f = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is considered. Compatibility conditions being no longer satisfied, exact model matching is no longer possible; the PIM provides the result

$$k_{f1} = 2, k_{f2} = 0$$

and therefore the optimal closed loop matrix is

$$A_f + B_f K_f = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$$

which is unstable. The modified approach proposed in (Gao and Antsaklis, 1991) provides

$$k_{f1} = 0.8, k_{f2} = 0$$

which results in

$$A_f + B_f K_f = \begin{pmatrix} -0.2 & 0 \\ -0.8 & -1 \end{pmatrix}$$

(poles are $\lambda_1 = -1$ and $\lambda_2 = -0.2$). Note that this approach indeed allows to stabilize the system but offers no real control over the behavior of the closed loop system which results from fault accommodation : the nearest closed loop matrix (under stability constraints) is selected, which in this example results in one unchanged pole, and the other one divided by a factor 10. Following the approach proposed in this paper, define the set of admissible closed loop matrices by

$$\mathcal{M} = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} s.t. \begin{array}{c} 2p^2 + 2s^2 - 5ps + 9rq = 0 \\ p + s \in [-3.3, -2.7] \end{array} \right\}$$

It can easily be checked that any matrix in $\mathcal{M}$ has eigenvalues

$$\lambda_1 = \tau \lambda_1^*$$
$$\lambda_2 = \tau \lambda_2^*$$

with $\tau \in [0.9, 1.1]$ (this is the way the set has been constructed). This means that, instead of trying to match the reference model M^*, the accommodated control tries to obtain an admissible closed loop matrix, such that its eigenvalues lie within a $\pm 10\%$ range of the eigenvalues of M^*. A solution is

$$k_{f1} = -1, k_{f2} = 0$$

which gives

$$A_f + B_f K_f = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}$$

6. CONCLUSION

Fault tolerance is the property that a system remains able to achieve a given objective (or enjoy a given property) in the presence of faults from a given fault set. In this paper, the model matching objective has been addressed, in the presence of parametric faults.

The classical and the modified pseudo-inverse methods have been extended, by using a set of admissible models, rather than searching for an optimal one which does not provide any guarantee about the post-fault system behavior. This approach applies to both the fault accommodation and the system reconfiguration strategies, and it has ben shown that it provides some degree of robustness of the adaptation scheme with respect to unanticipated faults, whose accommodation domain has been characterized.

7. REFERENCES

Blanke, M., M. Kinnaert, J. Lunze and M. Staroswiecki (2003). *Diagnosis and Fault Tolerant Control.* Springer-Verlag, ISBN 3-540-01056-4.

Caglayan, A. K., S.M. Allen and K. Wehmuller (1988). Evaluation of a second generation reconfiguration strategy for aircraft flight control systems subjected to actuator failure surface damage. In Proc. of the 1988 IEEE National Aerospace and Electronics Conference. pp. 520 – 590.

Gao, Z. and P.J. Antsaklis (1991). Stability of the pseudo-inverse method for reconfigurable control systems. *Int. J. of Control* **53**, 717–729.

Huang, R., F. and C. Y. Strangel (1990). Restructurable control using proportional-integral implicit model following. *J. Guidance, Control and Dynamics* **13**, 303–309.

Maciejowski, J. M. (1997). Reconfigurable control using constrained optimization. In: *Proc. Of European Control Conference ECC'97, Plenary Lectures and Mini-Courses.* Brussels. pp. 107 – 130.

Niemann, H. H. and J. Stoustrup (2002). Reliable control using the primary and dual youla parameterization. In: *Proc. Of 41st IEEE Conf. Decision and Control*.

Ostroff, A. J. (1985). Techniques for accommodating control effector failures on a midly statically unstable airplane. In: *Proc. Of the American Control Conference*. pp. 903–906.

Patton, R. J. (1997). Fault tolerant control : The 1997 situation.. In: *Proc. Of IFAC Symp. On Fault Detection Supervision and Safety of Technical Processes, Safeprocess'97*. pp. 1033–1055.

Staroswiecki, M. (2003*a*). Actuator faults and the linear quadratic problem. In: *Proc. Of the 42th IEEE , CDC'03*. Maui, Hawaii, USA. pp. 959–965.

Staroswiecki, M. (2003*b*). On reconfigurability with respect to actuator failures. In: *IFAC World Congress, Barcelona*. Spain.

Steward, G. W. (1973). *Introduction to Matrix Computation*. New York Academic Press.