ENTIRELY LEFT EIGENSTRUCTURE-ASSIGNMENT FOR FAULT DIAGNOSIS OBSERVERS

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Abstract: In this paper a unified geometrically-embedded methodology of synthesis of static decoupled residual generators is presented that leads to a sub-optimal solution to a robust eigenstructure assignment problem based on a predetermined set of eigenvalues. Robustness of this design means that its eigenvector matrix is sufficiently well conditioned. An original idea of a convenient parameterization of corresponding attainable eigensubspaces is developed and new conditions for disturbance decoupling are derived solely in terms of the left eigenvectors of a system observation state-transition matrix. The resolved completely-static disturbance-decoupling design problem can be treated as an effective preliminary-design stage of an integral synthesis of residual generators. A numerical example illustrates the effectiveness of the presented method.

Keywords: fault detection and isolation, state estimation, observers, discrete-time systems, eigenstructure, robust design.

1. INTRODUCTION

The problem of synthesis of robust state observers for dynamic plants is considered to be an essential point in the design of diagnostic systems for *FDI* (failure detection and isolation, Chen and Patton, 1999; Gertler, 1998; Kowalczuk and Suchomski, 2004; Liu and Patton, 1998).

Intelligible analytical judgment shows that beneficial attributes of such observers can be obtained by appropriately designing the eigenstructure of a system observation state-transition matrix represented by its eigenvalues and eigenvectors.

Most design procedures based on the eigenstructure assignment paradigm are principally concerned with eigenvalue insensitivity graded by a certain sensitivity measure (Kautsky, *et al.* 1985). Unfortunately, as has been shown by Sobel *et al.* (1994), even a small improvement in a practical fault-sensitivity measure may result in a significant degradation in a frequency-domain measure of robustness. Thus, any eigenstructure synthesis should be regarded as a multi-objective optimization problem (Liu and Patton, 1998).

In this paper a unified methodology for synthesis of static decoupled residual generators is presented that specifies how to achieve a sub-optimal solution of the robust eigenstructure assignment problem using a geometric approach and a predetermined set of eigenvalues. In the applied narrow sense, robustness means that an eigenvector matrix is constrained to be as well conditioned as possible (Kowalczuk and Suchomski, 2004). In particular, we propose a convenient parameterization of the corresponding attainable eigensubspaces and derive new conditions for disturbance decoupling solely in terms of left eigenvectors of the observation state-transition matrix.

2. SYSTEM MODELING

Let the monitored system be described as

$$x(k+1) = Ax(k) + B_u u(k) + B_d d_d(k) + Ef(k) \quad (1)$$

$$y(k) = Cx(k) + D_u u(k) + D_n d_n(k) + Ff(k) \quad (2)$$

where $x(k) \in \mathbb{R}^n$ denotes a plant state, $u(k) \in \mathbb{R}^{n_u}$ means an input (controlling) signal, $y(k) \in \mathbb{R}^m$ is an output, $d_d(k) \in \mathbb{R}^{n_d}$ stands for an immeasurable plant disturbance, $d_n(k) \in \mathbb{R}^{n_n}$ is measurement noise, while the vector $f(k) \in \mathbb{R}^v$ models pertinent faults. It is assumed that (A, C) is completely observable, B_u and B_d are of a full column rank and C has a full row rank. Since the number of independent disturbances that can be decoupled cannot be larger than the number of independent measurements, we infer that $m \geq n_d$. It is also presumed that the vector fault is modeled by an unknown time function $f(k)$, while its influence on the state evolution $x(k)$ and the plant output $y(k)$ is described by the constant matrices E and F, respectively (Chen and Patton, 1999).

3. PRIOR SYNTHESIS OF RESIDUALS

A full-order observer has the following form (Chen and Patton, 1999; Gertler, 1998)

$$\hat{x}(k+1) = A_o\hat{x}(k) + (B_u - K_oD_u)u(k) + K_oy(k) \quad (3)$$

where $\hat{x}(k) \in \mathbb{R}^n$ is a state estimate and K_o is an observer gain. Let $A_o = A - K_oC$ denote the observer state transition matrix. It is assumed that A_o is a stable matrix, i.e. all its eigenvalues $\sigma(A_o)$ belong to the open unit circle. The evolution of the state estimation error $x_e(k) = x(k) - \hat{x}(k)$ can be described by $x_e(k+1) = A_ox_e(k) + B_dd_d(k) - K_oD_nd_n(k) + (E - K_oF)f(k)$.

An output estimate $\hat{y}(k)$ takes the form of $\hat{y}(k) = C\hat{x}(k) + D_uu(k)$ leading to an original output error $y_e(k) = y(k) - \hat{y}(k)$. Let $y_w(k) = Wy_e(k)$ denote a weighted residual, where $W \in \mathbb{R}^{w\times m}$ is a design matrix of a full row rank, $\operatorname{rank}(W) = w \leq m$ (Liu and Patton, 1998; Chen and Patton, 1999).

The $z-$transform of $y_w(k)$ takes the form

$$\begin{aligned} \underline{y}_w(z) = G_{rf}(z)\underline{f}(z) + G_{rd}(z)\underline{d}_d(z) + \\ G_{rn}(z)\underline{d}_n(z) + G_{rx}(z)x_e(0) \end{aligned} \quad (4)$$

where: $G_{rf}(z) = WF + HT_0(z)(E - K_oF)$, $G_{rd}(z) = HT_0(z)B_d$, $G_{rn}(z) = WD_n - HT_0(z) K_oD_n$, $G_{rx}(z) = zHT_0(z)$, $T_0(z) = (zI_n - A_o)^{-1}$, and $H = WC \in \mathbb{R}^{w\times n}$. A computational form of the residual is $\underline{y}_w(z) = (W - HT_0(z)K_o)\underline{y}(z) - (WD_u + HT_0(z)(B_u - K_oD_u))\underline{u}(z)$.

4. EIGENVECTORS ATTAINABILITY

Since (A, C) is completely observable, $\forall\lambda \in \mathbb{R}$ there exists such a $K \in \mathbb{R}^{n\times m}$ for which $\det(\lambda I_n - (A - KC)) = 0$. Assume that K is settled and consider a real $\lambda \in \sigma(A - KC)$. A synopsis of basic notions associated with λ is given below.

- $\alpha_\lambda-$algebraic multiplicity of λ, $1 \leq \alpha_\lambda \leq n$;
- $\gamma_\lambda-$geometric multiplicity (degeneracy) of λ, $1 \leq \gamma_\lambda \leq \alpha_\lambda$, *i.e.* the number of Jordan blocks (eigenchains) associated with λ, $\gamma_\lambda = \dim(\operatorname{Ker}(\lambda I_n - (A - KC))) = n - \operatorname{rank}(\lambda I_n - (A - KC))$;
- $\eta_{\lambda,i}-$size of the ith Jordan block, $i \in \{1, \dots, \gamma_\lambda\}$, containing one left distinct eigenvector associated with λ and corresponding $(\eta_{\lambda,i}-1)$ left generalized eigenvectors (left principal vectors), $1 \leq \eta_{\lambda,i} \leq \alpha_\lambda$ and $\sum_{i=1}^{\gamma_\lambda} \eta_{\lambda,i} = \alpha_\lambda$, $\eta_{\lambda,i}$; are also called the partial multiplicities of λ and the largest of them is called the index of λ;
- $l_{\lambda,i} - i$th left distinct eigenvector associated with λ, $i \in \{1, \dots, \gamma_\lambda\}$, $l_{\lambda,i}^T(A - KC) = \lambda l_{\lambda,i}^T$;
- $l_{\lambda,i,j}$ $-$ jth left generalized eigenvector (left principal vector) corresponding to the ith left distinct eigenvector associated with λ, $j \in \{1, \dots, \eta_{\lambda,i} - 1\}$ and $\eta_{\lambda,i} > 1$, $l_{\lambda,i,j}^T(A - KC) = \lambda l_{\lambda,i,j}^T + l_{\lambda,i,j-1}^T$, where the following notational convention is used $l_{\lambda,i,0} = l_{\lambda,i}$.

4.1 Parameterization of attainable eigensubspaces

Let $\lambda \in \mathbb{R}$. A vector $l_\lambda \in \mathbb{R}^n$ belongs to an ***attainable*** (***assignable***) ***left*** $\lambda-$***eigensubspace*** of (A, C), denoted as $L_\lambda(A, C) \subset \mathbb{R}^n$, if and only if the equality $(\lambda I_n - A^T)l_\lambda = C^Tv_\lambda$ holds for a certain parameter $v_\lambda \in \mathbb{R}^m$ satisfying $v_\lambda = -K^Tl_\lambda$ with a $K \in \mathbb{R}^{n\times m}$. A convenient way for parameterizing $L_\lambda(A, C)$ can be shown with the use of a basis of the null space $\operatorname{Ker}(T_\lambda) \subset \mathbb{R}^{n+m}$ of the matrix $T_\lambda = [\,(\lambda I_n - A^T)\ -C^T\,] \in \mathbb{R}^{n\times(n+m)}$.

The assumed observability of (A, C) implies that $\forall\lambda \in \mathbb{R}$ we have $\operatorname{rank}(T_\lambda) = n$. It follows that $\dim(L_\lambda(A, C)) = m$, and that the maximal attainable geometric multiplicity of a given real eigenvalue of $A - KC$ is equal to m. Two ways of parameterizing $L_\lambda(A, C)$ are derived below.

Separation case: $\lambda \notin \sigma(A)$

Assume that for a given $\lambda \in \mathbb{R}$ we have $\lambda \notin \sigma(A)$. This implies that $\det(\lambda I_n - A) \neq 0$ and consequently $\operatorname{Ker}(T_\lambda) = \operatorname{Im}([\,S_\lambda^T\ I_m\,]^T)$, where $S_\lambda \in \mathbb{R}^{n\times m}$ is defined as a full column rank matrix $S_\lambda = (\lambda I_n - A^T)^{-1}C^T$. The attainable left $\lambda-$eigensubspace of the pair (A, C) can thus be recognized as the range subspace of

S_λ: $L_\lambda(A,C) = \text{Im}\,(S_\lambda)$. An orthonormal basis for the subspace $L_\lambda(A,C) = \text{Im}\,(\underline{U}_S)$ can easily be obtained by the singular value decomposition (*svd*) of S_λ: $S_\lambda = [\,\underline{U}_S\;\bar{U}_S\,]\Sigma_S V_S^T$, where $\underline{U}_S \in \mathbb{R}^{n\times m}$, $\Sigma_S = [\,\underline{\Sigma}_S\; 0_{m\times(n-m)}\,]^T$ with a non-singular diagonal submatrix $\underline{\Sigma}_S \in \mathbb{R}^{m\times m}$, and a unitary $V_S \in \mathbb{R}^{m\times m}$. On the basis of the above we conclude that by taking auxiliary parameters $\bar{v}_\lambda \in \mathbb{R}^m$ we obtain the following preliminary characterizations of the set of parameters v_λ and the set of corresponding attainable left distinct eigenvectors l_λ: $v_\lambda = V_S\underline{\Sigma}_S^{-1}\bar{v}_\lambda$ and $l_\lambda = \underline{U}_S\bar{v}_\lambda$.

Common case: $\lambda \in \sigma(A)$

Let $\lambda \in \sigma(A)$ for a given $\lambda \in \mathbb{R}$. This implies that $\det\,(\lambda I_n - A) = 0$. An orthonormal basis for $\text{Ker}\,(T_\lambda)$ can be derived from the *svd* of T_λ: $T_\lambda = U_T\,\Sigma_T\,[\underline{V}_T\;\bar{V}_T]^T$, where $\Sigma_T = [\,\underline{\Sigma}_T\; 0_{n\times m}\,]$ has a non-singular diagonal submatrix $\underline{\Sigma}_T \in \mathbb{R}^{n\times n}$, $\underline{V}_T \in \mathbb{R}^{(n+m)\times n}$, and $\bar{V}_T \in \mathbb{R}^{(n+m)\times m}$. Hence $\text{Ker}\,(T_\lambda) = \text{Im}\,(\bar{V}_T)$. The following lemma describes submatrices of $[\underline{V}_T\;\bar{V}_T]$ partitioned conformally to $n\times m$.

Lemma 1. *Submatrices of*

$$[\,\underline{V}_T\;\bar{V}_T\,] = \begin{bmatrix} \underline{V}_{T,1} & \bar{V}_{T,1} \\ \underline{V}_{T,2} & \bar{V}_{T,2} \end{bmatrix} \tag{5}$$

have the following properties:

$$\text{rank}\,(\underline{V}_{T,1}) = \begin{cases} n & if \quad \lambda \notin \sigma(A) \\ n_\lambda = n - \gamma_{A,\lambda} & if \quad \lambda \in \sigma(A) \end{cases}$$

where $\gamma_{A,\lambda}$ is the geometric multiplicity of λ as the eigenvalue of A, $\underline{V}_{T,2}$ has a full row rank, $\bar{V}_{T,1}$ has a full column rank, while $\bar{V}_{T,2}$ is singular. □

The range of $\bar{V}_{T,1}$ establishes the attainable left $\lambda-$eigensubspace $L_\lambda(A,C) = \text{Im}\,(\bar{V}_{T,1})$. An orthonormal basis for this subspace, $L_\lambda(A,C) = \text{Im}\,(\underline{U}_{\bar{T}})$, can easily be derived by the *svd* $\bar{V}_{T,1} = [\,\underline{U}_{\bar{T}}\;\bar{U}_{\bar{T}}\,]\Sigma_{\bar{T}}V_{\bar{T}}^T$, where $\underline{U}_{\bar{T}} \in \mathbb{R}^{n\times m}$, $\Sigma_{\bar{T}} = [\,\underline{\Sigma}_{\bar{T}}\; 0_{m\times(n-m)}\,]^T$ has a non-singular diagonal submatrix $\underline{\Sigma}_{\bar{T}} \in \mathbb{R}^{m\times m}$, and $V_{\bar{T}} \in \mathbb{R}^{m\times m}$. Taking auxiliary parameters as $\bar{v}_\lambda \in \mathbb{R}^m$, we obtain a preliminary description of the set of parameters v_λ and the set of attainable left distinct eigenvectors l_λ: $v_\lambda = \bar{V}_{T,2}V_{\bar{T}}\underline{\Sigma}_{\bar{T}}^{-1}\bar{v}_\lambda$ and $l_\lambda = \underline{U}_{\bar{T}}\bar{v}_\lambda$.

4.2 Parameterization of attainable eigenflats

Let $\lambda \in \mathbb{R}$. A vector $l_{\lambda,i,j} \in \mathbb{R}^n$ belongs to an *attainable* (*assignable*) *left* $\lambda-$*eigenflat* of (A,C), denoted as $L_{\lambda,i,j}(A,C) \subset \mathbb{R}^n$, if and only if the equality $(\lambda I_n - A^T)l_{\lambda,i,j} = C^T v_{\lambda,i,j} - l_{\lambda,i,j-1}$ holds for a certain parameter $v_{\lambda,i,j} \in \mathbb{R}^m$ satisfying $v_{\lambda,i,j} = -K^T l_{\lambda,i,j}$ with a $K \in \mathbb{R}^{n\times m}$, $i \in \{1,\ldots,\gamma_\lambda\}$, $j \in \{1,\ldots,\eta_{\lambda,i}-1\}$ and $\eta_{\lambda,i} > 1$. The flat $L_{\lambda,i,j}(A,C)$ can be described as

$$L_{\lambda,i,j}(A,C) = L_\lambda(A,C) \oplus \bar{\bar{v}}_{\lambda,i,j} \tag{6}$$

where for convenience it is assumed that the offset $\bar{\bar{v}}_{\lambda,i,j} \in \mathbb{R}^n$ is orthogonal to $L_\lambda(A,C)$. Below, two ways for parameterization of $L_{\lambda,i,j}(A,C)$ will be considered.

Separation case: $\lambda \notin \sigma(A)$

In the separation case $\bar{\bar{v}}_{\lambda,i,j} = -\bar{U}_S\bar{U}_S^T(\lambda I_n - A^T)^{-1}l_{\lambda,i,j-1}$ leading to the following preliminary characterization of the set of parameters $v_{\lambda,i,j}$ and the set of attainable left generalized eigenvectors $l_{\lambda,i,j}$: $v_{\lambda,i,j} = V_S\underline{\Sigma}_S^{-1}(\bar{v}_{\lambda,i,j} + \underline{U}_S^T(\lambda I_n - A^T)^{-1}l_{\lambda,i,j-1})$ and $l_{\lambda,i,j} = \underline{U}_S\bar{v}_{\lambda,i,j} + \bar{\bar{v}}_{\lambda,i,j}$ with an auxiliary parameter $\bar{v}_{\lambda,i,j} \in \mathbb{R}^m$.

Common case: $\lambda \in \sigma(A)$

In the common case $\bar{\bar{v}}_{\lambda,i,j} = -\bar{U}_{\bar{T}}\bar{U}_{\bar{T}}^T\underline{V}_1\underline{\Sigma}_T^{-1}U_T^T\, l_{\lambda,i,j-1}$, $v_{\lambda,i,j} = -\underline{V}_2\underline{\Sigma}_T^{-1}U_T^T l_{\lambda,i,j-1} + \bar{V}_2V_{\bar{T}}\underline{\Sigma}_{\bar{T}}^{-1}(\bar{v}_{\lambda,i,j} + \underline{U}_{\bar{T}}^T\underline{V}_1\underline{\Sigma}_T^{-1}U_T^T l_{\lambda,i,j-1})$, $l_{\lambda,i,j} = \underline{U}_{\bar{T}}\bar{v}_{\lambda,i,j} + \bar{\bar{v}}_{\lambda,i,j}$, where $\bar{v}_{\lambda,i,j} \in \mathbb{R}^m$ denotes an auxiliary parameter.

4.3 Observer gain

Let $L_n = [\cdots\; l_{\lambda,\cdot}\;\cdots\; l_{\lambda,\cdot,\cdot}\;\cdots] \in \mathbb{R}^{n\times n}$ denote a left modal matrix with columns constructed of linearly independent attainable left (distinct $l_{\lambda,\cdot}$ and, if necessary, generalized $l_{\lambda,\cdot,\cdot}$) eigenvectors of A_o and $V_n = [\cdots\; v_{\lambda,\cdot}\;\cdots\; v_{\lambda,\cdot,\cdot}\;\cdots] \in \mathbb{R}^{m\times n}$ denote a matrix with columns of their corresponding parameters. Pairs of the related columns of these matrices can be arbitrarily ordered. Having completed L_n and V_n, the observer gain can be determined as $K_o = -(V_nL_n^{-1})^T$ (Chen and Patton, 1999; Liu and Patton, 1998). Note that columns of L_n (and the corresponding columns of V_n) can be ordered with respect to various purposes, therefore the similarity relation $J_n = L_n^TA_oL_n^{-T} \in \mathbb{R}^{n\times n}$ generally yields a (permuted) Jordan canonical form of A_o.

5. DISTURBANCE DECOUPLING

5.1 Necessary condition for decoupling

Lemma 2. (Necessary condition for decoupling, Liu and Patton, 1998) *If $G_{rd}(z) \equiv 0_{w\times n_d}$ then $HB_d = WCB_d = 0_{w\times n_d}$ which can be interpreted as a design constraint on the weighting matrix W.* □

The above lemma advises the following convenient rule for the choice of w: $w = \dim\,(\text{Ker}\,(\,(CB_d)^T)) = m - r$, where $r = \text{rank}\,(CB_d)$. In the sequel, it is assumed that $w \geq 1$. To ensure $WCB_d = 0_{w\times n_d}$ we can take a linear combination of vectors from $\text{Ker}\,((CB_d)^T)$ as columns of W^T. An orthonormal basis of this subspace can be established by columns of a submatrix $\bar{U}_d \in \mathbb{R}^{m\times w}$ of rank $(\bar{U}_d) = w$ found by means of the *svd* $CB_d = [\underline{U}_d \; \bar{U}_d]\Sigma_d V_d^T$. Consequently, a useful rule for parameterizing W can be stated as $W = W_w^T \bar{U}_d^T$, where $W_w \in \mathbb{R}^{w \times w}$ of rank $(W_w) = w$ is a non-singular matrix parameter.

5.2 Sufficient conditions for decoupling

The following lemma is an extension of a generic lemma of Liu and Patton (1998), who considered a slightly less general case (solely distinct eigenvalues of A_o).

Lemma 3. (Sufficient condition for decoupling) *Disturbance decoupling $G_{rd}(z) \equiv 0_{w \times n_d}$ is achieved if the following triple condition is satisfied*

$$(w1) \begin{cases} HB_d = 0_{w \times n_d} \\ \mathrm{Im}\,(L_w) = \mathrm{Im}\,(H^T) \\ J_n = \mathrm{diag}\;\{J_w, \bar{J}_w\} \end{cases} \tag{7}$$

where $L_n = [\, L_w \; \bar{L}_w \,]$ is an attainable left modal matrix with $L_w \in \mathbb{R}^{n \times w}$, while $J_w \in \mathbb{R}^{w \times w}$ and $\bar{J}_w \in \mathbb{R}^{(n-w)\times(n-w)}$ are diagonal blocks of the corresponding (permuted) Jordan form of A_o. □

Remarks:

1) Let $J_w = 0_{w \times w}$. This means that all columns of L_w are composed of attainable left distinct eigenvectors corresponding to zero-valued eigenvalues. Hence, the corresponding sufficient condition for disturbance decoupling takes the form $HA_o = 0_{w \times n}$. Consequently, $HT_0(z) = z^{-1}H$, which means that $y_w(k)$ can be implemented as a first-order parity equation (Chen and Patton, 1999). Moreover, taking into account that $H = W_w^T Y_w L_w^T = [\, W_w^T Y_w \;\; 0_{w \times (n-w)} \,] L_n^T$ we have $HK_o = -W_w^T Y_w [\, v_1 \; \cdots \; v_w \,]^T$. It thus follows that for a decoupled residual generator $G_{rf}(z)$ and $G_{rn}(z)$ do not depend on the eigenstructure associated with $\bar{J}_w$.

2) Let $\chi'(L_w) = \|P_{\mathrm{Im}\,(H^T)^\perp} L_w\|$ denote a *disturbance decoupling index* of L_w for the condition (*w1*), where $P_{\mathrm{Im}\,(H^T)^\perp} \in \mathbb{R}^{n \times n}$ is a projection matrix. Considering the *svd* $C^T \bar{U}_d = [\, \underline{U}_c \; \bar{U}_c \,]\Sigma_c V_c^T$ with $\bar{U}_c \in \mathbb{R}^{n \times (n-w)}$ yields $\chi'(L_w) = \|\bar{U}_c \bar{U}_c^T L_w\|$. It follows that for (*w1*) one should minimize:

- the angular distance between a unity-norm attainable left eigenvector l_i corresponding to a given λ_i and the subspace Ker $(\bar{U}_c^T) \cap \mathrm{Im}\,(L_{i-1})^\perp = \mathrm{Im}([\, \bar{U}_c \; \underline{U}_{i-1} \,])^\perp$, $\forall i \in \{1, \ldots, w\}$, and

- the angular distance between a unity-norm attainable left eigenvector l_i corresponding to a given λ_i and the subspace $\mathrm{Im}\,(\underline{U}_{i-1})^\perp$, $\forall i \in \{w+1, \ldots, n\}$, where $\underline{U}_i \in \mathbb{R}^{n \times i}$ is obtained from the *svd* $L_i = [\, l_1 \; \cdots \; l_i \,] = [\, \underline{U}_i \; \bar{U}_i \,][\, \underline{\Sigma}_i^T \; 0_{i \times (n-1)} \,]^T V_i^T$. □

Consider yet another sufficient condition for such a left eigenstructure-assignment that nullifies the transfer function $G_{rd}(z)$.

Lemma 4. (Sufficient condition for decoupling) *Assume that $n - n_0$ eigenvalues of A_o are assigned to $\mu \in \mathbb{R}$ and the corresponding attainable left eigenvectors constituting columns of $\bar{L}_{n_0} \in \mathbb{R}^{n \times (n-n_0)}$ are distinct, where $n - m \leq n_0 \leq n - n_d$. Disturbance decoupling $G_{rd}(z) \equiv 0_{w \times n_d}$ is achieved if the following triple condition is satisfied*

$$(w2) \begin{cases} HB_d = 0_{w \times n_d} \\ \mathrm{Im}\,(L_{n_0}) \subset \mathrm{Ker}\,(B_d^T) \\ J_n = \mathrm{diag}\;\{J_{n_0}, \bar{J}_{n_0}\} \end{cases} \tag{8}$$

where $L_n = [\, L_{n_0} \; \bar{L}_{n_0} \,] \in \mathbb{R}^{n \times n}$, with $L_{n_0} \in \mathbb{R}^{n \times n_0}$, is an attainable left modal matrix, $J_{n_0} \in \mathbb{R}^{n_0 \times n_0}$ denotes a diagonal block of the corresponding (permuted) Jordan form of A_o, and $\bar{J}_{n_0} = \mu I_{n-n_0} \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$.

Remarks:

1) In general, there is no requirement for A_o to be diagonalisable (nondefective). However a diagonal structure of J_{n_0} should be recommended, mainly from the sensitivity point of view.

2) The minimal $n_0 = n - m$ seems to be the most convenient choice. In the case of $n_0 > n - m$, several $\mu-$valued eigenvalues can appear in J_{n_0}.

3) By assuming that $HB_d = 0_{w \times n_d}$, we can conclude that the condition (*w2*) can be achieved when the columns of B_d are assigned as attainable right distinct eigenvectors of A_o corresponding to the zero-valued eigenvalues, *i.e.* $A_o B_d = 0_{n \times n_d}$ (Liu and Patton, 1998). Condition (*w2*) can thus be regarded as a generalization of the above approach solely in terms of the left eigenstructure assignment.

4) The resulting transfer function $G_{rf}(z)$ is not of the dead-beat form. Consequently, the residual signal has a recursive structure and does not correspond to a parity relation. On the other hand, the response of the decoupled residual generator can be speeded up be shifting the free eigenvalues close to zero. This, however, can degrade the conditioning of the corresponding left modal matrix L_n.

5) Let $\chi''(L_{n_0}) = \|P_{\mathrm{Ker}\,(B_d^T)^\perp} L_{n_0}\|$ denote a *disturbance decoupling index* of L_{n_0} for the condition (*w2*), where $P_{\mathrm{Ker}\,(B_d^T)^\perp} \in \mathbb{R}^{n \times n}$ is a projection matrix. Performing the *svd* $B_d = [\, \underline{U}_b \; \bar{U}_b \,]\Sigma_b V_b^T$ with $\underline{U}_b \in \mathbb{R}^{n \times n_d}$ gives $\chi''(L_{n_0}) = \|\underline{U}_b \underline{U}_b^T L_{n_0}\|$. It follows (Kowalczuk and Suchomski, 2004) that we should minimize:

- the angular distance between a unity-norm attainable left eigenvector l_i corresponding to a given λ_i and the subspace Ker $(B_d^T) \cap \text{Im}\,(L_{i-1})^{\perp} = \text{Im}([\,\underline{U}_b\ \underline{U}_{i-1}\,])^{\perp}$, $\forall i \in \{1,\ldots,n_0\}$, and

- the angular distance between a unity-norm attainable left eigenvector l_i corresponding to the $\alpha-$ valued λ_i and the subspace Im $(\underline{U}_{i-1})^{\perp}$, $\forall i \in \{n_0+1,\ldots,n\}$. $\square$

Taking into account the above development of (*w2*), we are going to obtain a dead-beat residual generator. It can be done by constructing a suitable Jordan form of A_o with the zero-valued eigenvalue of the maximal algebraic multiplicity $\alpha_\lambda = n$.

Let us consider a class of systems described by (1) and (2) with $n \leq 2m$. Assume that m eigenvalues of A_o have been assigned to zero and the corresponding attainable left eigenvectors $\{l_i\}_{i=1}^{m}$ are distinct. Let for the properly ordered eigenvectors $\{l_i\}_{i=1}^{n_0} \subset \{l_i\}_{i=1}^{m}$ with $n_0 = n - m$ it hold that $l_i \in$ Ker (B_d^T), $\forall i \in \{1,\ldots,n_0\}$. Next, assume that each l_i forms a two-element eigenchain associated to the zero eigenvalue and let $l_{i,1} \in \mathbb{R}^n$ denote the corresponding first attainable left generalized eigenvector, $\forall i \in \{1,\ldots,n_0\}$. Taking into account the presumed orthogonality of the parameterization (6) and observing that $\{l_i\}_{i=1}^{m}$ forms a basis in $\mathbb{R}^m$, we can conclude that all auxiliary parameters $\{\bar{v}_{i,1}\}_{i=1}^{n_0}$ describing the set $\{l_{i,1}\}_{i=1}^{n_0}$ should be zeroed. It follows that each attainable left generalized eigenvector is uniquely determined solely by the offset of the corresponding flat $l_{i,1} = \bar{\bar{v}}_{i,1}$, $\forall i \in \{1,\ldots,n_0\}$. Clearly, $\{l_{i,1}\}_{i=1}^{n_0}$ can 'truly' be accepted as a set of attainable left generalized eigenvectors if and only if $\{\bar{\bar{v}}_{i,1}\}_{i=1}^{n_0}$ are all nonzero (*i.e.* if all required two-element eigenchains can be effectively established).

Lemma 5. (Sufficient conditions for a decoupled dead-beat observer) *Consider a class of systems with $n \leq 2m$. Assume that m eigenvalues of A_o have been assigned to zero and corresponding attainable left eigenvectors $\{l_i\}_{i=1}^{m}$ are distinct. Disturbance decoupling $G_{rd}(z) \equiv 0_{w\times n_d}$ is achieved if the following triple condition holds*

$$(w2') \begin{cases} HB_d = 0_{w\times n_d} \\ \text{Im}\,(L_{n_0}) \subset \text{Ker}\,(B_d^T) \\ \text{rank}\,(\tilde{L}_{n_0}) = n_0 \end{cases} \qquad (9)$$

where $n_0 = n - m$, $L_{n_0} = [\,l_1\ \cdots\ l_{n_0}\,]$ and $\tilde{L}_{n_0} = [\,l_{1,1}\ \cdots\ l_{n_0,1}\,]$ with

$$\tilde{L}_{n_0} = \begin{cases} \bar{U}_S \bar{U}_S^T A^{-T} L_{n_0} & \textit{if } 0 \notin \sigma(A) \\ -\bar{U}_{\bar{T}} \bar{U}_{\bar{T}}^T \underline{V}_1 \underline{\Sigma}_T^{-1} U_T^T L_{n_0} & \textit{if } 0 \in \sigma(A) \end{cases}$$

Attainable left generalized eigenvectors $\tilde{L}_{n_0}$ are characterized by parameters $\tilde{V}_{n_0} = [\,v_{1,1}\ \cdots\ v_{n_0,1}\,]$

$$\tilde{V}_{n_0} = \begin{cases} -V_S \underline{\Sigma}_S^{-1} \underline{U}_S^T A^{-T} L_{n_0} & \textit{if } 0 \notin \sigma(A) \\ (\bar{V}_2 V_{\bar{T}} \underline{\Sigma}_{\bar{T}}^{-1} \underline{U}_{\bar{T}}^T \underline{V}_1 - \underline{V}_2) \underline{\Sigma}_T^{-1} U_T^T L_{n_0} & \textit{if } 0 \in \sigma(A) \end{cases}$$

Remarks:

1) Considering the resulting left modal matrix $L_n = [\,L_{n_0}\ l_{n_0+1}\ \cdots\ l_m\ \tilde{L}_{n_0}\,]$, we obtain a permuted Jordan form of A_o

$$J_n = \begin{bmatrix} & 0_{m\times m} & 0_{m\times n_0} \\ I_{n_0\times n_0} & 0_{n_0\times(m-n_0)} & 0_{n_0\times n_0} \end{bmatrix} \qquad (10)$$

2) The objective disturbance decoupling index of L_{n_0} takes the form $\chi''(L_{n_0}) = \|\underline{U}_b \underline{U}_b^T L_{n_0}\|$.

3) In fact, the full row rankness of $\bar{U}_S^T A^{-T} L_{n_0}$ or $\bar{U}_{\bar{T}}^T \underline{V}_1 \underline{\Sigma}_T^{-1} U_T^T L_{n_0}$ should be tested for (9).

4) Since A_o is nilpotent, it follows that $T_0(z) = z^{-1} I_n + z^{-2} A_o$, which gives $G_{rf}(z) = WF + z^{-1} H (I_n + z^{-1} A_o)(E - K_o F)$ and $G_{rn}(z) = W D_n - z^{-1} H (I_n + z^{-1} A_o) K_o D_n$. Thus $y_w(k)$ takes the form of a second-order parity equation.

5) Similar results can easily be obtained for high order systems with $n > 2m$. For example, in the case of $2m < n \leq 3m$ the following three- and two-element eigenchains should be considered: $\{l_i, l_{i,1}, l_{i,2}\}_{i=1}^{n-2m}$ and $\{l_i, l_{i,1}\}_{i=n-2m+1}^{m}$, respectively. $\square$

6. EXAMPLE AND CONCLUSION

By taking $\|G\|_\infty = \sup_{\theta\in(-\pi,\pi]} \|G(e^{j\theta})\|$ we can define the following convenient indices of detecting abilities of the examined observers

$\eta_{d/f} = \frac{\|G_{rd}\|_\infty}{\bar{\sigma}(G_{rf}(e^{j\theta}))|_{\theta=0}}$ and $\eta_{n/f} = \frac{\|G_{rn}\|_\infty}{\bar{\sigma}(G_{rf}(e^{j\theta}))|_{\theta=0}}$

Consider the discrete model of an unstable plant

$$A = \begin{bmatrix} 1.5 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.2 \end{bmatrix}, B_u = \begin{bmatrix} 0.05 \\ -0.20 \\ 0.70 \end{bmatrix}, B_d = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, D_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

governed by a state controller $u(k) = -K_c \hat{x}(k)$ with the gain $K_c = [\,14.7973\ 0.2847\ -0.1893\,]$ corresponding to a set of closed-loop poles $\{0.7165, 0.7165, 0.7165\}$. The disturbance $d_d(k) \in \mathbb{R}$ and measurement noise $d_n(k) \in \mathbb{R}^2$ are both Gaussian processes obtained by shaping prototype white-noise processes of dispersions of 1.5 and 0.1, respectively, with the aid of first-order shaping filters of time constants of 3 s and 1 s, respectively.

The following model of a fault in the control channel is assumed: $E = B_u$, $F = D_u$, and

$$f(t) = \begin{cases} 0 & \text{for } t < 100\ \text{s} \\ 0.5 & \text{for } 100\ \text{s} \leq t \leq 120\ \text{s} \\ 0 & \text{for } t > 120\ \text{s} \end{cases}$$

6.1 Decoupled non-dead-beat residual generator

Employing the condition (*w2*) for $n_0 = 1$, $\lambda_1 = 0.4$ and $\lambda_2 = \lambda_3 = 0$ gives the gain

$$K_o = \begin{bmatrix} 1.3571 & -1.2143 \\ 0.0952 & 0.8095 \\ -0.0190 & 0.0381 \end{bmatrix}$$

The generator is described by: $\chi''(L_{n_0}) = 9.81 \times 10^{-18}$, $\kappa(L_n) = 7.31$, $\eta_{n/f} = 13.65$ dB, $\|HA_o\| = 0.77$ and $\|A_oB_d\| = 8.93 \times 10^{-16}$. As can be seen, the disturbance decoupling has been achieved but the generator does not possess the dead-beat attributes. Moreover: $\bar{\sigma}(G_{rf}(e^{j\theta}))\big|_{\theta=0} = 0.410$, $\|G_{rf}(z)\|_\infty = 0.910$, $\|G_{rn1}(z)\|_\infty = 0.958$, $\|G_{rn2}(z)\|_\infty = 1.725$, and $\|G_{rn}(z)\|_\infty = 1.973$. Some additional time and frequency properties of this decoupled non-dead-beat residual generator are shown in Fig. 1.

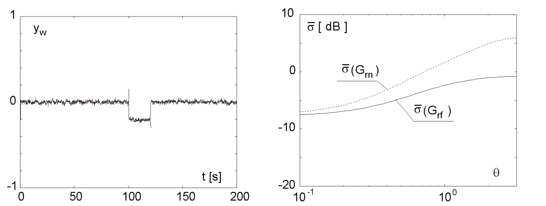

Fig. 1. Time and frequency domain characteristics of a decoupled non-dead-beat residual generator.

For this plant a decoupled dead-beat residual generator with $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 0.4$ can not be obtained by using (*w1*). This condition yields

$$K_o = \begin{bmatrix} 1.5233 & -1.1105 \\ -0.0155 & 0.7403 \\ 0.0031 & 0.0519 \end{bmatrix}$$

The resulting generator is now characterized by: $\chi'(L_w) = 0.47$, $\kappa(L_n) = 7.02$, $\eta_{n/f} = 14.49$ dB, and $\eta_{d/f} = 1.52$ dB. Hence, the disturbance decoupling has not been achieved. Moreover, we observe that $\|HA_o\| = 0.74$ and $\|A_oB_d\| = 0.53$. The plots given in Fig. 2 as well as the following numerical quantities $\bar{\sigma}(G_{rf}(e^{j\theta}))\big|_{\theta=0} = 0.382$, $\|G_{rf}(z)\|_\infty = 0.898$, $\|G_{rd}(z)\|_\infty = 0.455$, $\|G_{rn1}(z)\|_\infty = 1.144$, $\|G_{rn2}(z)\|_\infty = 1.669$, and $\|G_{rn}(z)\|_\infty = 2.024$ confirm the above criticism.

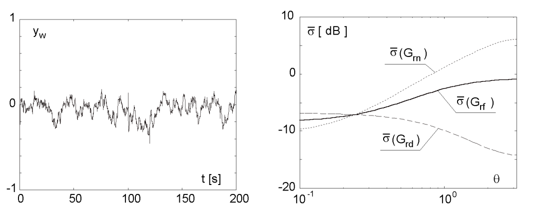

Fig. 2. Time and frequency domain characteristics of a non-decoupled non-dead-beat residual generator.

6.2 Decoupled dead-beat residual generator

Consider the same plant as in the previous section. Employing (*w2'*) with $n_0 = 1$ we get

$$K_o = \begin{bmatrix} 1.6429 & -1.7857 \\ -0.0952 & 1.1905 \\ 0.0190 & -0.0381 \end{bmatrix}$$

The resulting generator is now described by the indices: $\chi''(L_{n_0}) = 3.93 \times 10^{-17}$, $\kappa(L_n) = 2.17$, and $\eta_{n/f} = 21.01$ dB as well as by norms $\|HA_o\| = 1.13$ and $\|A_oB_d\| = 3.36 \times 10^{-16}$. It is clear that the disturbance decoupling has been achieved. Moreover: $\bar{\sigma}(G_{rf}(e^{j\theta}))\big|_{\theta=0} = 0.246$, $\|G_{rf}(z)\|_\infty = 1.275$, $\|G_{rn1}(z)\|_\infty = 1.342$, $\|G_{rn2}(z)\|_\infty = 2.415$ and $\|G_{rn}(z)\|_\infty = 2.763$. Some properties of the residual generator can be examined by inspecting the plots given in Fig. 3.

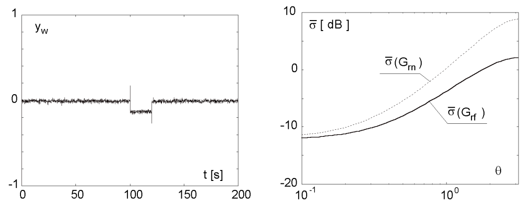

Fig. 3. Time and frequency domain characteristics of a decoupled and dead-beat residual generator.

In conclusion, new conditions on complete static disturbance decoupling have been formulated in terms of left eigenvectors. The issue of parameterization of the attainable subspaces (flats) of the eigenvectors of the state-transition matrix of the residual generator has been proposed.

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