LIMIT CYCLES IN FEEDBACK CONTROL SYSTEMS WITH HYSTERESIS

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Abstract: The paper presents a theoretical and numerical study of the existence problem of self-sustained periodic solutions in feedback control systems including a hysteresis nonlinearity with non-local memory. The presence of this type of nonlinearity in the feedback control system can cause undesired effects, e.g. limit cycles, instability, and thus must be taken into account in the design phase of the control system. In the present work the problem is addressed by resorting to functional analysis tools and to the Preisach operator theory.

Keywords: Hysteresis, Limit cycles, Feedback control, Preisach operators

1. INTRODUCTION

In traditional control systems, the problem of foreseeing limit cycles has been usually tackled by modelling the nonlinearities of the components, e.g. the saturation of the power amplifier, or the backlash of a mechanical transmission, with a memory-less function, or with simple nonlinear models with memory, e.g. ideal relay with hysteresis. In modern control systems the major source of nonlinearity can be imputed to the actuator when this device uses a so-called "smart material", e.g. magnetostrictive materials, electroactive polymers, piezoelectric ceramics and shape memory alloys. The principal use of such materials is the development of actuators and sensors for a number of different applications in, e.g. aeronautics both for fixed-wing and rotary-wing, naval and aerospace engineering, MEMS and nanoscale manufacturing, micropositioning systems, medical sensors. In this paper, the attention is focused on the problem of foreseeing limit cycles in control systems employing actuators based on Terfenol-D magnetostrictive materials (see (**?**)), which exhibit a strong hysteretic behaviour due to their magnetic nature caused by the loss phenomena taking place inside the active material. The problem of modelling such a behaviour can be carried out either on a physical ground, describing processes on a mesoscopic scale (**?**) or, from the phenomenological viewpoint, by defining mathematical operators with memory able to describe input/output relationships of systems with hysteresis (**?**).

The existence problem of periodic solutions in a closed-loop system containing a hysteretic component has been tackled by resorting to different mathematical approaches, i.e. the ideal relay with hysteresis (**?**), the Poincaré maps, the harmonic balance (**?**), and, more in general frequency methods (**?**). The relay hysteresis model is simple and describes the main characteristic of the oscillation phenomenon due to hysteresis. Also, more general relay models are adopted to study periodic oscillations in (**?**), where topological methods are applied. However, those models cannot accurately reconstruct real hysteretic behaviours, due to discontinuous outputs or local memory mechanism.

In order to overcome these limitations, in the present paper, the Preisach operator (**?**) is adopted to model the hysteresis, and the existence of a limit cycle is theoretically discussed by resorting to a suitable modification of the describing function method (**?**; **?**). Specifically, owing to the Lipschitz continuity of the Preisach operator, sufficient conditions for the existence and nonexistence of periodic solutions, based on fixed-point methods, are derived. Moreover, a constructive algorithm to compute the periodic solution is presented, but without resorting to the describing function tool, as proposed in a former paper (?). In fact, a case study is presented where the describing function method revisited in (?) and further extended in (?) cannot be applied at all. Then, a second case study presents an experiment with a commercial magnetostrictive actuator used in a feedback system where a limit cycle occurs.

2. PREISACH HYSTERESIS MODELS

The Preisach operator formalizes a classical model describing magnetization of materials at a macroscopic scale (?). After some decades it has been studied by a pure mathematical viewpoint in (?) where it has been put into a sort of spectral decomposition of operators. The model has been widely spread to researchers in (?) (see references therein for a comprehensive history of that model), and, according to the formalism in the above reference, the model is defined as

$$y = \Gamma x \stackrel{\triangle}{=} \iint_{\alpha\geq\beta} \mu(\alpha,\beta)\,\hat{\gamma}_{\alpha\beta}\,x\,\mathrm{d}\alpha\mathrm{d}\beta \qquad (1)$$

where x and y are the input and the output, respectively, of the hysteresis. The operator can be seen as the linear superposition, weighted by the so-called distribution function $\mu(\alpha,\beta)$, of ideal relays $\hat{\gamma}_{\alpha\beta}$ having "up" switching value α, and "down" switching value β. A correspondence between every relay $\hat{\gamma}_{\alpha\beta}$, with $\alpha \geq \beta$, and the half plane above the line $\alpha = \beta$, referred to as the Preisach plane sketched in Fig. 1, can be defined, resulting into a useful geometrical interpretation. Assume that every relay is in the "down" state (the negative saturation state). If now the input is increased up to a value x_{M1}, all the relays having "up" switching state below such extremum, will switch in the "up" state ($\hat{\gamma}_{\alpha\beta} = 1$). Assuming now that the input decreases down to the value x_{m2}, a portion of the relays switched just before (those having lower switching threshold fulfilling the condition $\beta \geq x_{m2}$), attains the value at $\hat{\gamma}_{\alpha\beta} = -1$. If now the input is again increased up to a value $x_{M3} \leq x_{M1}$ a new step of a "staircase" shaped line, $\psi(t)$, in the Preisach plane is formed. Such function "separates" the relays switched in the "up" state ($\hat{\gamma}_{\alpha\beta} = 1$) from those in the "down" state ($\hat{\gamma}_{\alpha\beta} = -1$) and takes into account the input past history experienced by the operator. Such function can be referred to as the state, or memory, of the Preisach operator and is shown in Fig. 1.

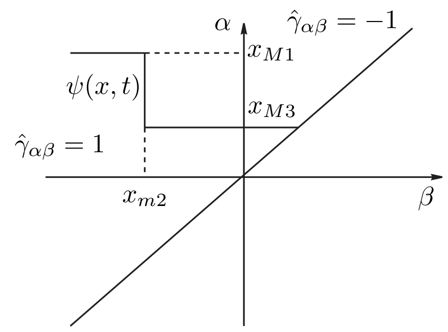

Fig. 1. Memory of the Preisach operator

In the following, only the properties of the Preisach operator explicitly exploited in the paper are recalled; for a comprehensive treatment of the argument see (?).

Rate-independence. A Preisach model is a rate-independent hysteresis operator, namely operator's output is not affected by the rate of variation of input. Formally:

$$\Gamma x(t) = \Gamma x(\tau),\ \tau = \alpha t,\ t \in [0,T] \qquad (2)$$

Lipschitz continuity. The Preisach operator defined in (1), is Lipschitz continuous if the Preisach distribution function $\mu(\alpha,\beta)$ is such that

$$L = 2\int_0^{+\infty} \sup_{s\in\Re} |\omega(r,s)|\,\mathrm{d}r < +\infty \qquad (3)$$

where $\omega(r,s) = \mu(\alpha,\beta)|_{\alpha=s+r,\beta=s-r}$, and L is the Lipschitz constant, i.e.

$$\|\Gamma x - \Gamma y\|_\infty \leq L\|x-y\|_\infty \qquad (4)$$

where $x, y \in C([0,T])$ and the adopted norm is

$$\|x\|_\infty = \sup_{t\in[0,T]} |x(t)| \qquad (5)$$

Moreover, the Preisach operator maps the Banach space $(C([0,T]), \|\cdot\|_\infty)$ into itself.

Saturation. If the Preisach distribution function is integrable in the region $\alpha \geq \beta$, defined the following constant

$$M = \frac{1}{2}\iint_{\alpha\geq\beta} |\mu(\alpha,\beta)|\,\mathrm{d}\alpha\mathrm{d}\beta \qquad (6)$$

the operator output saturates, i.e. is such that

$$\|\Gamma x\|_\infty \leq M\ , \quad \forall x \in C([0,T]) \qquad (7)$$

3. LIMIT CYCLES IN FEEDBACK SYSTEMS INCLUDING A PREISACH OPERATOR

The block scheme in Fig. 2 represents a typical feedback control system, where an actuator based on a smart material affected by hysteresis is adopted and its behaviour is modelled by a Preisach operator $\Gamma(\cdot)$. The controlled system and

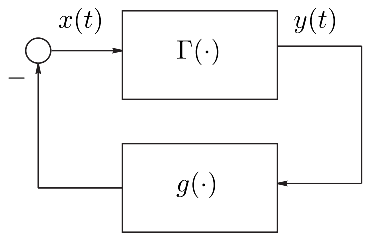

Fig. 2. Scheme of the feedback control system.

the control law are assumed linear and represented by a linear operator $g(\cdot)$, with a band-pass frequency response $G(j\omega)$ such that $|G(j\omega)| < +\infty$, for all $\omega \in \mathbb{R}^+$, i.e. no pure resonances are allowed.

A typical problem in such a control system is the rising of self-sustained oscillations, the so-called limit cycles. To prevent the occurrence of this phenomenon, it is necessary to find conditions under which these oscillations can exist. This problem can be formulated as the search of periodic solutions $x(t')$, with fundamental frequency ω and amplitude a_1. Following the basic ideas in (?), the time variable t' can be normalized as $t = \omega t'$, so that the solution $x(t)$ is $2\pi-$periodic. The normalization also affects the definition of the linear operator g, that in the normalized time will be denoted by g_ω, but does not affect Γ owing to its rate-independence. Therefore, the solution belongs to the space $C([0, 2\pi])$, which, equipped with the norm (5), is a Banach space, call it Π_∞.

In the following it will be useful the adoption of the norm

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} f(t)^2 \mathrm{d}t\right)^{1/2} \tag{8}$$

related to the norm in (5) through the following inequality

$$\|f\|_2 \leq \|f\|_\infty \ . \tag{9}$$

It is well-known that each $2\pi-$periodic signal can be described as the Fourier expansion

$$f(t) = \frac{f_0}{2} + \sum_{k=1}^{\infty} f_k \cos(kt + \phi_k) \tag{10}$$

where the Fourier coefficients are such that

$$f_k e^{j\phi_k} = \frac{1}{\pi}\int_0^{2\pi} f(t) e^{-jkt} \mathrm{d}t, \quad k = 0, 1, 2, \ldots \tag{11}$$

moreover, the Parseval equality holds

$$\|f\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty} f_k^2 + \frac{f_0^2}{4} \tag{12}$$

The space Π_∞ can be decomposed into the direct sum of two subspaces, namely $\Pi_\infty = \Pi_\infty^1 \oplus \hat{\Pi}_\infty$, where Π_∞^1 is the set of all the signals of the form $x_1(t) = a_1 \cos t$, while $\hat{\Pi}_\infty$ is its complement with respect to Π_∞. Let, moreover, $P_1 : \Pi_\infty \to \Pi_\infty^1$ be the continuous linear operator that projects signals in Π_∞ to the first harmonic, and $\hat{P} : \Pi_\infty \to \hat{\Pi}_\infty$ the projector to the remaining harmonics of the Fourier decomposition (including the dc component), then $\hat{P} = I - P_1$, being I the identity operator. To carry out the analysis, the solution is written as (?)

$$x(t) = P_1 x(t) + \hat{P}x(t) = x_1(t) + \hat{x}(t) \tag{13}$$

and, since the linear operator g_ω is a band-pass filter, $\hat{x}(t)$ can contain only higher harmonics, i.e. there is no dc component.

In order to be a solution of the nonlinear dynamic system in Fig. 2, $x(t)$ must satisfy the equation in the Banach space Π_∞

$$x = -g_\omega \Gamma x \triangleq Sx \ . \tag{14}$$

First of all, notice how the definition of the operator S depends on ω and a_1, next we show that the mapping S is contractive for all the frequencies belonging to the set

$$\Omega_1 = \left\{\omega \ : \ \rho'(\omega) < \frac{1}{\sqrt{2}L}\right\} \tag{15}$$

being

$$\rho'(\omega) = \left(\sum_{k=1}^{\infty} |G(jk\omega)|^2\right)^{1/2} \tag{16}$$

which is well defined owing to the band-pass nature of $G(j\omega)$, as shown in (?). In order to demonstrate that S is contractive, it must be shown that

$$\|Sz_1 - Sz_2\|_\infty \leq \lambda \|z_1 - z_2\|_\infty \ \ \forall z_1, \ z_2 \in \Pi_\infty, \ \lambda \in [0, 1) \tag{17}$$

To this aim, let

$$\Delta(t) = \Gamma(z_1)(t) - \Gamma(z_2)(t) = \frac{\gamma_0}{2} + \sum_{k=1}^{\infty} \gamma_k \cos(kt + \theta_k) \tag{18}$$

and consider the inequalities

$$\|g_\omega \Delta(t)\|_\infty = \left\|\sum_{k=1}^{\infty} |G(jk\omega)| \gamma_k \cos\bigl(kt + \theta_k + \angle G(jk\omega)\bigr)\right\|_\infty \leq \sqrt{2}\rho'(\omega)\|\Delta(t)\|_\infty \leq \sqrt{2}L\rho'(\omega)\|z_1 - z_2\|_\infty \tag{19}$$

where the Schwartz's inequality and the Parseval's equality (12) have been exploited, together with inequality (9) and Lipschitz continuity of the Preisach operator (4).

If $\rho'(\omega)$ is small enough, Eq. (14) is a contraction mapping and thus, since it admits $x(t) = 0$ as solution, the following proposition holds

Proposition 1. Equation (14) has the unique solution $x(t) = 0 \ \forall \ \omega \in \Omega_1$.

Once a set of frequencies for the limit cycle being sought has been excluded, then a set of admissible frequencies is searched for, by projecting Eq. (14) with the two operators P_1 and $\hat{P}$ defined above.

$$\hat{x} = -\hat{P} g_\omega \Gamma(x_1 + \hat{x}) \stackrel{\triangle}{=} T\hat{x} \quad (20)$$

$$x_1 = -P_1 g_\omega \Gamma(x_1 + \hat{x}) \stackrel{\triangle}{=} W x_1 \quad (21)$$

Again, note that the definition of the operators T and W depends on a_1 and ω, then Eq. (20) can be shown to be a contraction mapping in $\hat{\Pi}_\infty$. In fact, with the same steps followed form Eq. (18) to Eq. (19), the following inequality holds

$$\begin{aligned} &\|\hat{P} g_\omega \left(\Gamma(x_1 + z_1) - \Gamma(x_1 + z_2)\right)\|_\infty \\ &\leq \sqrt{2} L \rho(\omega) \|z_1 - z_2\|_\infty \end{aligned} \quad (22)$$

where the function

$$\rho(\omega) = \left(\sum_{k=2}^{\infty} |G(jk\omega)|^2 \right)^{1/2} \quad (23)$$

is well defined as already discussed for the function $\rho'(\omega)$ in Eq. (16). Once the following set has been defined

$$\Omega_2 = \left\{ \omega \ : \ \rho(\omega) < \frac{1}{\sqrt{2}L} \right\} \quad (24)$$

the mapping in (20) is contractive $\forall\, \omega \in \Omega_2$ and a unique fixed point $\hat{x}$ exists, for given ω and a_1. This means that a limit cycle can exist only with a fundamental frequency belonging to the set

$$\Omega = \Omega_2 - \Omega_1 \quad (25)$$

In order to establish the existence of the limit cycle, also a set containing the fundamental amplitude a_1 must be sought. Bounds on the possible amplitude a_1 as solutions of Eq. (21), can be obtained by searching for conditions that guarantee that Eq. (21) admits a fixed point in Π^1_∞. The problem of the existence of the solution is solved by the following

Theorem 2. At least a solution of Eq. (14) of the type (13) exists if $\omega \in \Omega$ and $a_1 < a_{\max}$, with

$$a_{\max} = \sqrt{2} M \sup_{\omega \in \Omega} |G(j\omega)| \quad (26)$$

PROOF. Define the closed convex subset of Π^1_∞

$$K = \left\{ x_1 \in \Pi^1_\infty : \|x_1\|_\infty \leq a_{\max} \right\} \quad (27)$$

and consider the inequalities

$$\begin{aligned} \|W x_1\|_\infty &= \|P_1 g_\omega \Gamma(x_1 + \hat{x})\|_\infty \\ &\leq \sqrt{2} M \sup_{\omega \in \Omega} |G(j\omega)| = a_{\max} \end{aligned} \quad (28)$$

which mean that W maps K into K. The idea is to apply Schauder's fixed point theorem to prove the existence of a sinusoidal signal x_1 solution of Eq. (21), hence compactness of the operator W has to be firstly shown. To this aim, its continuity must be proven. Fixed $\epsilon > 0$ and considering two signals x' and x'' belonging to K and such that $\|x' - x''\|_\infty < \delta_\epsilon$ with $\delta_\epsilon = \epsilon / (\sqrt{2} L \sup_{\omega \in \Omega} |G(j\omega)|)$, it results

$$\begin{aligned} \|Wx' - Wx''\|_\infty &= \|P_1 g_\omega \big(\Gamma(x' + \hat{x}) - \Gamma(x'' + \hat{x})\big)\|_\infty \\ &\leq \sqrt{2} L \sup_{\omega \in \Omega} |G(j\omega)| \delta_\epsilon = \epsilon \end{aligned} \quad (29)$$

so W is continuous. Now, fix $\epsilon > 0$ and consider two time instants t_a and t_b in $[0, 2\pi]$, then

$$\begin{aligned} &|Wx_1(t_a) - Wx_1(t_b)| = \\ &\eta |G(j\omega)| \left| \cos(t_b + \varphi + \angle G(j\omega)) - \cos(t_a + \varphi + \angle G(j\omega)) \right| \\ &\leq \sqrt{2} M \sup_{\omega \in \Omega} |G(j\omega)| \epsilon' = \epsilon \end{aligned} \quad (30)$$

where η and φ are amplitude and phase of the first harmonic of the signal $\Gamma(x_1 + \hat{x})$, and the continuity of the cosinusoidal function has been usefully exploited, i.e.

$$\begin{aligned} &\forall \epsilon' > 0 \quad \exists \delta > 0 : |t_b - t_a| < \delta \Rightarrow \\ &|\cos(t_b + \varphi + \angle G(j\omega)) - \cos(t_a + \varphi + \angle G(j\omega))| < \epsilon' \end{aligned}$$

with $\epsilon' = \epsilon / (\sqrt{2} M \sup_{\omega \in \Omega} |G(j\omega)|)$. By observing that Eq. (30) is valid for every $x_1 \in K$, equicontinuity of $W(K)$ is proved, and thus the Ascoli-Arzelà theorem (?) ensures compactness of $W : K \to K$. In conclusion, according to Schauder's theorem (?), Eq. (21) has at least one fixed point in K.

4. COMPUTING THE PERIODIC SOLUTION

In order to compute the periodic solution $x(t)$, the parameters a_1 and ω of the first harmonic have to be computed as well as the higher harmonics term $\hat{x}(t)$. In the previous section it has been shown that Eq. (20) admits a unique solution for each fixed couple (a_1, ω). Also, a solution of Eq. (21) can exist only for couples $(a_1, \omega) \in K \times \Omega$. As both members of Eq. (21) involve only sinusoidal signals, it is easy to see that it is equivalent to the phasor equation

$$\frac{1}{G(j\omega)} = N(a_1, \omega, \hat{x}) \quad (31)$$

where

$$N(a_1, \omega, \hat{x}) = \frac{\eta_1(a_1, \omega, \hat{x}) e^{j\varphi_1(a_1, \omega, \hat{x})}}{a_1} \quad (32)$$

being η_1, φ_1 amplitude and phase of the first harmonic of the signal $-\Gamma(x_1 + \hat{x})$. Therefore, the algorithm to compute the solution, could proceed as follows

1. fix a starting guess couple $(a_1, \omega) \in K \times \Omega$
2. compute $\hat{x}$ by solving Eq. (20) (e.g. by Picard's iteration method)

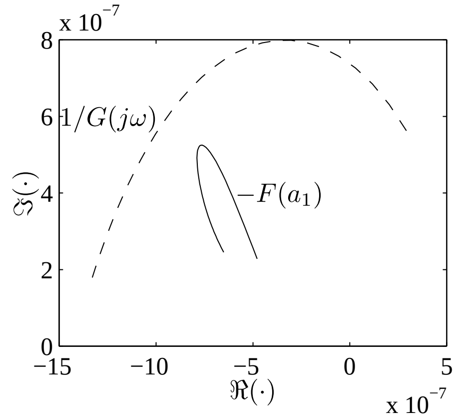

Fig. 3. First case study: diagram of the describing function

3. solve Eq. (31) by standard nonlinear optimization algorithm

Note that at each iteration of the optimization problem of the third step, for the new "guess-couple" (a_1, ω) the solution of Eq. (20) must be computed again. It is worth underlining that no computation of the describing function of the Preisach operator is necessary, as required, instead, by the algorithm proposed in (?). Furthermore, the proposed procedure works also in cases where the describing function method cannot be applied, as shown next.

5. CASE STUDIES

Two case studies will be presented. In the first one, a feedback system, of the type in Fig. 2, containing a magnetic transducer modelled as a Preisach operator is considered. The parameters of the Preisach model have been identified according to the procedure proposed in (?) based on the measurement of the first order reversals branches of the ferromagnetic material. The linear part of the feedback loop is modelled as the transfer function

$$G(s) = -4.8 \cdot 10^6 \frac{s^2 - 2s}{s^3 + 7s^2 + 6.5s + 3} \tag{33}$$

Before presenting the analysis with the proposed procedure, application of one of the methods based on the describing function is attempted. The describing function $F(a_1)$ of the Preisach operator is computed as described in (?), but the results of (?; ?), based on the degree theory, cannot be applied because the diagrams of $-F(a_1)$ and of the frequency response function $1/G(j\omega)$ do not intersect (Fig. 3).

To apply the method presented here, the set Ω is firstly computed. To this aim, the functions $\rho(\omega)$ and $\rho'(\omega)$ defined in Eq. (23) and Eq. (16) respectively, together with the Lipschitz constant in Eq. (3) are computed ($L \simeq 5 \cdot 10^{-7}$). From Fig. 4,

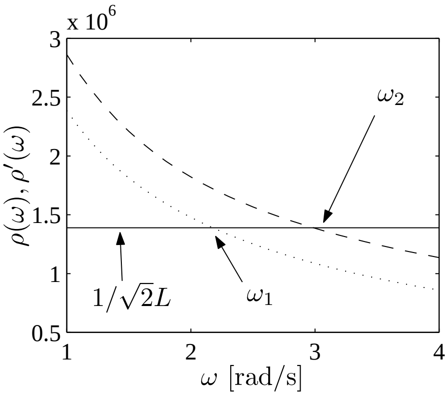

Fig. 4. First case study: functions $\rho(\omega)$ (dotted) and $\rho'(\omega)$ (dashed) for determination of the set Ω

it is easy to see that $\Omega = [\omega_1, \omega_2]$, where $\omega_1 = 2.17\,\text{rad/s}$ and $\omega_2 = 2.98\,\text{rad/s}$. To compute the fundamental frequency and amplitude of the periodic solution a starting guess is chosen for the optimization algorithm. The set K has to be selected by computing $a_{\max} = \sqrt{2}M \sup_{\omega\in\Omega} |G(j\omega)| \simeq 1542$. With the starting guess $a_1 = 1090$, $\omega = 2.34$, the algorithm computes a solution with $a_1 = 1097.8$, $\omega = 2.355$. A time period of the computed signals is reported in Fig. 5, where the computed solution is compared to the solution obtained by numerically simulating the feedback system. It is evident that the result obtained by applying the proposed algorithm is quite accurate.

The second case study concerns a micro-positioning system adopting a magnetostrictive actuator produced by Energen, Inc. and equipped with displacement and current sensors, and a current amplifier. The experimental setup comprises a dSPACE rapid prototyping system. For the implementation of the linear part of the control system, we have used a modular dSPACE system with a DS 1005 board, equipped with a Motorola PowerPC750 processor at 480 MHz, a DS 2001 ADC board with 16 bit channels and 5 µs sample time and a DS 2102 DAC board, with 16 bit channels and 2 µs settling time. The considered linear subsystem has transfer function

$$G(s) = \frac{1470s}{(s+8)(s+9)(s+10)(s^2+7s+49)}$$

while the hysteretic behaviour of the actuator is modelled through a Preisach operator whose Everett integral has been experimentally identified on the measured first order reversal curves and analytically expressed by means of a fuzzy universal approximator (?). The Lipschitz constant of this model is $L \simeq 54\,\mu\text{m}$, while $M = 29\,\mu\text{m}$. In Fig. 6 the typical butterfly-shaped hysteresis cycle of the actuator can be recognized, plotted as a dashed line. It is well-known that this particular shape causes the effect of frequency doubling typical of magnetostrictive actuators, and to avoid

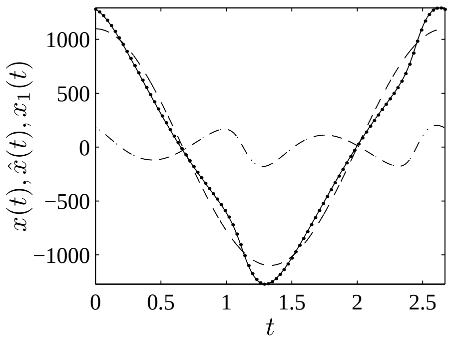

Fig. 5. First case study: $x(t)$ (solid: computed, dotted simulated) – $\hat{x}(t)$ (dash-dot) – $x_1(t)$ (dashed)

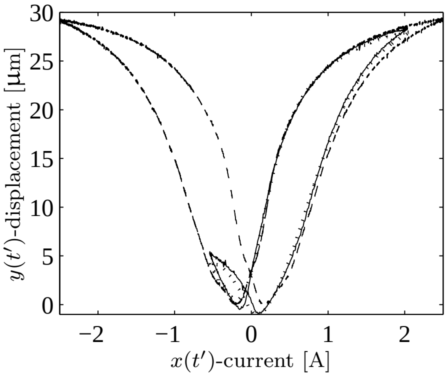

Fig. 6. Second case study: measured (-) and computed (:) limit cycle, and major hysteresis loop (–).

this, a dc component of 0.75 A has been added to the actuator input current. Note that this dc component does not play any special role owing to the band-pass nature of the linear filter and to the vertical congruency of the Preisach operator (see (**?**)). The same figure also reports the limit cycle exhibited by the closed-loop system, which is a minor hysteresis loop of the actuator. This periodic solution can be foreseen and computed by the method described so far by repeating the same steps followed in the first case study. The determination of the sets Ω_1 and Ω_2 leads to the limit frequencies $\omega_1 = 8.67$ rad/s and $\omega_2 = 17.11$ rad/s; therefore, $\Omega = [\omega_1, \omega_2]$. By applying the algorithm of Section 4, the computed periodic solution has fundamental frequency 9.219 rad/s and amplitude 1.289 A, in very good accordance with the experimentally measured limit cycle.

6. CONCLUSIONS

The paper presents a study of feedback control systems employing a component affected by hysteresis with non-local memory. The problem of the existence of limit cycles is tackled by resorting to Preisach hysteresis operators and to classical functional analysis tools. An algorithm to foresee and compute the parameters of the limit cycle is presented and tested in two case studies, both in simulation and experimentally.