EXTREMUM SEEKING CONTROL VIA SLIDING MODE WITH TWO SURFACES

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Abstract: In this paper, we present a new approach of extremum seeking control scheme, which applies two surfaces. Compared to other extremum seeking control algorithms, the new approach produces nearly no oscillation in the control results because the searching signal in this approach is no longer a periodic one. The control accuracy can be guaranteed by choosing two surfaces at a suitable distance. The proof of stability of the approach and several examples are also provided.

Keywords: sliding mode control, sliding surface, adaptive algorithm, SISO, interconnected systems, Nash games.

1. INTRODUCTION

In most control problems it is assumed that the reference value (set point) is given or easily determined. On other occasions it can be more difficult to find a suitable reference value or the best operating point of a process. In stead a performance index or cost function is employed. This function usually has an *extremum* and the objective is to select the set point to keep the output at the extremum value of the function. However, the function is not usually completely known either by function expression or with undetermined parameters. The uncertainty in the performance index makes it necessary to use some sort of adaptation to find the set point which minimizes or maximizes the output (Astrom and Wittenmark 1995). Extremum seeking control approaches have been proposed to find a set point or track a varying set point where the output or a cost function of the system reaches the extremum (Haskara, *et al*., 2002; Krstic and Wang 2000; Pan, *et al*., 2003; Yu and Ozguner 2002). A general plant for extremum seeking control is modeled as a static nonlinear map cascaded with a linear or nonlinear block. The approaches reported usually separate systems into a fast and a slow part, assuming that the plant dynamics and associated stabilizing controller are fast with respect to the outer-loop extremum seeking scheme. They all applied periodic searching signals added to the inputs of systems, and made the systems track the extremum by exploring an estimation of the derivative of the performance index.

Haskara, *et al*. (2002) and Pan, *et al*., (2003) have proposed extremum seeking controllers with sliding mode. The inherit problem of steady state oscillation by using periodic searching signals was discussed by Yu (2002). In this paper, we will propose a new extremum seeking control approach via sliding mode with two surfaces. A non-increasing or non-decreasing searching signal instead of periodic searching signal is used in the controller. This controller will drive a system to a boundary layer enclosed by the two surfaces before the extremum point is reached and to a sliding surface after that. So, the control accuracy can be guaranteed by choosing two surfaces very close to each other. We apply this extremum seeking control approach to a SISO system, an interconnected system and an n-person non-cooperative dynamic game. The simulation results show that the extremum points are reached and the steady state oscillation is successfully suppressed by using our approach.

2. EXTREMUM SEEKING FOR SISO SYSTEMS

A nonlinear single-input-single-output (SISO) system with a performance index function is given as

$$\dot{x} = F(x,u)$$
$$y = H(x) \tag{1}$$
$$z = Z(y)$$

where $x \in R^n, u \in R, y \in R$, $F: R^n \times R \to R^n$ and $H: R^n \to R$ are smooth functions. $z(t)$ is the value of the performance index of time, and is not completely known either as a function of expression or with undetermined parameters. Consider a smooth control law

$$u = P(x(t), \xi(t)) \tag{2}$$

where $\xi \in R$ is a variable satisfying

$$\dot{\xi}(t) = v(t) \tag{3}$$

which we shall call the auxiliary system.

After applying the control (2) to (1), the closed-loop system $\dot{x} = F(x, P(x,\xi))$ has an equilibrium manifold that is a function of ξ. We also make the following assumptions:

Assumption 1: There exists a unique smooth function $f: R \to R^n$ such that $F(x_e, P(x_e, \xi)) = 0$ where $x_e = f(\xi)$.

Assumption 2: For any $\xi \in R$, there exists a smooth control law $u^* = P^*(x(t), \xi(t))$ such that the equilibrium point $x_e = f(\xi)$ of (1) is locally exponentially stable.

Assumption 3: There exists a minimum $\xi^* \in R$ such that $(H \circ f)'(\xi^*) = 0$ and $(H \circ f)''(\xi^*) > 0$, where $(H \circ f)(\xi) = H(f(\xi))$ is a combined function. Then the output equilibrium map $y = H(f(\xi))$ is smooth and has a minimum at $\xi = \xi^*$. We further assume that the minimum point of the performance index is unique globally.

We can easily define a maximum for the combined function $(H \circ f)$. Without loss of generality, in the following we use only the minimum function in developing our control strategy and make stability proof since they can be easily used for maximum functions by putting a negative sign in front of the performance index.

Now we define a reference searching signal

$$\dot{g}(t) = \begin{cases} -\rho & s_2 < 0 \\ 0 & s_2 \ge 0 \end{cases} \tag{4}$$

where ρ is a positive constant value. Its initial value $g(0)$ and the meaning of s_2 will be given later. It is obvious that $g(t)$ is non-increasing with time. Then we have an error signal between the value of the performance index and the searching signal

$$e = Z(\xi) - g(t) \tag{5}$$

$$\dot{e} = \dot{Z} - \dot{g} = \frac{\partial Z}{\partial \xi}\dot{\xi} - \dot{g} \tag{6}$$

Two surfaces are then defined as below:

$$s_1 = e + \varepsilon_1 \tag{7}$$

$$s_2 = e - \varepsilon_2 \tag{8}$$

where ε_1 and ε_2 are the error tolerances and are very small positive numbers.

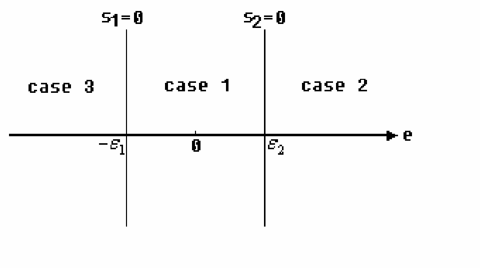

Fig. 1. Two surfaces and three cases

The variable structure control law for the auxiliary system is designed as

$$v(t) = \begin{cases} M & s_1 > 0, s_2 < 0 \\ 0 & s_1 \le 0 \\ -M sign[\dot{e}(t-\tau)v(t-\tau)] & s_2 \ge 0 \end{cases} \tag{9}$$

where $M > \rho$ is a positive constant value and τ is a small time delay.

Theorem 1: Under control (2), (3) and (9), the states of system (1) will go to the equilibrium, which minimizes the value of the performance index Z.

Proof: Before the minimum of the performance index is reached, *i.e.* $\partial Z/\partial \xi \ne 0$, three cases can happen as shown in Fig. 1.

Case 1: when $s_1 > 0$ and $s_2 < 0$, $v(t) = M$ and $\dot{g}(t) = -\rho$

$$\dot{e} = \frac{\partial Z}{\partial \xi}\dot{\xi} - \dot{g} = \frac{\partial Z}{\partial \xi}v + \rho = \frac{\partial Z}{\partial \xi}M + \rho \tag{10}$$

If $\frac{\partial Z}{\partial \xi} < -\frac{\rho}{M}$, then $\dot{e} < 0$ and e will go to the surface $s_1 = 0$;

If $\frac{\partial Z}{\partial \xi} > -\frac{\rho}{M}$, then $\dot{e} > 0$ and e will go to the surface $s_2 = 0$.

Case 2: when $s_2 \ge 0$, $v(t) = -M sign[\dot{e}(t-\tau)v(t-\tau)]$ and $\dot{g}(t) = 0$

$$\dot{e} = \frac{\partial Z}{\partial \xi}v - \dot{g} = -M\frac{\partial Z}{\partial \xi} sign[\dot{e}(t-\tau)v(t-\tau)] \tag{11}$$

We also get $sign[\dot{e}(t-\tau)v(t-\tau)] = sign\left(\frac{\partial Z}{\partial \xi}(t-\tau)\right)$ since $\dot{g} = 0$. According to the problem statement and Assumptions 1 ~ 3, $\partial Z/\partial \xi$ is a smooth and continuous function. So, when τ is small enough, we obtain the equation below before the minimum is reached:

$$sign[\dot{e}(t-\tau)v(t-\tau)] = sign\left(\frac{\partial Z}{\partial \xi}(t-\tau)\right)$$

$$= sign\left(\frac{\partial Z}{\partial \xi}(t)\right) \tag{12}$$

Substitute equation (12) into equation (11), we have $\dot{e} = -M\left|\partial Z/\partial \xi\right| < 0$ and then e will approach the surface $s_2 = 0$.

Case 3: when $s_1 \le 0$, $v(t) = 0$ and $\dot{g}(t) = -\rho$

$$\dot{e} = \frac{\partial Z}{\partial \xi} v - \dot{g} = \rho > 0 \quad (13)$$

e will go to the surface $s_1 = 0$.

Based on the discussion above, we conclude that $e = Z - g$ will converge to the boundary layer enclosed by two surfaces $s_1 = e + \varepsilon_1 = 0$ or $s_2 = e - \varepsilon_2 = 0$ before the minimum is reached. In this boundary layer, we have $g - \varepsilon_1 < Z < g + \varepsilon_2$ and g continuously decreases, so Z will decrease accordingly until its minimum is reached.

Now we consider the case when the minimum of the performance index is reached, i.e. $\partial Z / \partial \xi = 0$. If e is in the boundary layer ($s_2 < 0 < s_1$), we have $\dot{e} = \rho > 0$. If $s_2 > 0$, $\dot{e} < 0$ from case 2 in the above proof. Thus,

$$\dot{s}_2 = \dot{e} = -\text{sgn}(s_2) \quad (14)$$

and e will go to the surface $s_2 = 0$. When $s_2 = 0$ is reached, g will stop decreasing. Therefore, sliding mode can happen in the manifold $s_2 = 0$ and the reference signal g stops decreasing after the minimum of the performance index is reached. Because $\dot{e}(t-\tau)$ and $v(t-\tau)$ are unavailable at $t = 0$, we need pick the initial value $g(0)$ greater than the initial value $Z(\xi(0))$ to avoid case 2 happening at $t = 0$.

We will illustrate the proposed algorithm by an example of a second order system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) - 0.2x_2 + u \quad (15) \\ y &= x_1 \end{aligned}$$

And the performance index is $Z = 10(y-5)^2 + 10$.

First we design a sliding mode control, which can regulate x_1 to ξ quickly

$$u = -\sin(x_1) + 0.2x_2 - 2(x_2 - \dot{\xi}) - 3\text{sgn}(\lambda) \quad (16)$$

where $\lambda = 2(x_1 - \xi) + x_2$ is the selected sliding surface for the sliding mode control *u*.

The performance index function has only one minimum point at $\xi^* = 5$. We apply the extremum seeking control algorithm proposed above and pick $M = 0.6$, $\rho = 0.5$, $\varepsilon_1 = \varepsilon_2 = 0.03$, $g(0) = 300$ in (4)~(9). Fig. 2 and Fig. 3 show the simulation results with an initial value $\xi(0) = 0$. In fig. 2, we see that ξ converges to the minimum point without steady state oscillation. In fig. 3, we find that the error signal *e* is attracted to the boundary layer before the minimum point is reached and stays on the manifold $s_2 = 0$ after that.

Remarks: simulation results show that the size of the boundary layer or the selection of $\varepsilon_1, \varepsilon_2$ is very important to the performance of the algorithm. The smaller $\varepsilon_1, \varepsilon_2$ are, the higher accuracy and faster convergence we could obtain. But too small $\varepsilon_1, \varepsilon_2$ will make the system unstable.

3. EXTREMUM SEEKING FOR INTERCONNECTED SISO SYSTEMS

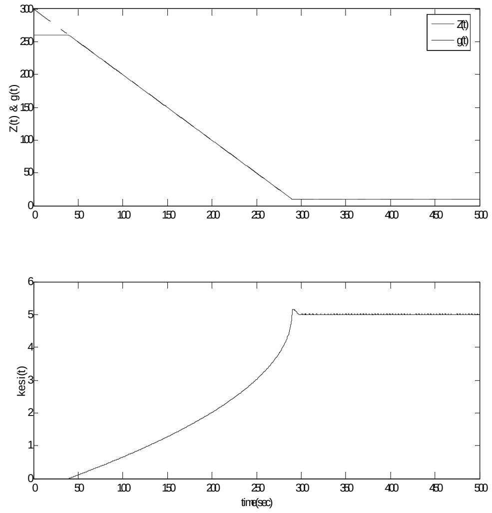

Fig. 2. Simulation results of *Z*, *g* and ξ

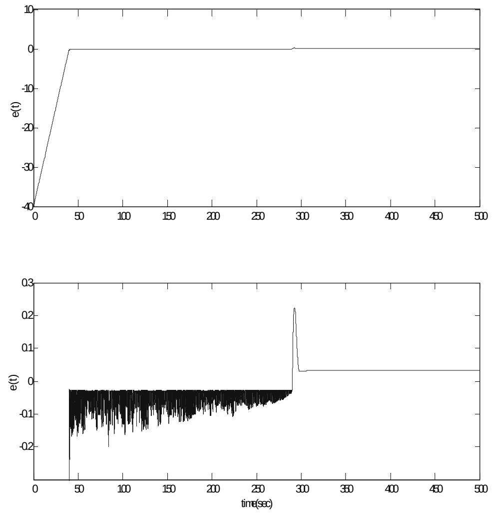

Fig. 3. Simulation result of *e* and its zoom in *e*-axis

Consider a nonlinear interconnected SISO system with each subsystem being affine in the control input as

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} f_1(X_1) + q_1(X_2, \ldots, X_n) + g_1(X_1)u_1 \\ f_2(X_2) + q_2(X_1, X_3, \ldots, X_n) + g_2(X_2)u_2 \\ \vdots \\ f_n(X_n) + q_n(X_1, \ldots, X_{n-1}) + g_n(X_n)u_n \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} h_1(X_1) \\ h_2(X_2) \\ \vdots \\ h_n(X_n) \end{bmatrix} \qquad \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} Z_1(y_1) \\ Z_2(y_2) \\ \vdots \\ Z_n(y_n) \end{bmatrix} \quad (17)$$

where $X_i = [x_{i1}, x_{i2}, \ldots, x_{ip_i}]^T \in R^{p_i}$, $u_i \in R$, $y_i \in R$; f_i, q_i, g_i, and h_i are smooth and bounded vector fields. z_i $(i = 1, \ldots, m)$ are values of the performance indices $Z_i(y_i)$ for each subsystem that are not completely known either by function of expression or with undetermined parameters. We take all the assumptions in Section 2. We set up a first order auxiliary searching system for each subsystem $\left(i = 1, \ldots, n\right)$

$$\dot{\xi}_i(t) = v_i(t) \tag{18}$$

$$e_i = Z_i(\xi_i) - g_i(t) \tag{19}$$

$$\dot{e}_i = \dot{Z}_i - \dot{g}_i = \frac{\partial Z_i}{\partial \xi}\dot{\xi}_i - \dot{g}_i = \frac{\partial Z_i}{\partial \xi} v_i - \dot{g}_i \tag{20}$$

Two surfaces for each subsystem are defined as below:

$$s_{i1} = e_i + \varepsilon_{i1} \tag{21}$$

$$s_{i2} = e_i - \varepsilon_{i2} \tag{22}$$

where ε_{i1} and ε_{i2} are the error tolerances and are very small positive numbers. The variable structure control laws for the auxiliary systems are designed as:

$$v_i(t) = \begin{cases} M_i & s_{i1} > 0, s_{i2} < 0 \\ 0 & s_{i1} \le 0 \\ -M_i sign(\dot{e}_i(t-\tau) v_i(t-\tau)) & s_{i2} \ge 0 \end{cases} \tag{23}$$

The searching signal is picked as:

$$\dot{g}_i(t) = \begin{cases} -\rho_i & s_{i2} < 0 \\ 0 & s_{i2} \ge 0 \end{cases} \tag{24}$$

where $0 < \rho_i < M_i$ is a positive constant value. Its initial value $g_i(0)$ is given greater than the initial value $Z_i(\xi_i(0))$. It is obvious that $g_i(t)$ is non-increasing with time.

Theorem 2: For each subsystem of the interconnected system (17) with control (23), the system states will converge to the equilibrium, which minimizes the value of its performance index z_i.

Proof: Following the lines of the proof of Theorem 1, we can deduce that the auxiliary variable ξ_i can reach its minimum point ξ_i^*, which makes the performance index $Z_i(\xi_i)$ to reach its minimum value.

Now we need to find a control such that each subsystem's output y_i can follow ξ_i very fast. Looking at (17), we know that each subsystem is a SISO nonlinear control system, which is affine in the control input if we consider the interconnected part q_i as interference and put it together with f_i. We can use exact input-output linearization and transform each subsystem in a new coordinates $\eta_i = \left[\eta_{i1}, \ldots, \eta_{ip_i}\right]^T$ as (suppose subsystem i has a relative degree r_i)

$$\begin{aligned} \dot{\eta}_{i1} &= \eta_{i2} \\ &\vdots \\ \dot{\eta}_{i(r_i-1)} &= \eta_{ir_i} \\ \dot{\eta}_{ir_i} &= L^{r_i}_{(f_i+q_i)} h_i(\eta_i) + L_{g_i} L^{r_i-1}_{(f_i+q_i)} h_i(\eta_i) u_i = a_i(\eta_i) + b_i(\eta_i) u_i \\ \dot{\eta}_{i(r_i+1)} &= d_{i(r_i+1)}(\eta_i) \\ &\vdots \\ \dot{\eta}_{ip_i} &= d_{ip_i}(\eta_i) \\ y_i &= \eta_{i1} \end{aligned} \tag{25}$$

where $L^{r_i}_{(f_i+q_i)}$ and $L_{g_i} L^{r_i-1}_{(f_i+q_i)}$ are Lie derivatives (Marquez, 2003).

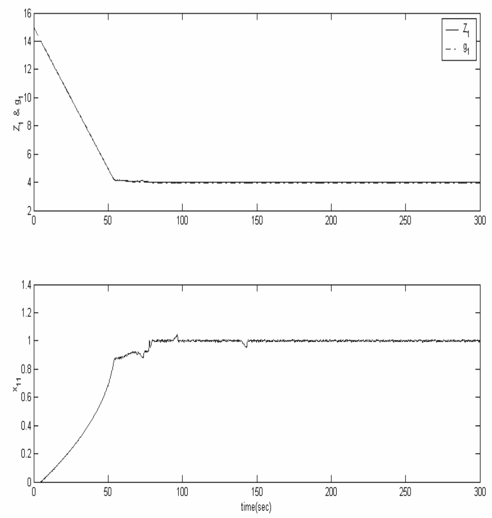

Fig. 4. Extremum Seeking for Subsystem 1

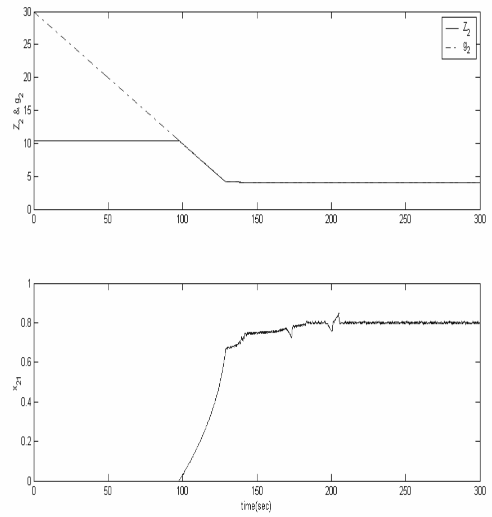

Fig. 5. Extremum Seeking for Subsystem 2

Define the tracking error as

$$\delta_i(t) = y_i(t) - \xi_i(t) = \eta_{i1}(t) - \xi_i(t) \tag{26}$$

and a surface as

$$\lambda_i = \delta_i^{(r_i-1)} + c_{i(r_i-1)}\delta_i^{(r_i-2)} + \cdots + c_{i1}\delta_i \tag{27}$$

where $c_{i1}, \ldots, c_{i(r_i-1)}$ are the controller's parameters and all positive. Design a sliding mode control for this subsystem as

$$u_i = \frac{1}{b_i(\eta_i)}[-a_i(\eta_i) - K_i \operatorname{sgn}(\lambda_i) + \xi_i^{(r_i)} - c_{i(r_i-1)}\delta_i^{(r_i-1)} - \cdots - c_{i1}\dot{\delta}_i] \tag{28}$$

Under the control (26) the subsystem will be induced onto the sliding surface $\lambda_i = 0$ after a finite time. Then we can choose $c_{i1}, \ldots, c_{i(r_i-1)}$ such that the output y_i converges to ξ_i by a desired dynamics.

We will illustrate the proposed algorithm by a two-machine swing equation model. Consider the following model of two-machine power systems; see (Guo and Salam, 1994)

$$\dot{x}_{11} = x_{12}$$

$$\dot{x}_{12} = -\sin(x_{11}) - 0.5x_{12} - 0.1\sin(x_{11} - x_{21}) + u_1$$

$$\dot{x}_{21} = x_{22} \tag{29}$$
$$\dot{x}_{22} = -\sin(x_{21}) - 0.5x_{12} - 0.1\sin(x_{21} - x_{11}) + u_2$$
$$y_1 = x_{11}$$
$$y_2 = x_{21}$$

The performance indices are

$$z_1 = 10(y_1 - 1)^2 + 4 \tag{30}$$
$$z_2 = 10(y_2 - 0.8)^2 + 4 \tag{31}$$

The only minimum point of (28) is $\xi_1^* = 1$ and the only minimum point of (29) is $\xi_2^* = 0.8$.

The extremum seeking control algorithm (18) ~ (24) and (28) are implemented for the above system with parameters as

$\rho_i = 0.2$, $M_i = 0.05$, $\varepsilon_{i1} = \varepsilon_{i2} = 0.03$, $K_i = 1$, $c_{i1} = 5$, $\tau = 0.01$ $(i = 1,2)$

The simulation results are shown in Fig. 4 and Fig. 5. Each subsystem converges to the minimum of its performance index under the proposed control. Note that the oscillation in the control results x_1 and x_2 are not caused by the extremum seeking control. It comes from the sliding mode control in u_1 and u_2, which drive the states to track the extremum points.

4. NASH SOLUTION BY EXTREMUM SEEKING CONTROL

For an *n*-person non-cooperative dynamic game, each player has a cost function and adjusts some of the control parameters to minimize his own cost function to find a Nash equilibrium solution. When the cost function is not completely known either by function expression or with undetermined parameters although it is measurable, extremum seeking control with sliding mode can be used to solve for the Nash solution. Extremum seeking control via sliding mode with two surfaces for the Nash solution will be proposed as follow.

Consider an *n*-person non-cooperative dynamic described by a nonlinear system

$$\frac{d}{dt}x(t) = f(x(t), u_1(t), u_2(t), \cdots, u_n(t)) \tag{32}$$

with a cost function for *i*th player

$$J_i(t) = J_i(x(t)), \quad (i \in N) \tag{33}$$

where *N* is the index set of player defined as

$N = \{1,2,\ldots,n\}$

$x(t) \in R^m$, $u_i(t) \in R$ $(i \in N)$, and $J_i(t) \in R$ $(i \in N)$ are the state variable, the *i*th player's control input, and the *i*th player's cost function, respectively. The functions, f and $J_i(x)$ $(i \in N)$ are smooth. Here we take all the assumptions given by Pan, *et al*., 2002.

Assumption 4: There exist smooth control laws $u_i(t) = \alpha_i(x(t), \theta_i)$ $(i \in N)$ for all players to stabilize the above nonlinear system (1), where θ_i $(i \in N)$ is a control parameter.

Assumption 5: There exist a smooth function $x_e : R \to R^n$ such that

$$f(x(t), \alpha_1(x(t), \theta_1), \ldots, \alpha_n(x(t), \theta_n)) = 0$$
$$\Updownarrow$$
$$x = x_e(\theta_1, \ldots, \theta_n)$$

Assumption 6: The static performance map at the equilibrium point $x_e(\theta_1, \ldots, \theta_n)$ to $J_i(t)$ represented by

$$J_i^e = J_i(x_e(\theta_1, \ldots, \theta_n)) = J_i(\theta_1, \ldots, \theta_n) \qquad (i \in N)$$

is smooth and has a unique Nash equilibrium solution $J_i^*(\theta_1^*, \ldots, \theta_n^*)$ at point $(\theta_1^*, \ldots, \theta_n^*)$ such that J_i^* is a minimum.

To design an extremum seeking controller using sliding mode for the *i*th player $(i \in N)$:

$$\dot{u}_i(t) = v_i(t) \tag{34}$$
$$e_i = J_i(t) - g_i(t) \tag{35}$$

Two surfaces are defined as below:

$$s_{i1} = e_i + \varepsilon_{i1} \tag{36}$$
$$s_{i2} = e_i - \varepsilon_{i2} \tag{37}$$

where ε_{i1} and ε_{i2} are the error tolerances and are very small positive numbers.

The variable structure control law is selected as:

$$v_i(t) = \begin{cases} M_i & s_{i1} > 0, s_{i2} < 0 \\ 0 & s_{i1} < 0 \\ -M_i sign(\dot{e}_i(t-\tau)v_i(t-\tau)) & s_{i2} \geq 0 \end{cases} \tag{38}$$

where M_i is a positive number and τ is a small time delay. And the searching signal is picked as:

$$\dot{g}_i(t) = \begin{cases} -\rho_i & s_{i2} < 0 \\ 0 & s_{i2} \geq 0 \end{cases} \tag{39}$$

where $0 < \rho_i < M_i$ is a positive constant value. Its initial value $g_i(0)$ is given greater than the initial value $J_i(0)$. It is obvious that $g_i(t)$ is non-increasing with time.

Theorem 3: For the dynamic non-cooperative game (30), the sliding mode controller with extremum seeking control approach for the *i*th player $(i \in N)$ as (32) ~ (37) ensures that the cost functions $J_i(t)$ $(i \in N)$ are minimized to get the Nash equilibrium solution.

Proof: from *theorem 2* we find that the proposed extremum seeking control approach will make each player to approach the minimum point of its own cost function under any circumstance. Thus, the sliding mode controller with extremum seeking control approach will ensure that the cost functions $J_i(t)$'s are minimized to get the Nash equilibrium solution.

To illustrate the proposed algorithm, consider a two person non-cooperative dynamic game described by a second order linear system with unknown parameters

$$\dot{x}(t) = \begin{bmatrix} -1 & 0.2 \\ 0.3 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5(u_1(t) - 2 - 0.1u_2(t))^2 + 1.0 \\ 0.7(u_2(t) - 1 - 0.2u_1(t))^2 + 0.5 \end{bmatrix} \tag{40}$$

The cost functions for two players are given by

$$J_i(t) = x_i(t) \qquad (i = 1,2)$$

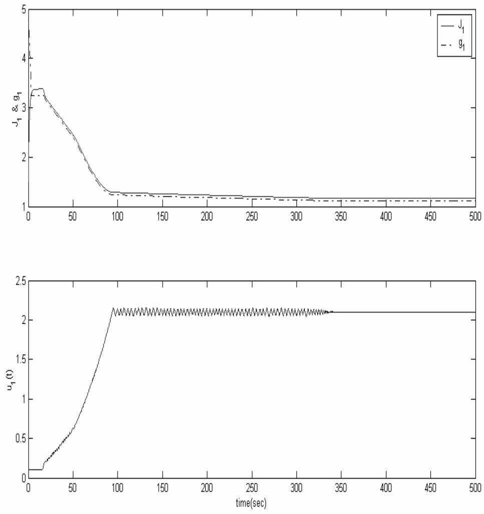

Fig. 6. Solution by Extremum Seeking Control for Player 1

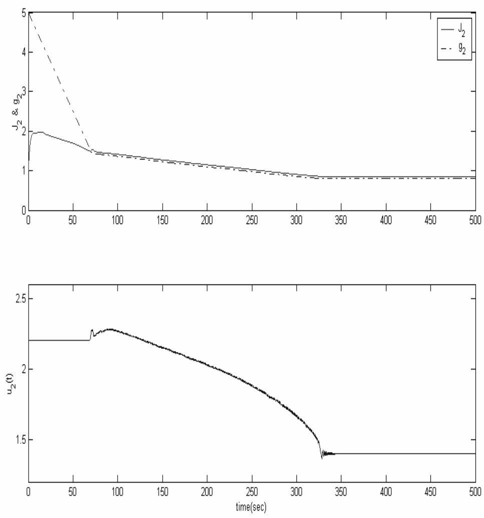

Fig. 7. Solution by Extremum Seeking Control for Player 2

It is clear that the Nash equilibrium point is ($u_1^* = 2.1429$, $u_2^* = 1.4286$). The extremum seeking control algorithm (32) ~ (37) is implemented for the above system with parameters as $\rho_i = 0.05$, $M_i = 0.05$, $\varepsilon_{i1} = \varepsilon_{i2} = 0.03$, $\tau = 0.01$ $(i = 1,2)$. The simulation results are shown in Fig. 6 and Fig. 7, which show that the system finally reaches the Nash equilibrium point.

5. CONCLUSION

The extremum seeking control approaches via sliding mode with two sliding surfaces were implemented in a SISO system, an interconnected system and an n-person non-cooperative dynamic game. With the proposed control algorithms, those systems all converge to the extremum points or the Nash equilibrium point without steady state oscillation. The simulation results show the effectiveness.

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