SOLVING OPTIMAL FEEDBACK CONTROL OF CHINESE POPULATION DYNAMICS BY VISCOSITY SOLUTION APPROACH

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Abstract: In this paper, the optimal birth feedback control of a McKendrick type age-structured population dynamic system based on the Chinese population dynamics is considered. Adopt the dynamic programming approach, to obtain the Hamilton-Jacobi-Bellman equation and prove that the value function is its viscosity solution. By the derived classical verification theorem, the optimal birth feedback control is found explicitly. A finite difference scheme is designed to solving numerically the optimal birth feedback control. Under the same constraint, by comparing with different controls, the validity of the optimality of the obtained control is verified numerically.

Keywords: Numerical solutions, Finite difference, Optimal control, Feedback control, Dynamic programming, Distributed-parameter systems, Optimality.

1. INTRODUCTION

Optimal control holds two vividly developing orientations: one is the abstract theory and the other the computational method. For the latter, essentially speaking, there are three approaches (Sargent, 2000). First one is that by the necessary condition of optimality, such as the Pontryagin's maximum principle, to solve a two-point boundary value problem mainly utilizing the multiple shooting method. Second is to convert the original problem into a finite-dimensional nonlinear program by the discretization for the problem. The last one is by the parameterization of the control trajectory to get a nonlinear program problem and then adopt proper steps to tackle it (von Stryk and Bulirsch, 1992). Unfortunately, for above three methods, each one has its fatal faults. The Pontryagin maximum principle provides only necessary conditions for the optimal control and it is usually not in feedback form. The big difficulty for shooting method is the "guess" for the initial data to start the iterative numerical process. It demands that the user understands the essential of the problem well in physics, which is often not a trivial task (Bryson, 1996). For other two methods, the simplification for the original problem leads to the fall of the reliability and accuracy, and when the degree of discretization and parameterization is very high, the work of computation stands out and the solving process gives rise to "curse of dimensionality" (Bryson, 1996). In this paper, differing from any one of above three numerical methods, a viscosity solution approach is adopted to get the optimal feedback control of a McKendrick type age-structured population

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dynamic system based on the Chinese population dynamics numerically. So, in some sense, our method can be seen as the fourth computational method to attack the optimal control problem.

The paper is organized as follows. In Section 2, the dynamic programming principle (DPP, for short) for the value function of the optimal control problem is established. Some other properties of the solution as well as the continuity of the value function are presented. Section 3 is devoted to show that the value function is just the viscosity solution of the corresponding HJB equation. In Section 4, the optimal feedback control is formulated by the value function under the smooth assumption. In Section 5, a finite difference scheme is formulated to the HJB equation of the population system with linear quadratic optimal control. The numerical solution of optimal feedback control is presented. The validity of the optimality of the obtained control is verified numerically in the last section.

2. PROBLEM FORMULATIN AND DPP

A McKendrick type model of age-structured population dynamics developed in (Song and Yu, 1988) is a first order partial differential equation with nonlocal boundary condition described by

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} + \dfrac{\partial p(r,t)}{\partial r} = -\mu(r)p(r,t), \\ \qquad\qquad 0 < r < a_m,\ t > 0, \\ p(r,0) = p_0(r),\ 0 \le r \le a_m, \\ p(0,t) = \beta(t) \displaystyle\int_{a_1}^{a_2} b(r)p(r,t)dr,\ t \ge 0 \end{cases} \tag{1}$$

where $p(r,t)$ denotes the age density distribution at time t and age r for a closed population. $\mu(r)$ is the relative mortality of the population, which satisfies $\int_0^r \mu(\rho)d\rho < \infty$ for $r < a_m$ and $\int_0^{a_m} \mu(\rho)d\rho = \infty$. a_m is the highest age even attained by individuals of the population. $b(r) = k(r)h(r)$. $k(r)$ is the ratio of females and $h(r)$ is the fertility pattern of females satisfying $\int_{a_1}^{a_2} h(r)dr = 1$. Assume that $b(r)$ is bounded and measurable in $[a_1, a_2]$, the fecundity period of females, $0 < a_1 < a_2 < a_m$. $p_0(r)$ is the initial distribution. $\beta(t)$ is the specific fertility rate of the females at time t, which is considered to be the birth control of the population in macro-level.

Let $\mathbf{H} = L^2(0, a_m)$ be the state space with the usual inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For any $t > s \ge 0$, let

$$\mathcal{U}[s,t] = \{\beta(\tau) \in [\beta_0, \beta_1] \subset \mathbf{R}^+ \mid \tau \in [s,t], \\ \beta(\tau) \text{ is measurable on } [s,t]\}.$$

Given $T > 0$ and $p_0 \in \mathbf{H}$, the optimal control problem is to find an optimal control $\beta^*(\cdot) \in \mathcal{U}[0,T]$ such that

$$\begin{cases} J(\beta^*) = \displaystyle\inf_{\beta(\cdot) \in \mathcal{U}[0,T]} J(\beta), \\ J(\beta) = \displaystyle\int_0^T \int_0^{a_m} L(p(r,t), \beta(t), r, t)drdt \\ \qquad\qquad + \displaystyle\int_0^{a_m} f_0(r, p(r,T))dr \end{cases} \tag{2}$$

where $p(r,t)$ is the solution of equation 1 corresponding to $\beta(\cdot)$ and L, f_0 satisfy

$$\begin{cases} \left| \displaystyle\int_0^{a_m} f_0(r, p(r))dr \right|, \left| \displaystyle\int_0^{a_m} L(p(r), \beta, r, t)dr \right| \\ \le C_1 + C_2 \|p\|, \\ \qquad \forall (t, \beta) \in [0,T] \times [\beta_0, \beta_1],\ p \in \mathbf{H}, \\ \left| \displaystyle\int_0^{a_m} [f_0(r, p_1(r)) - f_0(r, p_2(r))]dr \right|, \\ \left| \displaystyle\int_0^{a_m} [L(p_1(r), \beta, r, t) - L(p_2(r), \beta, r, t)]dr \right| \\ \le C_3 \|p_1 - p_2\|, \\ \qquad \forall\ (t, \beta) \in [0,T] \times [\beta_0, \beta_1],\ p_1,\ p_2 \in \mathbf{H}, \\ \displaystyle\int_0^{a_m} L(p(r), \beta, r, t)dr \text{ is continuous in} \\ \qquad\qquad (t, p, \beta) \in \mathbf{R}^+ \times \mathbf{H} \times [\beta_0, \beta_1] \end{cases} \tag{3}$$

for some constants C_i, $i = 1, 2, 3$.

Define the time-dependent operators $\mathbf{A}_\beta(t)$ as follows:

$$\begin{aligned} &\mathbf{A}_\beta(t)\phi(r) = -\phi'(r) - \mu(r)\phi(r), \forall\ \phi \in D(\mathbf{A}_\beta(t)), \\ &D(\mathbf{A}_\beta(t)) = \Big\{\phi(r) \big| \phi,\ \mathbf{A}_\beta(t)\phi \in \mathbf{H}, \\ &\qquad\qquad \phi(0) = \beta(t) \int_{a_1}^{a_2} b(r)\phi(r)dr \Big\}. \end{aligned} \tag{4}$$

Then the equation 1 can be written as a first order evolution equation in $\mathbf{H}$:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t} = \mathbf{A}_\beta(t)p(r,t), \\ p(r,0) = p_0(r). \end{cases} \tag{5}$$

From now on, use $\mathbf{A}_\beta$ to denote the operator $\mathbf{A}_\beta(t)$ when $\beta(t) \equiv \beta$ independent of time t, and the semigroup generated by $\mathbf{A}_\beta$ will be denoted as $\mathbf{T}_\beta(t)$. It is seen that $\mathbf{A}_\beta(t) = \mathbf{A}_{\beta(t)}$.

Define a family of evolution operators $\mathbf{T}(t,s;\beta), 0 \le s \le t < \infty$ by

$$\mathbf{T}(t,s;\beta)\phi(r) = \begin{cases} \phi(r-t+s)e^{-\int_{r-t+s}^{r}\mu(\rho)d\rho}, r \ge t-s, \\ \beta(t-r)\int\limits_{a_1}^{a_2} b(\tau)\phi(\tau-t+s+r) \\ \cdot e^{-\int_{\tau-t+s+r}^{\tau}\mu(\rho)d\rho}d\tau e^{-\int_0^r \mu(\rho)d\rho}, r < t-s, \\ \qquad \forall \phi \in \mathbf{H},\ 0 \le t-s \le a_1. \end{cases}$$

$$\mathbf{T}(t,s;\beta) = \mathbf{T}(t, s+\left[\frac{t-s}{a_1}\right]a_1;\beta) \cdot \prod_{n=1}^{[\frac{t-s}{a_1}]} \mathbf{T}(s+na_1, s+(n-1)a_1;\beta), \quad t-s > a_1 \tag{6}$$

where $[x]$ denotes the maximal integer not exceeding x. $\mathbf{T}(t,s;\beta)$ is uniquely determined by $\{\beta(\tau) \mid \tau \in [s,t]\}$.

Limited by the length, theorems in this paper are given without proof.

Theorem 1. *Let $\mathbf{A}_\beta^*$ be the adjoint operator of $\mathbf{A}_\beta$ in $\mathbf{H}$. Then there exists an operator $\mathbf{C}$ on $\mathbf{H}$ that is bounded, linear, self-adjoint and positive definite such that for each $\beta \in [\beta_0, \beta_1]$, $\mathbf{A}_\beta^*\mathbf{C}$ is a bounded linear operator on $\mathbf{H}$ and*

$$\sup_{\beta\in[\beta_0,\beta_1]} \|\mathbf{A}_\beta^*\mathbf{C}\| \le 1 + \beta_1\|b\|\sqrt{a_m}. \tag{7}$$

Furthermore, the set $\mathcal{D}^*$ defined by

$$\mathcal{D}^* = D(\mathbf{A}_\beta^*) = \{\phi(r) \big| \phi(r),\ \phi'(r) - \mu(r)\phi(r) \in \mathbf{H},\ \phi(a_m) = 0\} \tag{8}$$

is independent of $\beta \in [\beta_0, \beta_1]$ and dense in $\mathbf{H}$.

Theorem 2. *Let $q(\cdot) \in \mathcal{D}^*$. Then $\langle q(\cdot), p(\cdot,t)\rangle$ is differentiable almost everywhere on $[0,T]$ and*

$$\frac{d}{dt}\langle q(\cdot), p(\cdot,t)\rangle = \langle \mathbf{A}_\beta^*(t)q(\cdot), p(\cdot,t)\rangle,\ t \in [0,T]\ a.e. \tag{9}$$

where $p(r,t) = \mathbf{T}(t,0;\beta)p_0(r)$ for any $p_0 \in \mathbf{H}$ and $\beta(\cdot) \in \mathcal{U}[0,T]$. $\mathbf{A}_\beta^*(t) = \mathbf{A}_{\beta(t)}^*$ is the adjoint operator of $\mathbf{A}_\beta(t)$.

By virtue of theorem 2, $p(r,t) = \mathbf{T}(t,0;\beta)p_0(r)$ is considered as the (weak) solution of equation 1 in the sense of equation 9, which is obtained by integrating equation 1 along the characteristics.

The value function $V(t,p_0)$ for the optimal control problem is defined by

$$V(t,p_0) = \inf_{\beta(\cdot)\in\mathcal{U}[t,T]} \left\{ \int\limits_t^T \int\limits_0^{a_m} L(p(r,\tau),\beta(\tau),r,\tau)drd\tau + \int\limits_0^{a_m} f_0(r,p(r,T))dr \right\} \tag{10}$$

where $p(r,\tau) = \mathbf{T}(\tau,t;\beta)p_0(r)$ is the solution of equation 1 corresponding to $\beta(\cdot) \in \mathcal{U}[t,T]$.

Theorem 3 [DPP]. *For any $0 < \delta < T-t$,*

$$\begin{cases} V(t,p_0) = \\ \inf\limits_{\beta(\cdot)\in\mathcal{U}[t,t+\delta]} \Big\{ \int\limits_t^{t+\delta}\int\limits_0^{a_m} L(p(r,\tau),\beta(\tau),r,\tau)drd\tau \\ \qquad +V(t+\delta, p(\cdot,t+\delta))\Big\}, \\ V(T,p_0) = \int\limits_0^{a_m} f_0(r,p_0(r))dr \end{cases} \tag{11}$$

where $p(r,\tau) = \mathbf{T}(\tau,t;\beta)p_0(r)$.

Theorem 4. *With M and ω as in Lemma 1 of (Guo and Yao, 1996) and constants $C_i,\ i = 1,2,3$ in equation 3, the following assertions are true:*

(i). $V(t,p_0) \le (1+T)C_1 + MC_2(1+\omega^{-1})\ e^{\omega T}\|p_0\|$, for all $t \in [0,T]$, $p_0 \in \mathbf{H}$.

(ii). $|V(t,p_1) - V(t,p_2)| \le (1+M\omega^{-1})C_3\ e^{\omega T}\|p_1 - p_2\|$, for all $t \in [0,T]$, p_1, $p_2 \in \mathbf{H}$.

(iii). For any fixed p_0, $V(t,p_0)$ is continuous in t.

3. VISCOSITY SOLUTIN TO HJB EQUATION

For brevity in notation, rewrite equation 5 as

$$\begin{cases} \dfrac{d\mathbf{P}(t)}{dt} = \mathbf{A}_\beta(t)\mathbf{P}(t), \\ \mathbf{P}(0) = P \end{cases} \tag{12}$$

where $\mathbf{P}(t) = p(\cdot,t)$, $P = p_0(\cdot)$. Let $\mathbf{P}(\tau) = \mathbf{T}(\tau,t;\beta)P$, $\psi(\mathbf{P}(T)) = \int_0^{a_m} f_0(r,p(r,T))dr$, $f(\tau,\mathbf{P}(\tau),\beta(\tau)) = \int_0^{a_m} L(p(r,\tau),\beta(\tau),r,\tau)dr$.

Theorem 5. *If $V(t,P) \in C^1([0,T]\times\mathbf{H})$, then V satisfies the following HJB equation*

$$\begin{cases} V_t(t,P) + \\ \inf\limits_{\beta\in[\beta_0,\beta_1]} \Big\{ \langle V_p(t,P), \mathbf{A}_\beta P\rangle + f(t,P,\beta) \Big\} = 0, \\ V(T,P) = \psi(P), \\ \qquad \forall t \in [0,T], P \in \bigcap\limits_{\beta\in[\beta_0,\beta_1]} D(\mathbf{A}_\beta). \end{cases} \tag{13}$$

From now on, always assume that $\mathbf{A}_\beta(t)$ is dissipative for all $\beta \in [\beta_0, \beta_1]$. First, give a definition for the solution of the HJB equation 13 in the "viscosity sense" (Bardi and Capuzzo-Dolcetta, 1997). Let Ω be an open set of $\mathbf{H}$ and set $\text{USC}([0,T]\times\Omega) = \{\text{upper-semicontinuous mappings } u : [0,T]\times\Omega \to \mathbf{R}\}$, $\text{LSC}([0,T]\times\Omega) = \{\text{lower-semicontinuous mappings } u : [0,T]\times\Omega \to \mathbf{R}\}$.

Definition 1. *$U(t,P) \in \text{USC}([0,T]\times\Omega)$ (respectively $U(t,P) \in \text{LSC}([0,T]\times\Omega)$) is a subsolution (respectively, supersolution) of equation 13 on $[0,T]\times\Omega$ if for every test function $\Phi = \varphi + g$, φ, $\varphi_p \in C([0,T]\times\Omega;\mathbf{R})$, $g \in C^1(\Omega;\mathbf{R})$ satisfying equation 14 and equation 15 below:*

$$\begin{cases} \text{Range}(\varphi_p) \subset \mathcal{D}^*, \textit{the mapping} \\ (t,P) \to \langle \mathbf{A}_\beta^* \varphi_p(t,P), P\rangle \textit{from } [0,T]\times\Omega \\ \textit{to } \mathbf{R} \textit{ is equicontinuous in } \beta \in [\beta_0,\beta_1]; \end{cases} \quad (14)$$

$$\begin{cases} \textit{there exists a } \tilde{h} : [0,\infty) \to \mathbf{R} \textit{ such that} \\ \tilde{h} \textit{ is nondecreasing, } \tilde{h}'(0) = 0 \textit{ and} \\ g(P) = \tilde{h}(\|P\|),\ \forall\, P \in \mathbf{H} \end{cases} \quad (15)$$

and for the local maximum point (respectively, the minimum point) (t,P) of $U - \Phi$ (respectively $U + \Phi$), there is

$$\varphi_t(t,P) + \inf_{\beta\in[\beta_0,\beta_1]} \left\{ \langle \mathbf{A}_\beta^* \varphi_p(t,P), P\rangle + f(t,P,\beta) \right\} \geq 0. \quad (16)$$

(respectively,

$$-\varphi_t(t,P) + \inf_{\beta\in[\beta_0,\beta_1]} \left\{ -\langle \mathbf{A}_\beta^* \varphi_p(t,P), P\rangle + f(t,P,\beta) \right\} \leq 0.) \quad (17)$$

$U(t,P) \in C([0,T]\times\Omega)$ is a viscosity solution of equation 13 if it is both a subsolution and a supersolution on $[0,T]\times\Omega$.

Theorem 6. *The value function $V(t,P)$ is a viscosity solution of the HJB equation 13.*

4. QUEST FOR OPTIMAL FEEDBACK CONTROL

Theorem 7. *Let $V(t,P)$ be the value function. Then for any trajectory-control pair $(\mathbf{P}^*(t), \beta^*(t))$, $\mathbf{P}^*(t) = \mathbf{T}(t,0;\beta^*)P$, the function*

$$t \to V(t, \mathbf{P}^*(t)) - \int_t^T f(\tau, \mathbf{P}^*(\tau), \beta^*(\tau))d\tau \quad (18)$$

is nondecreasing in $[0,T]$. Moreover, $(\mathbf{P}^*(\cdot), \beta^*(\cdot))$ is an optimal pair if and only if the above function is constant on $[0,T]$. Consequently, if $V(t,P) \in C^1([0,T]\times\mathbf{H})$, $\beta^*(\cdot) \in C^1[0,T]$, $\mathbf{P}^*(0) \in D(\mathbf{A}_{\beta^*(0)})$, then $(\mathbf{P}^*(\cdot), \beta^*(\cdot))$ is an optimal pair if and only if

$$\begin{cases} V_t(t,\mathbf{P}^*(t)) + \langle V_p(t,\mathbf{P}^*(t)), \mathbf{A}_{\beta^*(t)}\mathbf{P}^*(t)\rangle \\ \qquad\qquad + f(t,\mathbf{P}^*(t),\beta^*(t)) = 0, \\ V(T,\mathbf{P}^*(T)) = \psi(\mathbf{P}^*(T)),\ \forall\, t \in [0,T]. \end{cases} \quad (19)$$

The last assertion of Theorem 7 will lead to the classical verification theorem. Suppose the value function $V(t,P)$ is smooth. Let

$$\begin{aligned} S(Z) :=& \\ \arg &\min_{\beta\in[\beta_0,\beta_1]} \left\{ \langle V_z(t,Z), \mathbf{A}_\beta Z\rangle + f(t,Z,\beta) \right\} \\ &= \{\beta \in [\beta_0,\beta_1] \mid \mathcal{H}(Z, V_z(t,Z)) = \\ &\qquad \langle V_z(t,Z), \mathbf{A}_\beta Z\rangle + f(t,Z,\beta)\}. \end{aligned} \quad (20)$$

Then equation 19 says that $\beta^*(\cdot) \in \mathcal{U}[0,T]$ is an optimal control for the initial state p_0 if and only if

$$\beta^*(t) \in S(\mathbf{P}^*(t)) \text{ for } t \in [0,T] \text{ a.e.} \quad (21)$$

where $\mathbf{P}^*(t) = \mathbf{T}(t,0;\beta^*)p_0$. Equation 21 is the formula for finding the optimal feedback control.

5. ALGORITHM AND STIMULATION

In this section, one will be limited to the following linear quadratic optimal control problem:

$$J(\beta^*) = \inf_{\beta(\cdot)\in\mathcal{U}[0,T]} J(\beta) = \inf_{\beta(\cdot)\in\mathcal{U}[0,T]} \frac{1}{2} \cdot \left\{ \int_0^T [\beta(t) - \bar{\beta}(t)]^2 dt + \int_0^{a_m} [p(r,T) - \bar{p}(r)]^2 dr \right\} \quad (22)$$

where $\bar{\beta}(t)$ denotes the mean critical fertility rate of females in an ideal society and $\bar{p}(r)$ denotes the stable age density distribution of the population with the zero increasing rate.

To solve the above equation numerically, let $t_j = T + j\Delta t,\ j = 0,1,2,\cdots,N$ in which $\Delta t = -T/N$ and N is an integer. For given $\varepsilon > 0$, use the following approximation:

$$\langle V_p(t,p), \mathbf{A}_\beta p\rangle = \left\langle V_p(t,p), \varepsilon \frac{\mathbf{A}_\beta p}{\|\mathbf{A}_\beta p\|} \right\rangle \frac{\|\mathbf{A}_\beta p\|}{\varepsilon} \approx \left[V\left(t, p + \varepsilon \frac{\mathbf{A}_\beta p}{\|\mathbf{A}_\beta p\|}\right) - V(t,p) \right] \frac{\|\mathbf{A}_\beta p\|}{\varepsilon}.$$

For the initial state p_0 let $p_i = p_{i-1} + \frac{\mathbf{A}_\beta p_{i-1}}{\|\mathbf{A}_\beta p_{i-1}\|}\varepsilon,\ i = 1,2,\cdots,M$.

The following sufficient condition for the stability of the difference scheme will be assumed:

$$\frac{|\Delta t|}{\varepsilon} \max_{1\leq i\leq M} \|\mathbf{A}_\beta p_i\| \leq 1. \tag{23}$$

Set $\alpha_i^j = \frac{\Delta t}{\varepsilon}\|\mathbf{A}_{\beta_i^j} p_i\|$. Although the convergence of the described numerical algorithm has not been sated theoretically, the algorithm for the numerical solution of equation 19 is given as follows.

Step 1: initialization. Set

$$\begin{cases} V_i^0 = V(T, p_i) = \psi(p_i), \\ \qquad\qquad p_i = p_{i-1} + \dfrac{\mathbf{A}_\beta p_{i-1}}{\|\mathbf{A}_\beta p_{i-1}\|}\varepsilon, \\ \beta_i^0 \in \arg \inf\limits_{\beta\in[\beta_0,\beta_1]} \Bigg\{ \dfrac{\psi(p_i) - \psi(p_{i-1})}{\varepsilon} \\ \qquad\qquad \cdot \|\mathbf{A}_\beta p_i\| + \dfrac{a_m}{2}[\beta - \bar{\beta}^0]^2 \Bigg\} \\ := \arg \inf\limits_{\beta\in[\beta_0,\beta_1]} \{H(\beta, p)\}, i = 1, 2, \cdots, M. \end{cases} \tag{24}$$

Step 2: iteration.

$$\begin{cases} V_i^{j+1} = \\ \quad (1-\alpha_i^j)V_i^j + \alpha_i^j V_{i-1}^j - \dfrac{a_m}{2}(\beta_i^j - \bar{\beta}^j)^2 \Delta t, \\ \beta_i^{j+1} \in \arg \inf\limits_{\beta\in[\beta_0,\beta_1]} \Bigg\{ \dfrac{V_i^{j+1} - V_{i-1}^{j+1}}{\varepsilon} \|\mathbf{A}_\beta p_i\| \\ \qquad\qquad + \dfrac{a_m}{2}[\beta - \bar{\beta}^{j+1}]^2 \Bigg\} \end{cases} \tag{25}$$

for $i = 1, 2, \cdots, M$ and $j = 0, 1, 2, \cdots, N-1$.

From equation 21, the optimal feedback control is

$$\beta_{p_0}^*(t, p^*(\cdot, t)) \in \arg \inf_{\beta\in[\beta_0,\beta_1]} \Bigg\{ \langle V_p(t, p^*(\cdot, t)), \\ \mathbf{A}_\beta p^*(\cdot, t)\rangle + \frac{a_m}{2}[\beta - \bar{\beta}(t)]^2 \Bigg\} \tag{26}$$

in which $p^*(\cdot, t)$ is the optimal trajectory of the system. Because it involves the optimal trajectory in equation 26, finding the solution of equation 1 is necessary. Utilize the difference scheme to compute the approximate solution of the initial value problem equation 1, see (Song and Yu, 1988). Adopt the equally spaced grid method, to obtain

$$\begin{cases} \dfrac{p_{i,j} - p_{i-1,j}}{\Delta s} + \dfrac{p_{i,j} - p_{i,j-1}}{\Delta s} = -\mu_i p_{i,j}, \\ \qquad\qquad 1 \leq j \leq J,\ 1 \leq i \leq 2K, \\ p_{i,0} = p_0(i\Delta s),\ 0 \leq i \leq 2K, \\ p_{0,j} = \beta_j \Delta s \sum\limits_{i=a_1}^{a_2} b_i p_{i,j},\ 1 \leq j \leq J \end{cases} \tag{27}$$

in which $b_i = b(i\Delta s)$, $\beta_j = \beta(j\Delta s)$, $i = a_1, \cdots, a_2$, $2K\Delta s = a_m$, $J\Delta s = T$.

Next steps of finding the optimal feedback control are given in detail. It focuses on the solving the difference scheme equation 27 to obtain the optimal trajectory. Moreover, in the process, the algorithm of solving the HJB equation will be called frequently to get the corresponding feedback control function.

Algorithm of finding the optimal feedback control.

- Step 1: Call the algorithm of solving the HJB equation to get the feedback control function $\beta(t, p_0)$. Substitute $\beta(0, p_0)$ into equation 27 to get the optimal trajectory $p^*(\cdot, s_1)$, $s_1 = \Delta s$.
- Step 2: Take $p^*(\cdot, s_1)$ as the initial data p_0 as in the first step, and call the algorithm of solving the HJB equation again to get the feedback control $\beta(s_1, p^*(\cdot, s_1))$. Substitute $\beta(s_1, p^*(\cdot, s_1))$ into equation 27 to get the optimal trajectory $p^*(\cdot, s_2)$, $s_2 = 2\Delta s$.
- Step 3: Repeat the processes above, to get all feedback control functions $\beta(s_j, p^*(\cdot, s_j))$, $s_j = j\Delta s$, $j = 0, 1, \cdots, J$, that is to say,

$$\beta_{p_0}^*(t, p^*(\cdot, t)) = \Big\{ \beta(0, p_0(\cdot)), \\ \beta(s_1, p^*(\cdot, s_1)), \cdots, \beta(T, p^*(\cdot, T)) \Big\} \tag{28}$$

  that is the optimal feedback control.

Now it is the time to find the numerical solution of linear quadratic optimal control for the Chinese population based on the scheme equation 24, 25 and 27. The initial data are extracted from the Chinese population sampling census in 1989 (DDC, 1990), which are plotted by MATLAB 6.1 as figure (3) (age-structure of females, the age-structure of whole population, the relative mortality); The age-structure of an ideal society taken from (Song and Yu, 1988) are listed in table 1, which is used to get $\bar{p}(r)$ by multiplying the proportion in the table with the total ideal population $N_{sum} = 1,400,000$. The fertility pattern $h(r)$ is approximated by the Gamma density distribution curve in statistics (Song and Yu, 1988) in which the peak value of the fertility age is assumed to be 24. Other parameters are listed as follows: $T = 25, \beta_0 = 1, \beta_1 = 3, \bar{\beta}(t) = 2, a_1 = 15, a_2 = 49, a_m = 99, \varepsilon = 0.01$. All computations are performed in Visual C++ 6.0 and numerical results are plotted by MATLAB 6.1. Figure (1) is the value function $V(t, p^*(\cdot, t))$ and the optimal feedback control $\beta_{p_0}^*(t, p^*(\cdot, t))$.

Table 1 The age-structure of an ideal society

Age	Proportion	Age	Proportion
0	0.013	46 - 50	0.063
1 - 5	0.065	51 - 55	0.062
6 - 10	0.065	56 - 60	0.061
11 - 15	0.065	61 - 65	0.057
16 - 20	0.065	66 - 70	0.053
21 - 25	0.065	71 - 75	0.047
26 - 30	0.065	76 - 80	0.037
31 - 35	0.065	81 - 85	0.024
36 - 40	0.064	> 85	0.003
41 - 45	0.064		

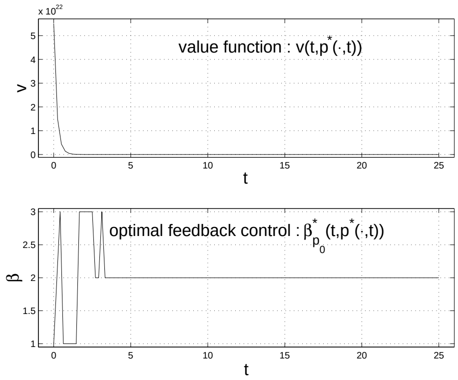

Fig. 1. The value function $V(t, p^*(\cdot, t))$ and the optimal feedback control $\beta^*_{p_0}(t, p^*(\cdot, t))$.

6. CONCLUSIONS

To end the paper, check the optimality of the numerical solution of optimal feedback control is necessary. This is done by comparing the cost functional $J(\beta^*_{p_0}(t, p^*(\cdot, t)))$ of obtained optimal control-trajectory pair with that of arbitrarily chosen control $\beta_{p_0}(t)$, $J(\beta_{p_0}(t))$ under the same initial condition, that is,

$$J(\beta^*_{p_0}(t, p^*(\cdot, t))) \leq J(\beta_{p_0}(t)). \quad (29)$$

Refer to figure (2), to compute these costs $J(\beta_i)$ $(i = 1, 2, \cdots, 7)$ for seven different controls $\beta_i \in \mathcal{U}[0, T]$ respectively. The cost value $J(\beta^*_{p_0}(t, p^*(\cdot, t)))$ is also computed. These results are listed in Table 2. It is seen that for the optimal feedback control $\beta^*_{p_0}(t, p^*(\cdot, t))$, $J(\beta^*_{p_0}(t, p^*(\cdot, t))) = 11641607717$, which is evidently less than other costs $J(\beta_i)$, $i = 1, 2, \cdots, 7$. In other words, the numerical solution of optimal birth feedback control for the Chinese population from 1989-2014 is indeed got.

Table 2 Different β and its corresponding cost $J(\beta)$.

β	$J(\beta)$	β	$J(\beta)$
$\beta_1(t)$	11641608005	$\beta_5(t)$	11641608172
$\beta_2(t)$	11641608005	$\beta_6(t)$	11641608172
$\beta_3(t)$	11641607901	$\beta_7(t)$	11641608172
$\beta_4(t)$	11641608172	$\beta^*_{p_0}(t, p^*(\cdot, t))$	11641607717

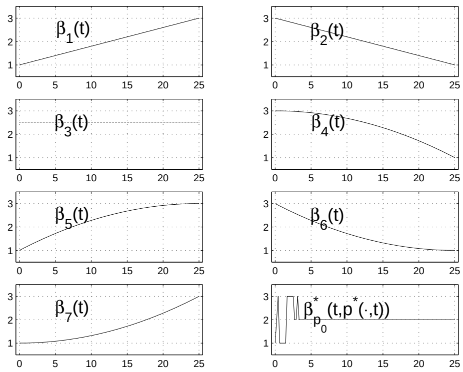

Fig. 2. Seven different arbitrarily chosen control β_i and the optimal feedback control $\beta^*_{p_0}(t, p^*(\cdot, t))$.

Fig. 3. Age-structures of females and whole population plus the relative mortality of Chinese population in 1989.

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