TUNING OF LEAD COMPENSATORS WITH GAIN AND PHASE MARGIN SPECIFICATIONS

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Abstract: For a servo plant with an integrator, analytical tuning formulas for phase lead compensators with both gain and phase margin specifications are derived in this paper. Comparing with other tuning methods, this method is simple with a short tuning time. Simulation results under two typical cases show that this method is general, too.

Keywords: phase lead compensator, auto-tuning, servo plant

1. INTRODUCTION

Phase lead compensators are widely used in industrial servo applications for their simple structures and efficient improvements to the transient performance. However, simple auto-tuning methods to satisfy both gain and phase margin specifications are not yet available both in the literature or in practice. The manual tuning method of Ogata (2002) has been applied broadly in such cases. It is a trial-and-error method and both the gain and phase margin specifications may not be satisfied very well. Loh *et al.* (2004) proposed an auto-tuning method based on a hysterisis relay. As it follows the similar way to Ogata's method, very long tuning time is inevitable. On the other hand, Yeung and Lee (1998; 2000) proposed some chart methods to satisfy both gain and phase margin specifications exactly, but crossover frequencies still need to be specified a priori. Ho and Wang proposed some more straightforward methods (Ho *et al.*, 1995; Fung *et al.*, 1998; Wang *et al.*, 1999*b*) for PID controllers design with exact gain and phase margin specifications, respectively. Unfortunately, these methods can not be extended to phase lead compensator design directly because lead/lag compensators have some parameters in the denominator of its transfer function, unlike PID controllers where all the parameters appear linearly. Up to now, no analytical solution exists for the tuning of phase lead compensators.

In this paper, a new tuning method of phase lead compensators to meet both gain and phase specifications are proposed. From the viewpoint of engineering practice, analytical tuning formulas for all three unknown parameters of phase lead compensators are derived. The procedure of the new method is quite simple and it requires short tuning time. Simulation results show that our solution is also very accurate.

The paper is organized as follows. In Section 2 and 3, the derivation of tuning formulas and frequency identification principles are presented, respectively. Examples and simulation results will follow in Section 4 to illustrate the method. Conclusions will be finally drawn in Section 5.

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2. TUNING METHOD

Consider a servo plant $G(s)$ with the frequency response $G(j\omega)$. Its Nyquist curve is shown as Figure 1. A_m, ϕ_m and ω_g are gain margin, phase margin and corresponding gain crossover frequency, respectively. A phase lead compensator with the following form,

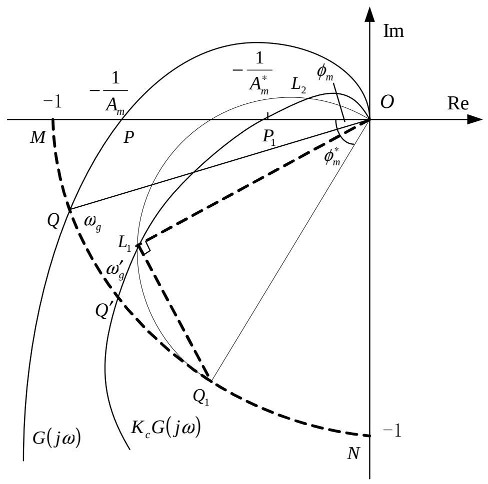

Fig. 1. Nyquist curve of systems

$$K(s) = K_c \frac{Ts+1}{\alpha Ts+1}, \quad 0 < \alpha < 1, \tag{1}$$

is to be inserted in series with the plant in a unit output feedback configuration. Our objective is to make $K(s)G(s)$ satisfy the desired gain margin A_m^* and phase margin ϕ_m^* simultaneously, i.e. the Nyquist curve of $K(j\omega)G(j\omega)$ should pass through P_1 and Q_1.

Noting that in engineering practice, the gain margin specification is not necessary to be exactly satisfied. It is good enough when the gain margin of the compensated system is equal or greater than the specification. So we have

$$K_c \frac{1+j\omega_p T}{1+j\alpha\omega_p T} G(j\omega_p) \geq -\frac{1}{A_m^*}. \tag{2}$$

For the proportional controller, $K(s) = K_c$ with

$$\frac{K_c}{A_m} = \frac{1}{A_m^*} \Rightarrow K_c = \frac{A_m}{A_m^*}, \tag{3}$$

the Nyquist curve of $K_cG(j\omega)$ will pass through P_1, shown as Figure 1. For a lead compensator with the gain K_c, the gain margin of the compensated system will usually become greater, which implies that gain margin specification (2) already holds if (3) is true. So, only phase margin specification needs to be satisfied, i.e.

$$K_c \frac{1+j\omega_g T}{1+j\alpha\omega_g T} G(j\omega_g) = -e^{j\phi_m^*}. \tag{4}$$

Define the complex function:

$$f(\omega) = -\frac{e^{j\phi_m^*}}{K_cG(j\omega)} = \mathrm{Re}(\omega) + j\mathrm{Im}(\omega), \tag{5}$$

where Re and Im denote the real and imaginary components of f. From (4) we have

$$\begin{aligned} 1 + j\omega_g T &= (1 + j\alpha\omega_g T)[\mathrm{Re}(\omega_g) + j\mathrm{Im}(\omega_g)] \\ &= \mathrm{Re}(\omega_g) - \alpha\omega_g T\mathrm{Im}(\omega_g) \\ &\quad + j[\mathrm{Im}(\omega_g) + \alpha\omega_g T\mathrm{Re}(\omega_g)]. \end{aligned} \tag{6}$$

The complex equation (6) is equivalent to two real equations as follows,

$$\mathrm{Re}(\omega_g) - \alpha\omega_g T\mathrm{Im}(\omega_g) = 1, \tag{7}$$

$$\mathrm{Im}(\omega_g) + \alpha\omega_g T\mathrm{Re}(\omega_g) = \omega_g T. \tag{8}$$

Now, we have three unknowns α, T and ω_g but two equations (7) and (8) only. It seems that infinity solutions exist. If ω_g can be determined, then we may find finite solutions of α and T in the following way.

From (7), we have

$$\alpha T = \frac{\mathrm{Re}(\omega_g) - 1}{\omega_g \mathrm{Im}(\omega_g)}. \tag{9}$$

Substituting (9) into (8) yields

$$T = \frac{r^2(\omega_g) - \mathrm{Re}(\omega_g)}{\omega_g \mathrm{Im}(\omega_g)}. \tag{10}$$

Substituting (10) back into (9), we obtain

$$\alpha = \frac{\mathrm{Re}(\omega_g) - 1}{r^2(\omega_g) - \mathrm{Re}(\omega_g)}, \tag{11}$$

where $r = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$ is the module of f.

Assume that the frequency response $K_cG(j\omega)$ is known, ω_g can be determined by the following lemma.

Lemma: If the gain margin of $K_cG(j\omega_g)$ is equal to the specification A_m^* and $A_m^* > 1/\cos\phi_m^*$ holds true, we can always find the range of ω_g given by

$$\omega_g \in \{\omega \,||K_cG(j\omega)| < \cos(\phi_m^* - \phi)\}, \tag{12}$$

where $\phi = \angle K_cG(j\omega) + \pi$.

The geometrical meaning is shown as Figure 1: the above range of ω_g is the frequency segment between L_1 and L_2, which are intersection points of $K_cG(j\omega)$ and the semi circle with diameter of $\overline{OQ_1}$.

Proof: Suppose $A = |K_cG(j\omega)|$ and $K_cG(j\omega) = x + jy$, then from (5) we have

$$
\begin{aligned}
f(\omega) &= -\frac{e^{j\phi_m^*}}{x+jy} \\
&= \frac{(-x+jy)(\cos\phi_m^* + j\sin\phi_m^*)}{A^2} \\
&= \frac{-x\cos\phi_m^* - y\sin\phi_m^*}{A^2} \\
&\quad + j\frac{y\cos\phi_m^* - x\sin\phi_m^*}{A^2} \\
&= \mathrm{Re}(\omega) + j\mathrm{Im}(\omega). \qquad (13)
\end{aligned}
$$

Therefore,

$$
\mathrm{Re}(\omega) = \frac{-x\cos\phi_m^* - y\sin\phi_m^*}{A^2}, \qquad (14)
$$

$$
\mathrm{Im}(\omega) = \frac{y\cos\phi_m^* - x\sin\phi_m^*}{A^2}. \qquad (15)
$$

Assume gain crossover frequency of $K_cG(j\omega)$ is ω_g', then $\omega_g > \omega_g'$ because phase lead angle added will shift the ω_g to the right of ω_g'. Thus

$$
r(\omega) = |f(\omega)| = \frac{1}{A} > 1. \qquad (16)
$$

So that

$$
r^2(\omega) - \mathrm{Re}(\omega) > r(\omega) - \mathrm{Re}(\omega) > 0. \qquad (17)
$$

The last “$>$” holds according to the triangle inequality.

On the other hand,

$$
\begin{aligned}
\frac{y}{x} < \frac{\sin\phi_m^*}{\cos\phi_m^*} &\Rightarrow y\cos\phi_m^* - x\sin\phi_m^* > 0 \\
&\Rightarrow \mathrm{Im}(\omega) > 0, \qquad (18)
\end{aligned}
$$

since $x < 0$, $0 < \phi_m^* < \pi/2$, $\cos\phi_m^* > 0$ and $\sin\phi_m^* > 0$.

From (10), (11), (17) and (18) we can see that $T(\omega) > 0$ is always true but $\alpha(\omega) > 0$ iff

$$
\mathrm{Re}(\omega) > 1. \qquad (19)
$$

Substituting (14) into (19) yields

$$
\begin{aligned}
A &< -\frac{x}{A}\cos\phi_m^* - \frac{y}{A}\sin\phi_m^* \\
&= \cos\phi\cos\phi_m^* + \sin\phi\sin\phi_m^* \\
&= \cos(\phi_m^* - \phi). \qquad (20)
\end{aligned}
$$

From Figure 1 we can see that all points on the semi circle with diameter $\overline{OQ_1} = 1$ satisfy $A = \cos(\phi_m^* - \phi)$. Points inside circle satisfy $A < \cos(\phi_m^* - \phi)$ and points outside circle satisfy $A > \cos(\phi_m^* - \phi)$. If $A_m^* > 1/\cos\phi_m^*$, then $\cos\phi_m^* > 1/A_m^*$, which implies there exists some frequency such that their corresponding frequency response are inside the circle $\overline{OQ_1}$. This frequency range can be denoted as the segment between intersection points of $K_cG(j\omega)$ and semi circle $\overline{OQ_1}$. □

If the frequency response $K_cG(j\omega)$ is unknown, then it should be identified a priori by auto-tuning based on FFT techniques and relay feedback (Wang *et al.*, 2001). Principles of this auto-tuning method will be demonstrated in the next section.

Once ω_g is determined, K_c, T and α can be easily calculated from (3), (11) and (10).

The tuning method can be summarized as follows:

Step 1. Calculate the frequency response of the plant. If the transfer function of the plant is unknown, estimate it with FFT-based auto-tuning method;
Step 2. Determine K_c by (3), compute $K_cG(j\omega)$ and choose ω_g by (12);
Step 3. Calculate α and T from (11) and (10), respectively.

3. FREQUENCY RESPONSE IDENTIFICATION

If the transfer function $G(s)$ of the plant is known, its frequency response $G(j\omega)$ can be easily obtained by converting $s = j\omega$. However, when $G(s)$ is unknown, its frequency response can still be obtained by relay feedback and FFT-based techniques(Wang *et al.*, 2001; Wang *et al.*, 1999*a*).

Consider a standard relay feedback system shown as Figure 2. If stable oscillation can be con-

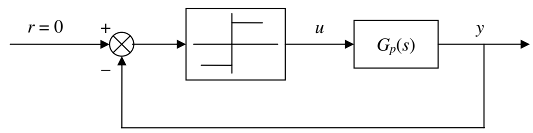

Fig. 2. Relay feedback system

structed, $y(t)$ or $u(t)$ can be decomposed into the periodic stationary cycle parts $y_s(t)$ or $u_s(t)$ and the transient parts $\Delta y(t)$ or $\Delta u(t)$ as

$$
\begin{aligned}
y(t) &= \Delta y(t) + y_s(t), \\
u(t) &= \Delta u(t) + u_s(t).
\end{aligned}
$$

Thus

$$
G(s) = \frac{Y(s)}{U(s)} = \frac{\Delta Y(s) + Y_s(s)}{\Delta U(s) + U_s(s)}. \qquad (21)
$$

where $\Delta Y(s)$ and $\Delta U(s)$ are the Laplace transforms of the transient parts $\Delta y(t)$ and $\Delta u(t)$, respectively; $Y_s(s)$ and $U_s(s)$ are the Laplace transforms of the periodic parts $y_s(t)$ and $u_s(t)$, respectively.

For the periodic parts, their Laplace transforms can be calculated by the following lemma(Kuhfittig, 1978).

Lemma: Suppose that $f(t)$ is a periodic function with period T_c for $t \geq 0$, i.e.,

$$f(t) = \begin{cases} f(t+T_c), & t \in [0, +\infty), \\ 0, & t \in (-\infty, 0). \end{cases} \quad (22)$$

Assume that $\mathcal{L}f(t) = F(s)$ exists, then

$$F(s) = \frac{1}{1-e^{-sT_c}} \int_0^{T_c} f(t)e^{-st}dt. \quad (23)$$

Thus

$$Y_s(j\omega) = \frac{1}{1-e^{-j\omega T_c}} \int_0^{T_c} y_s(t)e^{-j\omega t}dt. \quad (24)$$

For the transient parts, we suppose that after $t = T_f$, both $\Delta y(t)$ and $\Delta u(t)$ are approximately zero, then

$$\Delta Y(j\omega) = \int_0^{\infty} \Delta y(t)e^{-j\omega t}dt \approx \int_0^{T_f} \Delta y(t)e^{-j\omega t}dt. \quad (25)$$

After discretization, we have

$$\begin{aligned} \Delta Y(\omega_i) &= T\sum_{k=0}^{N-1} \Delta y(kT)e^{-j\omega_i kT}, \\ Y_s(j\omega_i) &= \frac{1}{1-e^{-j\omega_i T_c}} \sum_{k=0}^{N_c} y_s(kT)e^{-j\omega_i kT}T, \\ &\quad i = 1, 2, \cdots, m. \end{aligned}$$

where $m = N/2, \omega_i = 2\pi i/(NT), N_c = (T_c - T)/T, T_f = (N-1)T$ and T is the sampling interval.

$\Delta U(j\omega_i)$ and $U_s(j\omega_i)$ can be calculated in the same way, so that the frequency response of the plant is obtained as

$$G(j\omega_i) = \frac{\Delta Y(j\omega_i) + Y_s(j\omega_i)}{\Delta U(j\omega_i) + U_s(j\omega_i)}, \quad i = 1, 2, \cdots, m. \quad (26)$$

After the frequency response of the plant is obtained, ω_g^* and ϕ_m^* can be easily obtained by numerical interpolation and searching. For example, suppose that $\hat{G}_p(\omega_1)$ and $\hat{G}_p(\omega_2)$ are consecutive frequency response of $G(j\omega)$ by auto-tuning method, which satisfy $|\hat{G}_p(\omega_1)| > 1 > |\hat{G}_p(\omega_2)|$ for $\omega_1 < \omega_g^* < \omega_2$, then we have

$$\frac{\log|\hat{G}_p(\omega_1)| - \log|\hat{G}_p(\omega_2)|}{\log\omega_1 - \log\omega_2} \approx \frac{0 - \log|\hat{G}_p(\omega_2)|}{\log\omega_g^* - \log\omega_2}. \quad (27)$$

4. ILLUSTRATIVE EXAMPLE

Example 1 A lead compensator of the form

$$K(s) = K_c\frac{Ts+1}{\alpha Ts+1}, \quad \alpha < 1, \quad (28)$$

is designed for the plant

$$G(s) = \frac{4}{s(s+2)}\mathrm{e}^{-0.35s}, \quad (29)$$

with the desired gain margin $A_m^* = 3$ and phase margin $\phi_m^* = 60°$, respectively.

The auto-tuning procedures are shown as Figure 3. The first part of Figure 3 for $t \in [0, 11.8]$ is the relay test, then a phase lead compensator is tuned and commissioned. After settling down, at $t = 18$, a step set point change of one is introduced and the process output is quite good.

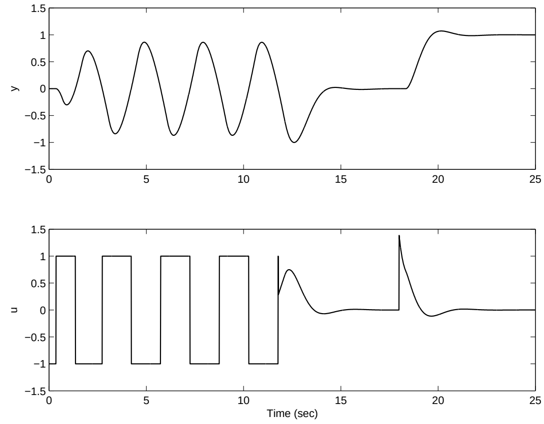

Fig. 3. Auto-tuning performance

The frequency response of the plant can be obtained from (26), shown as Figure 4, where "+" denotes the estimation of frequency response of plant. Its gain margin $A_m = 1.5549$. From (3), $K_c = 0.5183$; while from (12), $\omega_g \in (1.0669, 12.448)$.

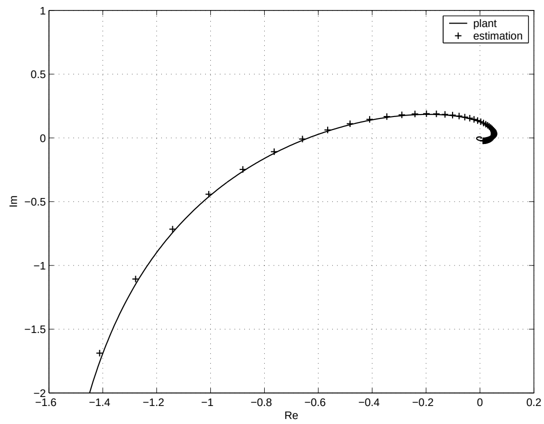

Fig. 4. Frequency response identification by auto-tuning

Let $\omega_g = 1.0669$, from (10) and (11)

$$\alpha = 0.3725,$$
$$T = 0.6252,$$
$$K(s) = \frac{0.324s + 0.5183}{0.2329s + 1}.$$

The Bode diagram of the compensated system $K(s)G(s)$ is given in Figure 5; its comparison with the uncompensated system is also presented. The gain margin, phase margin and corresponding crossover frequencies are shown as Table 1.

Table 1. Comparison of G and KG.

System	A_m	ϕ_m	ω_p	ω_g
G	1.572	20.3°	2.14	1.57
KG	3.016	60.3°	3.02	1.07

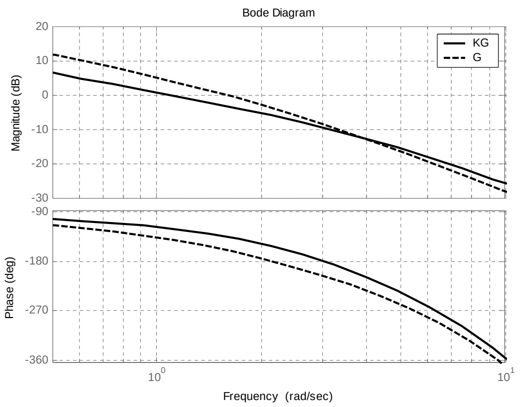

Fig. 5. Bode diagram of open-loop systems

From Figure 5 we can see that phase margin specification is well satisfied and gain margin can be guaranteed that it is greater than the specification. So, the lead compensator obtained is good enough and can be accepted.

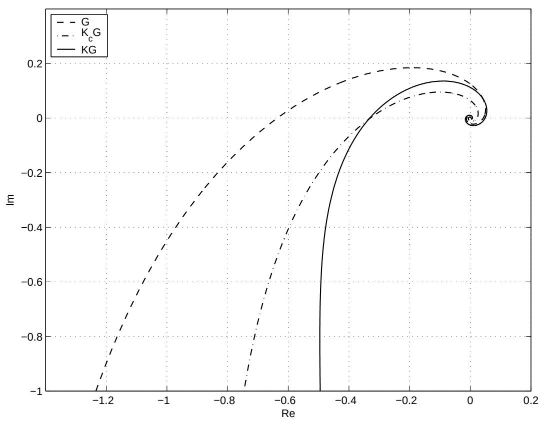

Fig. 6. Frequency response of open-loop systems

The open-loop frequency responses of processes: $G(j\omega), K_cG(j\omega)$ and $K(j\omega)G(j\omega)$ are shown as Figure 6. To evaluate improvements for the plant

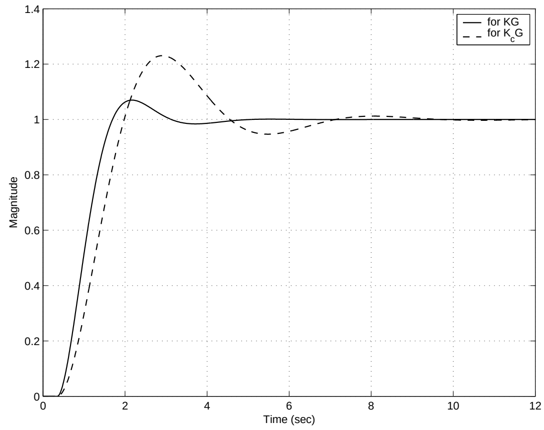

Fig. 7. Step response of closed-loop systems

by a phase lead compensator, we choose proportional control with K_c as a reference because it satisfies the gain margin specification. The closed-loop unit step response of $K(s)G(s)$ and $K_cG(s)$ are shown is Figure 7. From the figure we can see that the transient behavior of the lead compensated system is greatly improved.

Example 2 Design a compensator for the plant

$$G(s) = \frac{0.25}{s(0.5s + 1)(2.5s + 1)(5s + 1)}, \qquad (30)$$

with the desired gain margin $A_m^* = 3$ and phase margin $\phi_m^* = 60°$, respectively.

By auto-tuning, the gain margin of the plant $A_m = 1.8515$. Follow the same approach as what we used in Example 1, $K_c = 0.6172$ and $\omega_g \in (0.1437, 0.3232)$.

Let $\omega_g = 0.1437$, from (10) and (11) we have

$$\alpha = 0.1349,$$
$$T = 4.9436,$$
$$K(s) = \frac{3.051s + 0.6172}{0.6668s + 1}.$$

The Bode diagram of the compensated system is shown as Figure 8 and its comparison with the proportional control system is also presented. The gain margin, phase margin and corresponding crossover frequencies are shown as Table 2.

Table 2. Comparison of G and KG.

System	A_m	ϕ_m	ω_p	ω_g
G	1.875	20.8°	0.248	0.173
KG	6.839	60.4°	0.553	0.144

From the figure we can see that the phase margin specification is well satisfied but gain margin of the compensated system is sure to be greater than specification. So the lead compensator obtained is good enough and acceptable.

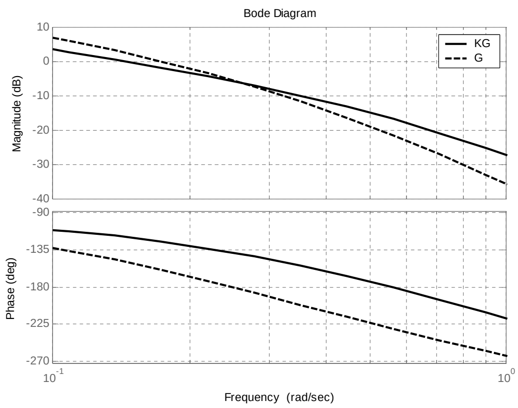

Fig. 8. Bode diagram of open-loop systems

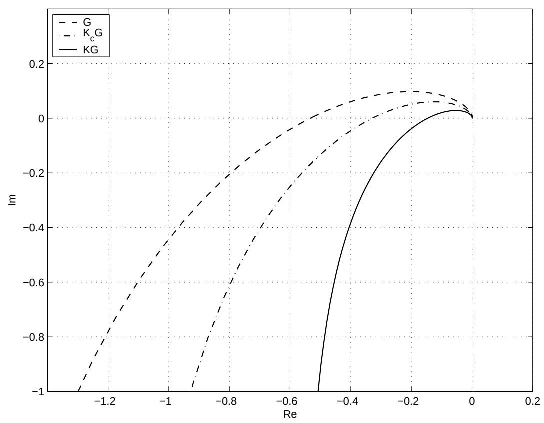

Fig. 9. Frequency response of open-loop systems

The open-loop frequency responses of processes: $G(j\omega), K_cG(j\omega)$ and $K(j\omega)G(j\omega)$ are shown in Figure 9. To evaluate improvements to the plant by a phase lead compensator, we choose proportional control with K_c as a reference since it satisfies the gain margin specification. The closed-loop unit step response of $K(s)G(s)$ and $K_cG(s)$ are shown as Figure 10, it is obvious that the transient performance of lead compensated plant is greatly improved.

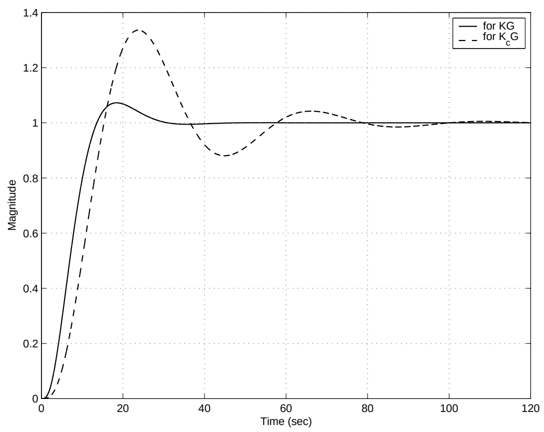

Fig. 10. Step response of closed-loop systems

5. CONCLUSION

With our method, it becomes simple and straightforward to tune a phase lead compensator for exact phase margin specification and acceptable gain margin. For the wide class of servo applications, we have been able to determine the proper parameters K_c, α and T by analytical formulas while facilitating simple and fast auto-tuning. Simulation results show that our method is very general.

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