REAL INTERPOLATION POINTS IN MODEL REDUCTION: JUSTIFICATION, TWO SCHEMES AND AN ERROR BOUND

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Abstract: A new mathematical justification for using real interpolation points in model reduction is given, with the help of optimal time function approximations by transformed Legendre polynomials. Based on that, two reduction schemes are proposed: The first one applies a projection to the original model and matches $2q$ moments, similar to known rational Krylov methods. The second one matches q moments while preserving stability and ensuring an optimal approximation of the step response in a weighted $\mathcal{L}_2$ norm sense. This new method also provides an error bound.

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1. INTRODUCTION

Nowadays, model reduction attracts a growing attention over a broad variety of science and engineering fields due to its contribution to reduced order controller design and improvement of simulation efficiency. Different methods have been developed and surveyed by both mathematicians and engineers. The most famous methods are based on Singular Value Decomposition (SVD), e.g. balanced truncation, proper orthogonal decomposition and moment matching using Krylov subspace methods. These methods are successfully applied to different technical systems, however, there exist one or more issues in each method concerning stability, global error bound and computational efficiency.

It is encouraging to find new methods that not only produce accurate reduced order systems, but also guarantee the stability of the reduced system, possess high computational efficiency for large scale systems or even provide some global error bound.

While evaluating the quality of reduction, it is very common to compare bode diagrams which illustrates the frequency response of a system at different imaginary points, $\mathbf{G}(j\omega)$. Once one gets used to such analysis, it becomes natural to approximate an original transfer function by interpolating the function value or derivatives at some imaginary points $j\omega_i$. However, approximation at some real points can lead to better results in practice. (Grimme 1997) endeavored to explain the advantage of real interpolation points by studying the mapping behavior of eigenvalues via Krylov subspace. His conclusion was:

- Pure imaginary interpolation points lead to excellent results locally, but can result in extremely slow convergence at all frequencies away from the predefined interpolation points.

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- Real interpolation points tend to yield a broader, but courser convergence to the true frequency response.

Although this conclusion is true for most of the technical systems, the comparison of the original and reduced system is based on the complex-valued transfer function found by Laplace transform. In all papers like (Anderson *et al.* 1990), there is no mathematical explanation about the connection between a dynamical system and the real rational interpolation.

To find such a connection, a time function approximation method using orthogonal polynomials, e.g. Legendre (Paraskevopoulos 1985) or Laguerre polynomials (Wahlberg and Makila 1996), as basis functions can be used. These methods are based on *time domain synthesis* that finds rational functions in s-domain whose inverse transformation approximates the given time functions (Horowitz 1963).

After applying such approximations by Legendre polynomials to the state space equations of a dynamical system, an algebraic representation is obtained leading to a real-valued transfer function which can be further approximated by an interpolation over some real points. It is also shown that if some equidistant points are chosen, some of the first coefficients of the original and reduced transfer functions are the same (Makila 1990)(Franke *et al.* 1993).

Because a set of orthogonal functions is used for the purpose of reduction, the optimality in the sense of weighted $\mathcal{L}_2$ norm can be achieved if the poles of the reduced system are set at some distinct points on the negative real axis.

In this paper, by using a set of transformed Legendre polynomials, it is shown how to match the first coefficients of the corresponding series expansion and two related reduction schemes are introduced that utilize projection technique and explicit pole placement. The second method ensures stability of the reduced model and provides an error bound.

2. FUNCTION APPROXIMATION USING TRANSFORMED LEGENDRE POLYNOMIALS

The Legendre polynomials are defined as

$$p_{k+1}(\tau) = \frac{1}{2^k k!}\frac{d^k}{d\tau^k}\left(\tau^2-1\right)^k,\ k = 0,1,2,\cdots,$$

for $\tau \in [-1,1]$. The first 3 Legendre polynomials are,

$$p_1(\tau) = 1,\ p_2(\tau) = \tau,\ p_3(\tau) = \frac{3}{2}\tau^2 - \frac{1}{2}. \quad (1)$$

Legendre polynomials possess orthogonality and there exist recurrence formulas for computing an arbitrary Legendre polynomial (Sansone 1959, Bell 1968).

It is known that by applying the transformation $\tau = 1 - 2e^{-\alpha t}$, where α is a positive real number, the

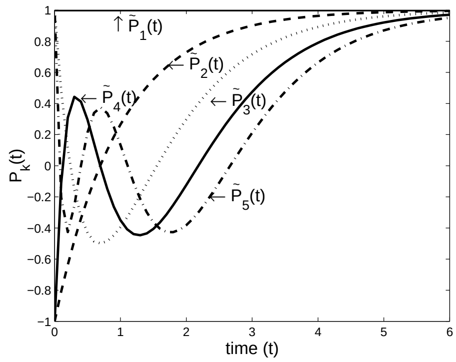

Fig. 1. Transformed Legendre polynomials for $\alpha = 1$.

Legendre polynomials are transformed into orthogonal exponentials, which can well approximate the impulse response of a physical system. This transformation increases monotonically from $t = 0$ (at $\tau = -1$) to $t = \infty$ (at $\tau = 1$). The transformed polynomials in time domain are linear combinations of descending exponential functions, as shown below,

$$\tilde{p}_1(t) = 1,\ \tilde{p}_2(t) = 1 - 2e^{-\alpha t}$$
$$\tilde{p}_3(t) = 1 - 6e^{-\alpha t} + 6e^{-2\alpha t}$$

where $t \in [0,\infty)$. The first 5 functions of transformed Legendre series with $\alpha = 1$ are shown in figure 1.

By changing the variable, the orthogonality of the new set of time functions can be expressed as follows,

$$\int_0^\infty \tilde{p}_j(t)\tilde{p}_k(t)e^{-\alpha t}dt = \begin{cases} 0 & j \neq k \\ \tilde{c}_k = \dfrac{1}{(2k-1)} & j = k \end{cases} \quad (2)$$

where $j,k = 1,2,\cdots$ and it includes a weighting function $e^{-\alpha t}$. A recursive formulation to compute these polynomials are,

$$2\alpha e^{-\alpha t}\dot{\tilde{p}}_{k+1} = (2k-1)\tilde{p}_k + 2\alpha e^{-\alpha t}\dot{\tilde{p}}_{k-1} \quad (3)$$
$$(2k-1)(1-2e^{-\alpha t})\tilde{p}_k = k\tilde{p}_{k+1} + (k-1)\tilde{p}_{k-1} \quad (4)$$

for $k = 2,3,\cdots$. By using the property (2) and the recursive formulation (3) and (4), it can be concluded,

$$\int_0^\infty e^{-\alpha t}\dot{\tilde{p}}_j(t)\tilde{p}_k(t)dt = \begin{cases} 0 & j < k \\ \dfrac{1-k}{2k-1} & j = k \\ -(-1)^{j+k} & j > k \end{cases}$$

where $j,k = 1,2,\cdots$.

2.1 Function approximation

Any n dimensional continuous real-valued time function with finite weighted $\mathcal{L}_2^n[0,\infty)$ norm,

$$\int_0^\infty \mathbf{f}^T(t)\mathbf{f}(t)e^{-\alpha t}dt < \infty, \quad (5)$$

can be expressed as a linear combination of the transformed Legendre polynomials,

$$\mathbf{f}(t) = \sum_{k=1}^{\infty} \tilde{\mathbf{f}}_k \tilde{p}_k(t),$$

where the transformed generalized Fourier coefficients $\tilde{\mathbf{f}}_\mathbf{k}$ are defined as follows,

$$\tilde{\mathbf{f}}_k = \frac{1}{\tilde{c}_k} \int_0^{\infty} \mathbf{f}(t)\tilde{p}_k(t)e^{-\alpha t}dt, \quad k = 1, 2, \cdots . \tag{6}$$

Any function with bounded $\mathcal{L}_2$ norm over $[0, \infty)$ fulfills the condition (5), because,

$$\int_0^{\infty} \mathbf{f}^T(t)\mathbf{f}(t)e^{-\alpha t}dt \le \int_0^{\infty} \mathbf{f}^T(t)\mathbf{f}(t)dt,$$

Theorem 1. If the continuous function $\mathbf{f}(t)$, which satisfies (5), is approximated using only the first N transformed Legendre polynomials,

$$\hat{\mathbf{f}}(t) = \sum_{k=1}^{N} \gamma_k \tilde{p}_k(t).$$

then the approximation is optimal in the sense of weighted $\mathcal{L}_2[0, \infty]$ norm; i.e. the cost function

$$J = \int_0^{\infty} \left(\mathbf{f}(t) - \hat{\mathbf{f}}(t)\right)^T \left(\mathbf{f}(t) - \hat{\mathbf{f}}(t)\right) e^{-\alpha t}dt$$

is minimized by choosing the transformed generalized Fourier coefficients, $\gamma_k = \tilde{\mathbf{f}}_k$ with the optimal value,

$$J^* = \alpha \int_0^{\infty} \mathbf{f}^T(t)\mathbf{f}(t)e^{-\alpha t}dt - \sum_{k=1}^{N} \tilde{c}_k \tilde{\mathbf{f}}_k^T \tilde{\mathbf{f}}_k \ .$$

2.2 Algebraic representation of dynamical systems

Consider the linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{7}$$

and approximate its input, output and state vectors using the first N transformed Legendre polynomials,

$$\begin{cases} \mathbf{x}(t) \simeq \bar{\mathbf{x}}(t) = \sum_{j=1}^{N} \tilde{\mathbf{x}}_j \tilde{p}_j(t), \\ \mathbf{u}(t) \simeq \bar{\mathbf{u}}(t) = \sum_{j=1}^{N} \tilde{\mathbf{u}}_j \tilde{p}_j(t), \\ \mathbf{y}(t) \simeq \bar{\mathbf{y}}(t) = \sum_{j=1}^{N} \tilde{\mathbf{y}}_j \tilde{p}_j(t) \end{cases} \tag{8}$$

with the initial condition

$$\mathbf{x}_0 = \mathbf{x}(0) = \sum_{j=1}^{\infty} \tilde{\mathbf{x}}_j \tilde{p}_j(0).$$

It is known that the value of the transformed polynomials satisfies the relation $\tilde{p}_j(0) = (-1)^{(k-1)}$.

The derived algebraic representation is

$$\begin{cases} \tilde{\mathbf{X}}\boldsymbol{\Delta} - \mathbf{X}_a = \mathbf{A}\tilde{\mathbf{X}} + \mathbf{B}\tilde{\mathbf{U}}, \\ \tilde{\mathbf{Y}} = \mathbf{C}\tilde{\mathbf{X}}. \end{cases} \tag{9}$$

where,

$$\begin{aligned} \tilde{\mathbf{X}} &= \{\tilde{\mathbf{x}}_\mathbf{1}, \cdots, \tilde{\mathbf{x}}_\mathbf{N}\} \\ \tilde{\mathbf{U}} &= \{\tilde{\mathbf{u}}_\mathbf{1}, \cdots, \tilde{\mathbf{u}}_\mathbf{N}\} \\ \tilde{\mathbf{Y}} &= \{\tilde{\mathbf{y}}_\mathbf{1}, \cdots, \tilde{\mathbf{y}}_\mathbf{N}\} \\ \mathbf{X}_a &= \alpha \begin{bmatrix} \mathbf{x}_0 & -3\mathbf{x}_0 & \cdots & (-1)^N(1-2N)\mathbf{x}_0 \end{bmatrix} \\ \boldsymbol{\Delta} &= \alpha \begin{bmatrix} 1 & -3 & 5 & \cdots & (-1)^N(1-2N) \\ 0 & 2 & -5 & \cdots & -(-1)^N(1-2N) \\ 0 & 0 & 3 & \cdots & (-1)^N(1-2N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \end{bmatrix} \end{aligned}$$

The matrix $\boldsymbol{\Delta}$ is called *differential matrix* (Franke *et al.* 1993). The matrix $\boldsymbol{\Delta}$ has N distinct eigenvalues $\alpha, 2\alpha, \cdots, N\alpha$ and therefore can be diagonalized,

$$\begin{aligned} \exists \mathbf{M} \in \mathbb{R}^{N\times N} &: \mathbf{M}\boldsymbol{\Delta}\mathbf{M}^{-1} = \boldsymbol{\Lambda}, \\ \boldsymbol{\Lambda} &= diag(\alpha, 2\alpha, \cdots, N\alpha). \end{aligned}$$

Now, by considering that the initial values are zero, the transformation $\tilde{\mathbf{X}} = \mathbf{X}\mathbf{M}$ is applied to the algebraic state equation (9),

$$\begin{cases} \mathbf{X}\mathbf{M}\boldsymbol{\Delta} = \mathbf{A}\mathbf{X}\mathbf{M} + \mathbf{B}\tilde{\mathbf{U}}, \\ \tilde{\mathbf{Y}} = \mathbf{C}\mathbf{X}\mathbf{M}. \end{cases}$$

By applying the same transformation to the coefficients of the input and output as $\tilde{\mathbf{U}} = \mathbf{U}\mathbf{M}$ and $\tilde{\mathbf{Y}} = \mathbf{Y}\mathbf{M}$, the final transformed system is,

$$\begin{cases} \mathbf{X}\boldsymbol{\Lambda} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}, \\ \mathbf{Y} = \mathbf{C}\mathbf{X}. \end{cases} \tag{10}$$

To calculate the coefficients in the new transformed space in (10), it is then concluded that,

$$\begin{cases} k\alpha\check{\mathbf{x}}_k = \check{\mathbf{x}}_k + \mathbf{B}\check{\mathbf{u}}_k, \\ \check{\mathbf{y}}_k = \mathbf{C}\check{\mathbf{x}}_k. \end{cases} \tag{11}$$

where $k = 1, \cdots, N$ and,

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} \check{\mathbf{x}}_1 & \cdots & \check{\mathbf{x}}_N \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_1 & \cdots & \tilde{\mathbf{x}}_N \end{bmatrix} \mathbf{M}^{-1}, \\ \mathbf{U} &= \begin{bmatrix} \check{\mathbf{u}}_1 & \cdots & \check{\mathbf{u}}_N \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{u}}_1 & \cdots & \tilde{\mathbf{u}}_N \end{bmatrix} \mathbf{M}^{-1}, \\ \mathbf{Y} &= \begin{bmatrix} \check{\mathbf{y}}_1 & \cdots & \check{\mathbf{y}}_N \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{y}}_1 & \cdots & \tilde{\mathbf{y}}_N \end{bmatrix} \mathbf{M}^{-1}. \end{aligned}$$

The transfer function from $\check{\mathbf{u}}_k$ to $\check{\mathbf{y}}_k$ is,

$$\check{\mathbf{y}}_k = \mathbf{C}\left(k\alpha\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}\check{\mathbf{u}}_k = \mathbf{G}(k\alpha)\check{\mathbf{u}}_k, \quad (12)$$

where $\mathbf{G}(s)$ is the transfer function of the state space equation (7) by Laplace transform.

After diagonalizing the matrix $\boldsymbol{\Delta}$, it turns out that if the transformed Legendre polynomials is applied to approximate the time functions $\mathbf{x}(t)$, $\mathbf{y}(t)$ and $\mathbf{u}(t)$ as in (8), the resulted algebraic representation of the system (9) not only has a similar form to the original state space equation (7), but also its transfer function from $\check{\mathbf{u}}_k$ to $\check{\mathbf{y}}_k$ matches the original transfer function at N equidistant real points! Dealing with the real valued transfer function (12) provides a theoretical background for the choice of real interpolation points in order reduction based on moment matching.

3. REALIZATION OF REDUCED SYSTEM BY PROJECTION

By selecting proper projection matrices, a reduced order system in the form of *internal representation* (Antoulas *et al.* 2001) can easily be found. Here the projection matrix $\mathbf{V}$ is constructed such that the reduced order system interpolates the original system at the points $\alpha, 2\alpha, \cdots, q\alpha$,

$$colspan\{\mathbf{V}\} = colspan\left[(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{B},\right. \\ \left.(\mathbf{A}-2\alpha\mathbf{E})^{-1}\mathbf{B}\cdots\ (\mathbf{A}-q\alpha\mathbf{E})^{-1}\mathbf{B}\right] \quad (13)$$

Similarly, the matrix $\mathbf{V}$ can be defined as

$$colspan\{\mathbf{V}\} = colspan\left[(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{C}^T\right. \\ \left.(\mathbf{A}-2\alpha\mathbf{E})^{-T}\mathbf{C}^T\ \cdots\ (\mathbf{A}-q\alpha\mathbf{E})^{-T}\mathbf{C}^T\right] \quad (14)$$

The reduced order approximation of the system (7) is then

$$\begin{cases}\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x_r}(t) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t)\\ \hat{\mathbf{y}}(t) = \mathbf{C}\mathbf{V}\mathbf{x_r}(t)\ .\end{cases} \quad (15)$$

where $\mathbf{W}$ is an arbitrary full rank matrix with the same size as $\mathbf{V}$. Equivalently, the matrices of the reduced order system can be computed as,

$$\begin{cases}\mathbf{E_r} = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{A_r} = \mathbf{W}^T\mathbf{A}\mathbf{V},\\ \mathbf{B_r} = \mathbf{W}^T\mathbf{B}, \quad \mathbf{C_r} = \mathbf{C}\mathbf{V}.\end{cases} \quad (16)$$

If $\mathbf{V}$ is set as in (13) and $\mathbf{W} = \mathbf{V}$, the projection method is denoted as *one-sided projection*. As shown below, with one-sided projection, the reduced order system interpolates the original system at q real points $\alpha, \cdots, q\alpha$ matching the first $N = q$ coefficients of the transformed Legendre polynomials. The first *moments* at these points are matched,

$$\begin{aligned} m_{r0}^{(i\alpha)} =& \mathbf{C_r}(\mathbf{A}_r - i\alpha\mathbf{E}_r)^{-1}\mathbf{B}_r \\ \overset{(16)}{=}& \mathbf{C}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V} - i\alpha\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B} \\ =& \mathbf{C}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V} - i\alpha\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\times \\ & \mathbf{W}^T\underbrace{(\mathbf{A}-i\alpha\mathbf{E})(\mathbf{A}-i\alpha\mathbf{E})^{-1}}_{\mathbf{I}}\mathbf{B} \end{aligned}$$

Because $(\mathbf{A}-i\alpha\mathbf{E})^{-1}\mathbf{B}$ is spanned by the columns of the projection matrix $\mathbf{V}$, there exists a matrix $\mathbf{R}_i$ such that $(\mathbf{A}-i\alpha\mathbf{E})^{-1}\mathbf{B} = \mathbf{V}\mathbf{R}_i$, and therefore,

$$\begin{aligned} m_{r0}^{(i\alpha)} =& \mathbf{C}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V} - i\alpha\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\times \\ & \mathbf{W}^T(\mathbf{A}-i\alpha\mathbf{E})\mathbf{V}\mathbf{R}_i \\ =& \mathbf{C}\mathbf{V}\mathbf{R}_i = \mathbf{C}(\mathbf{A}-k\alpha\mathbf{E})^{-1}\mathbf{B} = m_0^{(i\alpha)}. \end{aligned}$$

The number of matched parameters with the same order of the reduced system can be increased from q to $2q$ by keeping $\mathbf{V}$ as (13) and choosing $\mathbf{W}$ as

$$colspan(\mathbf{W}) = colspan\left[\ (\mathbf{A}-(q+1)\alpha\mathbf{E})^{-T}\mathbf{C}^T\right. \\ \left.(\mathbf{A}-(q+2)\alpha\mathbf{E})^{-T}\mathbf{C}^T\ \cdots\ (\mathbf{A}-2q\alpha\mathbf{E})^{-T}\mathbf{C}^T\ \right]$$

which introduces further information of the original system at $(q+1)\alpha, (q+2)\alpha, \cdots, 2q\alpha$ and $N = 2q$. Since the approximation seems to be stretched along the real axis, such a projection is denoted as *extended projection*. To enforce matching of DC gain, the interpolation points can be chosen as $0, \alpha, \cdots, (2q-1)\alpha$ instead of $\alpha, 2\alpha, \cdots, 2q\alpha$.

In this way, not only the transfer function of the original system is interpolated over a set of equidistant real points, but also some of the first coefficients of the series expansion by the transformed Legendre polynomials are matched. However, the approximation is not optimal due to the remaining nonzero coefficients of the transformed Legendre series expansion and no error bound can be given. In practice, the above scheme can perform excellently.

4. STABILITY BY EXPLICIT POLE PLACEMENT

Since the general stability of a reduced order system can not be guaranteed by applying the above projection methods, an option is to ensure the stability of the reduced order system (only for SISO case) by setting all poles of the reduced system explicitly to the left half complex plane. It is referred as "*explicit moment-matching*" (Grimme 1997). In this case, all the single equidistant poles are placed on the negative real axis to achieve also *optimality*. By this choice, the step response can be expanded by some of the first transformed Legendre polynomial and all the rest corresponding coefficients are zero. The transfer function of the reduced order SISO system then becomes

$$\begin{aligned} G_r(s) =& \frac{b_0 + b_1 s + \cdots + b_{q-1}s^{q-1} + b_q s^q}{(1+\frac{1}{\alpha}s)\,(1+\frac{1}{2\alpha}s)\ \cdots\ (1+\frac{1}{q\alpha}s)} \\ =& \frac{b(s)}{a(s)}, \end{aligned}$$

By applying the partial fractions, the transfer function $G_r(s)$ can equivalently be written as,

$$G_r(s) = r_0 + \frac{r_1}{1+\frac{1}{\alpha}s} + \cdots + \frac{r_q}{1+\frac{1}{q\alpha}s}\ . \quad (17)$$

The coefficients of $b(s)$ or the residuals $r_0 \cdots, r_q$ are chosen to match the rational moments, i.e.

$$G_r(i\alpha) = G(i\alpha), \quad i = 1, 2, \cdots, q+1.$$

Thus the numerator and denominator part of $G_r(s)$ fulfill the following relation

$$a(i\alpha) \cdot G(i\alpha) = b(i\alpha), \quad i = 1, 2, \cdots, q+1.$$

Because the calculation in $b_0, \cdots, b_q$ is ill-conditioned, in order to improve the calculation, the partial fraction of the transfer function in from (17) is used. The matrix form can be further presented as

$$\begin{bmatrix} 1 & \frac{1}{1+1} & \frac{1}{1+\frac{1}{2}} & \cdots & \frac{1}{1+\frac{1}{q}} \\ 1 & \frac{1}{1+2} & \frac{1}{1+1} & \cdots & \frac{1}{1+\frac{2}{q}} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \frac{1}{1+q} & \frac{1}{1+\frac{q}{2}} & \cdots & \frac{1}{1+1} \\ 1 & \frac{1}{1+q+1} & \frac{1}{1+\frac{q+1}{2}} & \cdots & \frac{1}{1+\frac{q+1}{q}} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_q \end{bmatrix} =$$

$$\left[\, G(\alpha), G(2\alpha), \cdots, G(q\alpha)\ G((q+1)\alpha) \,\right]^T \quad (18)$$

After calculating the coefficients in $b(s)$, the TF of the reduced order system $G_r(s)$ is determined.

This method improves the projection method in finding a stable reduced system and achieving optimality. In fact, by means of explicit pole placement, the poles of the reduced system are at $-\frac{1}{\alpha}, -\frac{1}{2\alpha}, \cdots, -\frac{1}{q\alpha}$ and the stability is guaranteed (that can not be offered by projection method). Furthermore, from theorem 1, the approximation is optimal in the sense of weighted $\mathcal{L}_2$ norm, with respect to the set of basis functions.

5. ERROR BOUND OF REDUCTION

Using the reduction method introduced above, a global error bound can be found for the transfer function. According to theorem 1, by choosing the first N generalized Fourier coefficients, the optimal cost function becomes

$$J^* = \alpha \int_0^\infty \mathbf{g}^T(t)\mathbf{g}(t)e^{-\alpha t}dt - \sum_{k=1}^{N} \tilde{c}_k \tilde{\mathbf{g}}_k^T \tilde{\mathbf{g}}_k. \quad (19)$$

The integral part can be implemented by means of *Parseval's theorem* for SISO systems,

$$\int_0^\infty g^2(t)e^{-\alpha t}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\frac{\alpha}{2}+j\omega)G(\frac{\alpha}{2}-j\omega)d\omega$$
$$= \left\| G(\frac{\alpha}{2}+s) \right\|_2^2. \quad (20)$$

The calculation of the second part of (19) which approximates the original transfer function is introduced in the next subsection.

5.1 Calculating the coefficients of expansion

Starting with an arbitrary time function $\mathbf{f}(t)$, the coefficients of the series expansion of the function $\mathbf{f}(t)$ can be calculated using equation (6),

$$\tilde{\mathbf{f}}_k = \alpha(2k-1)\int_0^\infty \mathbf{f}(t)p_k(t)e^{-\alpha t}dt, \quad (21)$$

where $k = 1, 2, \cdots$. The k-th transformed Legendre polynomial is a sum of some exponential functions of the form $e^{-i\alpha t}$, where $i = 0, 1, \cdots, k-1$. Consequently the function $\tilde{p}_k(t)e^{-\alpha t}$ is a sum of $e^{-i\alpha t}$, where $1 \leq i \leq k$. Thus, the coefficients $\tilde{\mathbf{f}}_k$ is a sum of $\mathbf{F}(i\alpha)$ for $1 \leq i \leq k$. It is reminded that the Laplace transform of the function $\mathbf{f}(t)$ is,

$$\mathbf{F}(s) = \int_0^\infty \mathbf{f}(t)e^{-st}dt. \quad (22)$$

The first 3 Fourier coefficients are

$$\begin{aligned} \tilde{\mathbf{f}}_1 &= \alpha\int_0^\infty \mathbf{f}(t)e^{-\alpha t}dt = \alpha\mathbf{F}(\alpha), \\ \tilde{\mathbf{f}}_2 &= 3\alpha\int_0^\infty \mathbf{f}(t)\left(1-2e^{-\alpha t}\right)e^{-\alpha t}dt \\ &= 3\alpha\mathbf{F}(\alpha) - 6\alpha\mathbf{F}(2\alpha), \\ \tilde{\mathbf{f}}_3 &= 5\alpha\int_0^\infty \mathbf{f}(t)\left(1-6e^{-\alpha t}+6e^{-2\alpha t}\right)e^{-\alpha t}dt \\ &= 5\alpha\mathbf{F}(\alpha) - 30\alpha\mathbf{F}(2\alpha) + 30\alpha\mathbf{F}(3\alpha). \end{aligned}$$

In general case, any $\tilde{\mathbf{f}}_k$ are linear combinations of its Laplace transformations at $i\alpha$, i.e.

$$\tilde{\mathbf{f}}_k = \alpha\sum_{i=1}^{k} c_{ik}\mathbf{F}(i\alpha) \quad (23)$$

where $c_{11} = 1, c_{12} = 3$ and $c_{22} = -6$. To calculate the rest of the Fourier coefficients, equations (3) and (4) are used,

$$\begin{aligned} (k-1)\tilde{p}_k(t) = (2k-3)(1-2e^{-\alpha t})\tilde{p}_{k+1}(t) \\ +(2-k)\tilde{p}_{k-1}(t). \quad (24) \end{aligned}$$

Multiplying equation (24) by $\mathbf{f}(t)e^{-\alpha t}$ and integrate from 0 to ∞ leads to,

$$\begin{aligned} &\alpha(2k-1)\int_0^\infty \mathbf{f}(t)\tilde{p}_k(t)e^{-2\alpha t}dt \\ &= \alpha\frac{2k-1}{k-1}\Big[(2k-3)\int_0^\infty \mathbf{f}(t)\tilde{p}_{k-1}(t)e^{-\alpha t}dt \\ &\qquad - 2(2k-3)\int_0^\infty \mathbf{f}(t)\tilde{p}_{k-1}(t)e^{-2\alpha t}dt \\ &\qquad + \frac{2-k}{2k-5}(2k-5)\int_0^\infty \mathbf{f}(t)\tilde{p}_{k-2}(t)e^{-2\alpha t}dt\Big] \end{aligned}$$

$$
\overset{(21)(23)}{=} \alpha \frac{2k-1}{k-1} \left(\sum_{i=1}^{k-1} c_{i(k-1)} \mathbf{F}(i\alpha) - 2 \sum_{i=1}^{k-1} c_{ik} \mathbf{F}((i+1)\alpha) + \frac{2-k}{2k-5} \sum_{i=1}^{k-2} c_{i(k-2)} \mathbf{F}(i\alpha) \right)
$$

$$
= \alpha \frac{2k-1}{k-1} \left(\sum_{i=1}^{k-1} c_{i(k-1)} \mathbf{F}(i\alpha) - 2 \sum_{i=2}^{k} c_{(i-1)(k-1)} \mathbf{F}(i\alpha) + \frac{2-k}{2k-5} \sum_{i=1}^{k-2} c_{i(k-2)} \mathbf{F}(i\alpha) \right) \tag{25}
$$

Since equations (23) and (25) are identical, after matching the coefficients of both sides the following recurrence formulas are found,

$$
\begin{aligned}
&c_{11} = 1,\ c_{12} = 3,\ c_{22} = -6 \\
&c_{1k} = \frac{2k-1}{k-1} \left(c_{1(k-1)} + \frac{2-k}{2k-5} c_{1(k-2)} \right), \\
&c_{ik} = \frac{2k-1}{k-1} \left(c_{i(k-1)} - 2c_{(i-1)(k-1)} + \frac{2-k}{2k-5} c_{i(k-2)} \right),
\end{aligned}
$$

for $i = 2, \cdots, k-1$ and $k = 3, 4, \cdots$. The diagonal elements for $i = k$ can be calculated by

$$
c_{kk} = -2 \frac{2k-1}{k-1} c_{(k-1)(k-1)},\ k = 2, 3, \cdots
$$

If the values of c_{ik} are put in a matrix, then the relationship between the Laplace transform at the points $\alpha, 2\alpha, \cdots, N\alpha$ and the first N coefficients of the transformed Legendre polynomials are found,

$$
\left[\tilde{\mathbf{f}}_1\ \tilde{\mathbf{f}}_2\ \cdots\ \tilde{\mathbf{f}}_N \right] = \alpha \left[\mathbf{F}(\alpha)\ \mathbf{F}(2\alpha)\ \cdots\ \mathbf{F}(N\alpha) \right] \times \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ 0 & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{NN} \end{bmatrix}}_{\mathbf{L}}.
$$

Now, the coefficients of the transformed Legendre polynomial of the step response of the system (7) are,

$$
\left[\tilde{\mathbf{g}}_1\ \tilde{\mathbf{g}}_2\ \cdots\ \tilde{\mathbf{g}}_N \right] = \mathbf{C} \left[(\alpha\mathbf{I} - \mathbf{A})^{-1} (2\alpha\mathbf{I} - \mathbf{A})^{-1} \cdots (N\alpha\mathbf{I} - \mathbf{A})^{-1} \right] \mathbf{BL}. \tag{26}
$$

With (20) and (26), the error bound of the reduction procedure can directly be implemented. It offers a criterion to evaluate the accuracy of the reduced order system in time domain. By fixing some parameters, e.g. the order after reduction, this error bound can be used to find an optimal parameter α through an optimization process, which determines the location of interpolation points. As a rule of thumb, $\frac{3.5}{\alpha}$ should be in the range of the settling time of the system.

6. CONCLUSION

Based on a series expansion using the transformed Legendre polynomials, a real valued transfer function is found. It turned out that if a reduced transfer function interpolates the original one at N equidistant real points, then the first N generalized Fourier coefficients of the output signals match.

Two different reduction methods have been introduced matching the coefficients of the corresponding series expansion. By the first method, only some of the corresponding coefficients are matched via projection, whereas the second method utilizes explicit pole placement and provides an error bound. The reduced model found by the second method is stable and its step response is an optimal approximation of the original one in a weighted $\mathcal{L}_2$ norm sense, by matching $q+1$ generalized Fourier coefficients and setting the others to zero. However, because the poles are fixed on real axis, the approximation of weakly damped systems requires relatively large value of q, in practice. From this point of view, the first method may be superior, though without error bound and guarantee of stability. Thereby, both methods are suitable for integrated *automatic* reduction implementations.

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