RELAY CONTROL WITH PARALLEL COMPENSATOR FOR NONMINIMUM PHASE PLANTS

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Abstract: Following the Smith compensator the parallel compensator designed for difficult, e.g. nonminimum phase plants is applied to systems with relay control. The compensator connected in parallel to the plant changes its properties so that the replacement plant model becomes simpler and may be shaped dependently upon the goal of the control. In the case of regulation on a constant level a first order lag may be chosen for the replacement plant model. In the case of tracking or disturbance rejection of signals with frequencies belonging to some working frequency band, the replacement plant model should have its frequency response close to that of the plant, in the working frequency band. The proposed approach simplifies the design and improves accuracy of the control.

Keywords: Relay control, compensators, regulation, tracking, stability.

1. INTRODUCTION

It is known that relay control such as on–off control, or sliding mode control is very robust i.e it has insensitive steady state error to relatively large plant parameter changes. However this observation concerns only minimum phase plants for which initial slope of the step response is positive.

The robustness property of the relay control may be explained verbally as follows. In continuous control with proportional regulator the gain of the regulator influences both the accuracy in steady state and stability. Higher gain would improve accuracy if the system would be stable. However higher gain usually causes instability. Therefore a trade off must be applied in choosing the gain. This creates a constraint for accuracy.

In contrary to that in relay control we resign from the demand for stability. The high frequency oscillations are generated which are filtered by the dynamics of the plant. The on–off relay works on the vertical part of its characteristic which is related with very high (close to infinite) gain. Therefore the filtered by the plant "steady state" is very accurate independently of some plant parameter changes.

It is also known that for nonminimum phase plant with negative initial slope of the step response it is impossible to implement a relay control assuring appropriate accuracy in steady state. This is related with high amplitude and not high frequency of oscillations appearing then in the system. Therefore in the case of the nonminimum phase plants a special approach is needed.

For the plants with pure time delay Smith (1958) proposed a compensator with effectively takes the delay outside the loop and allows a feedback design based on the plant dynamics without delay. The result is that the system designed in this manner is faster and assures higher accuracy. Now this compensator is commonly called Smith compensator (Franklin *et al.*, 1994) (or predictor

(Goodwin *et al.*, 2001)) and may be also applied to the systems with relay control.

In the present paper, following the idea of the Smith compensator a parallel compensator is proposed, which may be applied to nonminimum phase plants. Using this approach, the relay regulator may be designed for the replacement plant with appropriately chosen minimum phase model. Similarly as for the Smith compensator the assumption is that the plant is stable (in the case of Smith compensator this is not exactly formulated in literature). This approach may be also applied for the system with usual continuous P/PI/PID regulator. The preliminary idea applied to the sliding mode control with decreased chattering effect was presented in (Gessing, 2002).

The contribution of the paper is in proposing to the systems with relay and nonminimum phase plants the parallel compensator which improves the accuracy of control and in showing that the compensator may be applied both, for the case of regulation and tracking.

2. PARALLEL COMPENSATOR

Consider the linear plant described by the transfer function (TF)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{L(s)}{M(s)} \tag{1}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the plant output and input, respectively, while $L(s)$ and $M(s)$ are polynomials of m-th and n-th order, respectively and $m < n$. Assume that the plant is stable, that is its poles $p_i,\ i = 1, 2, ..., n$ have negative real parts i.e. $Rep_i < 0$.

In the case of difficult plant (e.g nonminimum phase, or with higher order dynamics), when it is difficult to design the relay regulator assuring an appropriate accuracy, a parallel compensator may be applied. The closed loop system with relay and parallel compensator, as well as the characteristic of the relay is shown in Fig. 1. The idea of parallel compensator, described by the TF

$$G_c(s) = \frac{Y_c(s)}{U(s)} = G_1(s) - G(s) \tag{2}$$

is similar to that of the Smith compensator. Here $Y_c(s)$ is the Laplace transform of the output y_c of the compensator, while $G_1(s)$ is the TF which should be appropriately chosen.

Note that in the proposed structure shown in Fig. 1 the TF $G_r(s)$ of the replacement plant is described by

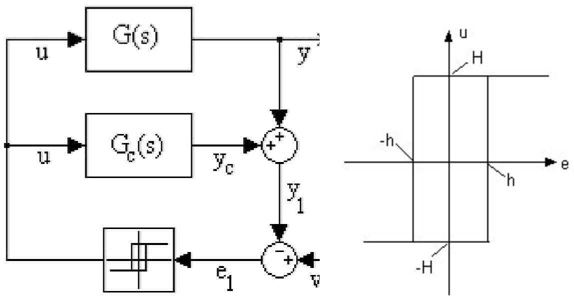

Fig. 1. CL system with parallel compensator and relay implementation; characteristic of the relay.

$$G_r(s) = \frac{Y_1(s)}{U(s)} = G(s) + G_c(s) =$$
$$= G(s) + G_1(s) - G(s) = G_1(s) \tag{3}$$

Thus the replacement plant is described by the TF $G_1(s)$ and the relay regulator should be designed for the replacement plant. Therefore the crucial point in the proposed method is the choice of the TF $G_1(s)$.

We will distinguish two cases dependently upon the goal of the control.

I. Regulation on a constant level in steady state under stepwise excitations;
II. Tracking and disturbance rejection with some accuracy for frequencies belonging to a working frequency band $[0, \omega_{mx}]$.

3. REGULATION ON A CONSTANT LEVEL

In this case we are mainly interested in the accuracy of the constant steady state, appearing after some time from occurrence of stepwise excitation (set point or disturbance). Since in the case of nonminimum phase plants we have a limited possibility of shaping transient response, which is dependent upon the placement of zeros and poles of the plant we do not formulate some special demands concerning transient response, though it should be acceptable. In this case the model $G_1(s)$ may be chosen in the form of a first order lag i.e

$$G_1(s) = \frac{k_0}{Ts+1}, \qquad k_0 = G(0), \tag{4}$$

so that $$G_1(0) = G(0) \tag{5}$$

Since the gain of constant signals ($\omega = 0$) is the same for both the models $G_1(s)$ and $G(s)$, then in steady state for constant signals, as results from (2) and (5), it is

$$y_c = 0 \quad \text{and} \quad e_1 = w - y = e \tag{6}$$

where e is the error signal.

Thus we have obtained the CL system composed of the simplified replacement plant with first order lag model (4) and the relay with the parameters H and h. For sufficiently small hysteresis h the frequency of the relay switchings is so high that it is filtered by the dynamics of the plant $G(s)$ and the oscillations are not seen in the output y.

Note that the relation (6) is valid even then when the plant parameters of $G(s)$ are changed. For $n - m \geq 2$ and small h it is easy to note that the change of the plant $G(s)$ parameters giving plant TF $G^*(s)$ does not change the initial slope of the step response of the replacement plant $G_1^*(s) = G_1(s) - G(s) + G^*(s)$ essentially, because initial slopes of $G(s)$ and $G^*(s)$ are zero (as $n - m \geq 2$). Thus the frequency of switchings is changed insignificantly which means that the system may be insensitive to the plant $G(s)$ parameter changes.

3.1 *Example 1*

Consider the nonminimum phase plant described by the following TF

$$G(s) = k_p \frac{-3s + 5}{s^3 + 2s^2 + 3.5s + 2.5}, \quad k_p = 1 \quad (7)$$

Assume that the goal of the control is regulation of the plant output on a constant value determined by the set point w. A stepwise change of the set point (or output disturbance) may occur. The relay control with parallel compensator is used, as shown on Fig. 1.

Assume $G_1(s)$ in the form (4) with $T = 0.1$ and $k_0 = G(0) = 2$.

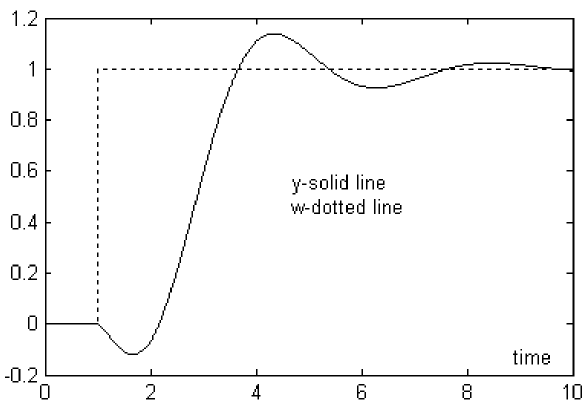

Fig. 2. Plots of y and w for Example 1.

In Fig. 2 the step response y of the CL system, with relay parameters $h = 0.1$, $H = 2$ and parallel compensator (4), to the set point $w = \mathbf{1}(t-1)$ is shown (here $\mathbf{1}(t) = 1$ for $t \geq 0$ and $\mathbf{1}(t) = 0$ for $t < 0$). It was obtained from simulations performed in SIMULINK. Some changes of the plant parameters do not influence the step response significantly. For instance, increase of the plant gain to $k_p = 1.5$ (without change of the parallel compensator) gives stable response with overshot ~ 1.66 undershot ~ -0.19 and higher decaying oscillations. Decrease the gain to $k_p = 0.5$ gives stable, aperiodic response.

4. TRACKING AND DISTURBANCE REJECTION

In this case we are mainly interested in the accuracy during tracking or disturbance rejection of varying signals with frequencies belonging to some working frequency band $[0, \omega_{mx}]$. Similarly as in the case of regulation systems, in the case of nonminimum phase systems we have limited possibility of shaping transient response, which however should be acceptable.

Choosing the model $G_1(s)$ we should take into account the fact that in the proposed system with the parallel compensator, the replacement plant has the model $G_1(s)$ and to this plant the relay should be designed. Therefore the model $G_1(s)$ should be minimum phase and it is recommended that the relative degree of the rational TF $G_1(s)$ is equal to one, since for this kind of the replacement plant the initial slope of the step response is nonzero and positive which gives high frequency switchings of the relay if h is small. The high frequency oscillations are filtered by the dynamics of the plant (1) so that in the output signal they are not noticeable.

Additional demand is that the frequency responses of $G(j\omega)$ and $G_1(j\omega)$ in the working frequency band $[0, \omega_{mx}]$ should be close one to other or

$$G_1(j\omega) \approx G(j\omega) \quad \text{for} \quad \omega \in [0, \omega_{\text{mx}}] \qquad (8)$$

The demand (8) is justified for linear systems. The justification of this demand for the systems with relay is based on the fact that in the case when the fast frequency oscillations resulting from the relay switchings are generated, there appear linearization and the system with relay works approximately as a linear system (Slotine and Li, 1991).

Thus the process of design contains the following steps

(1) choose the rational TF $G_1(s)$ with relative degree equal to one;
(2) find the values of the coefficients of polynomials in the numerator and denominator of $G_1(s)$ so that the approximation (8) is fulfilled in some interval $[0, \omega_{mx}]$;
(3) design the relay for which the CL system has negligible oscillations in the plant output sig-

nal. The choice of the parameters H and h of the relay may be performed experimentally.

4.1 *Algorithm for coefficients finding*

Denote by $\bar{G}(s) = G(s)/G(0)$ the normalized model of the plant with the gain $\bar{G}(0) = 1$.

Assume the normalized model $\bar{G}_1(s)$ in the form

$$\bar{G}_1(s) = \frac{b_1 s^{p-1} + b_2 s^{p-R} + ... + b_{p-1}s + 1}{a_1 s^p + a_2 s^{p-1} + ... + a_p s + 1} \quad (9)$$

We wont to find the coefficients $a_i, i = 1, 2, ..., p, b_j$, $j = 1, ..., p-1, p \leq n$, for which the frequency response $\bar{G}_1(j\omega)$ in some interval $[0, \omega_{mx}]$ approximates $\bar{G}(j\omega)$, and the model $\bar{G}_1(s)$ is stable and minimum phase. We have

$$\bar{G}_1(j\omega) = Re\bar{G}_1(j\omega) + jIm\bar{G}_1(j\omega) \quad (10)$$
$$\bar{G}(j\omega) = Re\bar{G}(j\omega) + jIm\bar{G}(j\omega)$$

where Re and Im are the real and imaginary parts of appropriate frequency response. Denote by

$$d(\omega) = ||\bar{G}_1(j\omega) - \bar{G}(j\omega)|| = \quad (11)$$
$$\sqrt{[Re\bar{G}_1(j\omega) - Re\bar{G}(j\omega)]^2 + [Im\bar{G}_1(j\omega) - Im\bar{G}(j\omega)]^2}$$

the distance between appropriate points of frequency responses $\bar{G}_1(j\omega)$ and $\bar{G}(j\omega)$. Of course it should be

$$d(\omega) \leq \Delta \quad \text{for} \quad \omega \in [0, \omega_{mx}] \quad (12)$$

where Δ is a given small positive number determining the accuracy of the approximation (e.g. $\Delta = 0.01$ or 0.05), and $[0, \omega_{mx}]$ determines the working frquency band in which the characteristics $\bar{G}_1(j\omega)$ and $\bar{G}(j\omega)$, for $\omega \in [0, \omega_{mx}]$ are close one to other.

Let N denotes an assumed number of points ω_i equally distributed in the interval $[0, \omega_{mx}]$ (eg. $N = 10$). Then

$$\omega_i = \frac{i}{N}\omega_{mx} \qquad i = 1, 2, ..., N \quad (13)$$

Denote by Ω the set of admissible values of the coefficients $a_i, \ \ i = 1, ..., p, \ \ b_j, \ \ j = 1, ..., p-1$, for which the polynomials of the numerator and denominator of the transfer function (9) are stable (i.e. their zeros have negative real parts). The set Ω may be determined using for instance the Hurwitz stability criterion for the coefficients of the numerator and denominator polynomials of the transfer function (9).

The distance between the characteristics $\bar{G}_1(j\omega)$ and $\bar{G}(j\omega)$ in the interval $[0, \omega_{mx}]$ may be determined from the dependence

$$d = \max_i d(\omega_i), \quad i \in \{1, 2, ..., N\}. \quad (14)$$

To find the values of the polynomial coefficients the following algorithm may be used

Algorithm

(1) choose ω_{mx}, N and Δ and determine ω_i from (13);
(2) find the coefficients $a_i, \ i = 1, 2, ..., p \ \ b_j = j = 1, 2, ..., p-1$ from minimizing the expression $d_{min} = \min_\Omega d$;
(3) if $d_{min} < \Delta$ end;
(4) if $d_{min} > \Delta$ decrease ω_{mx} and repeat the points 1 and 2 of the *Algorithm.*

As the result of applying the algorithm we obtain the coefficients $a_i, b_j, \ i = 1, 2, ..., p, \ j = 1, 2, ...p-1$ and d_{min} and ω_{mx} for the assumed transfer function $\bar{G}_1(s)$ and numbers N, Δ.

The sought transfer function $G_1(s)$ results from the dependence

$$G_1(s) = G(0)\bar{G}_1(s) \quad (15)$$

We may have difficulty with solving the minimization problem mentioned in point 2 of the *Algorithm* since the distance d as a function of the coefficients a_i, b_j usually has many local solutions. Therefore, it is reasonable to apply a random walk within some appropriately restricted set Ω.

4.2 *Example 2*

Consider the plant described by the TF (7). We would like to design CL system with relay and parallel compensator, which for some frequency band $[0, \omega_{mx}]$ tracks varying set point and/or rejects varying output disturbance.

We decide to use the second order model $\bar{G}_1(s)$ in the form

$$\bar{G}_1(s) = \frac{b_1 s + 1}{a_0 s^2 + a_1 s + 1} \quad (16)$$

Since $G(0) = 2$ then $\bar{G}(s = G(s)/2)$.

To find the coefficients of the TF (16) we have applied the described *Algorithm*, assuming $N = 10, \Delta = 0.3$ and $\omega_{mx} = 0.4 rad/sec$. To find the minimum of the function d we have sought in the set of coefficients restricted to the form $\Omega = \{0.1 < b_1 < 7, \ 0.1 < a_i < 7, \ i = 1, 2\}$, generating in the mentioned intervals random values $b_j, \ a_i$ equally distributed and next decreasing the intervals in the vicinity of the preliminary found values of the coefficients.

Using this approach we have found the following solution: $b_1 = 0.1016$, $a_1 = 2.038$, $a_2 = 1.898$. Accounting (15) and the value $G(0) = 2$ we obtain

$$G_1(s) = \frac{0.2032s + 2}{2.038s^2 + 1.898s + 1} \tag{17}$$

The Nyquist plots of the characteristics $G(jw)$ and $G_1(s)$ determined by (7) and (17), respectively are shown on Fig. 3, where $\omega_{mx} = 0.4\ rad/sec$.

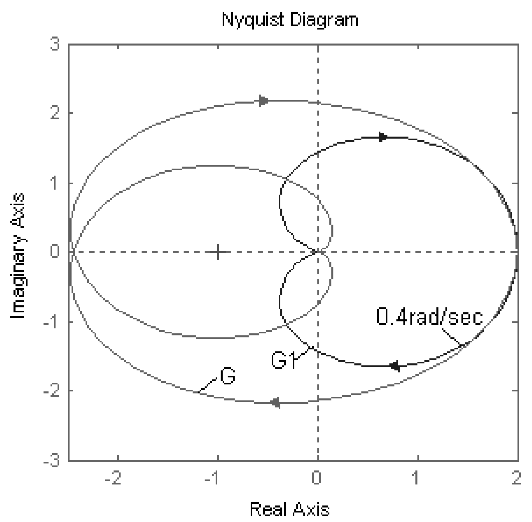

Fig. 3. Nyquist plots of $G(j\omega)$ and $G_1(j\omega)$ for Example 2.

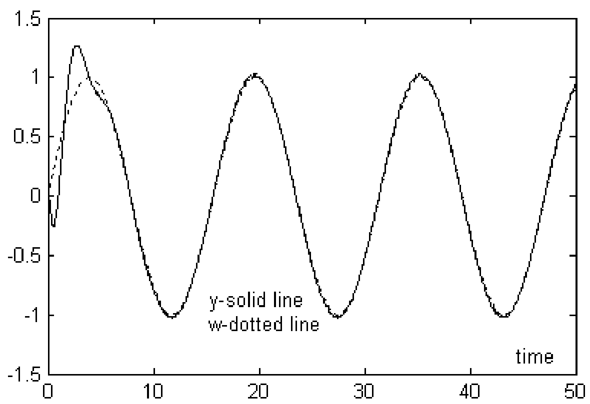

Fig. 4. Plots of y and w for Example 2.

In Fig. 4 the time response y of the CL system with relay parameters $h = 0.01$, $H = 2$ and parallel compensator (determined by (7) and (17)), to the set point $w = \sin(0.4t)\mathbf{1}(t)$, is shown. It is seen that beyond the initial, transient period the plots of y and w are almost the same. The hysteresis parameter h is now 10–times smaller than that in Example 1 to avoid switching oscillations in the output y. This is caused by the fact that the initial slope of the step response of $G_1(s)$ determined by $0.2032/2.038 = 0.0997$, which for given h decides about frequency of switching, is now $2/0.0997 = 20.0602$ – times smaller than that of $G_1(s)$ for Example 1. The system now is more sensitive to plant parameter changes than in Example 1. For instance the system is stable (neglecting switching oscillations not notable in the output y) for k_p from 0.75 to 1.24 without change of the compensator parameters.

5. APPROXIMATE DESCRIPTION OF THE SYSTEM

Let us notice that the block diagram of the CL system shown in Fig. 1 may be transformed to the form shown in Fig. 5.

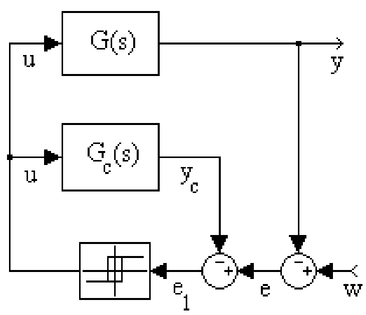

Fig. 5. The transformed block diagram of the system from Fig. 1

Assume that the hysteresis h of the relay is small and high frequency oscillations generated by the fast switchings of the relay are filtered by the dynamics of the plant $G(s)$, as well as by $G_1(s)$ and by $G_c(s)$. Let $\bar{y}(t)$ and $\bar{y}_c(t)$ be the outputs of the plant $G(s)$ and parallel compensator $G_c(s)$, respectively, in which the high frequency oscillations are neglected. Since the amplitudes of these oscillations are small then it is

$$\bar{y}(t) \approx y(t), \qquad \bar{y}_c(t) \approx y_c(t) \tag{18}$$

During fast switchings the relay works on vertical segment of its characteristic, therefore in approximate description we may treat the relay as the linear static element with very high gain $k \to \infty$.

Let $\bar{u}(t)$ is the averaged control signal containing slowly varying component such that $\bar{Y}(s) = G(s)\bar{U}(s)$, where $\bar{Y}(s) = \mathcal{L}[\bar{y}(t)]$, $\bar{U}(s) = \mathcal{L}[\bar{u}(t)]$ and $\mathcal{L}$ denotes Laplace transform. Let $\bar{Y}_c(s) = \mathcal{L}[\bar{y}_c(t)]$ and $\bar{E}(s) = W(s) - \bar{Y}(s)$, $W(s) = \mathcal{L}[w(t)]$. Then the variables $\bar{U}(s)$ and $\bar{E}(s)$ are related with the following TF

$$\frac{\bar{U}(s)}{\bar{E}(s)} = \frac{k}{1 + kG_c(s)} \approx \frac{1}{G_c(s)} \tag{19}$$

as k is high. Therefore we have

$$\frac{\bar{Y}(s)}{W(s)} \approx \frac{G(s)/G_c(s)}{1+G(s)/G_c(s)} = \\ = \frac{G(s)}{G_c(s)+G(s)} = \frac{G(s)}{G_1(s)} \quad (20)$$

Thus the CL system with relay is described approximately by the linear model with TF equal to the ratio of $G(s)$ to $G_1(s)$. The formula (20) may be also used for choosing appropriate replacement plant $G_1(s)$.

5.1 *A particular case*

Denote by
$$G_1(s) = \frac{L_1(s)}{M_1(s)} \quad (21)$$
a stable replace replacement plant with minimum phase zeros. Thus the polynomials $M_1(s)$ nad $L_1(s)$ are Hurwitz polynomials. In the considered particular case we may choose $M_1(s) = M(s)$ and $L_1(s)$ as an appropriate Hurwitz polynomial of $(n-1)$-th order, such that the CL system composed of the plant $G_1(s)$ and the proportional regulator with high gain k is stable. Then from (21) we obtain

$$\frac{\bar{Y}(s)}{W(s)} = \frac{L(s)}{L_1(s)} \quad (22)$$

Thus the numerator of TF (22) contains the polynomial appearing in the numerator of (1), while the denominator of (22) contains the polynomial appearing in the numerator of (21). From these considerations it results that in the considered case the choice of $L_1(s)$ influences essentially the dynamics of the researched CL system with relay. Really its characteristic equation takes the form

$$L_1(s) = 0 \quad (23)$$

These observation may help in choosing the polynomial $L_1(s)$ basing on linear theory (Goodwin *et al.*, 2001).

6. CONCLUSIONS

In the present paper, following the Smith compensator (Smith, 1958) we apply a similar compensator to relay systems with nonminimum phase plants. The compensator, connected in parallel to the plant, changes its model which becomes minimum phase. For the changed replacement plant model it is easy to design relay parameters which assures appropriate accuracy. The kind of the replacement plant model depends upon our choice and the goal of the control.

If the main goal of the control is the accuracy of regulation in constant steady state, then the replacement plant model may take the form of a first order lag with the gain equal to that of the plant. The time constant of this model has also a limited influence on under– and over–shot of the step response.

If the main goal of the control is tracking or disturbance rejection of signals with frequencies belonging to some working frequency band, then the replacement plant model in the form of rational transfer function with relative order equal to one, should be chosen in this manner that it is minimum phase and in the working frequency band its frequency response is approximately the same as that of the plant.

Especially in the case of regulation the proposed system structure is robust since the frequency response of the replacement plant model lies in the first negative quadrant of the Nyquist plane (first order lag). In the case of tracking or disturbance rejection the demand of closing the frequency response of the replacement plant to that of the plant causes some decrease of robustness, since the frequency response of the replacement plant may lay now in the first and second negative quadrants of Nyquist plane (closer to the critical point $(-1, j0)$). This consideration is based on the fact that the relay system, during fast oscillations of the relay may be treated as linear system in which the relay is replaced with a static linear element with high gain k.

It seems that the described idea of parallel compensator may be also used for other difficult plants improving accuracy at least in steady state and also robustness of the control.

ACKNOWLEDGMENT

The paper was realized in the first half of 2004 and was partially supported by the Science Research Committee (KBN), grant no. 4 T11A 012 23.

7. REFERENCES

Franklin G. F., J. D. Powell and A. Emami Naeini. (1994). *Feedback Control of Dynamic Systems*. Addison Wesley, N.Y.

Gessing R. (2002). Sliding Mode Control With Decreased Chattering for Nonminimum Phase Plants. *Proceedings of 11-th Mediterranean Conference on Control and Automation MED2002*, June 17–20, Rodoce, Greece, CD–Rom.

Goodwin G. C., S. F. Graebe and M. E. Salgado. (2001). *Control Systems Design*. Prentice Hall, N. J.

Slotine J.J.E. and W. Li. (1991). *Applied Nonlinear Control.* Prentice Hall Int., Inc.

Smith O. (1958). *Feedback Control Systems.* Mc Graw–Hill N.Y.