A COMPARISON AMONG PERFORMANCE MEASURES IN PORTFOLIO THEORY

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Abstract: This paper examines some performance measures to be considered as an alternative of the Sharpe Ratio. More specifically, we analyze allocation problems taking into consideration portfolio selection models based on different performance ratios. For each allocation problem, we compare the maximum expected utility observing all the portfolio selection approaches proposed here. We also discuss an ex-post multi-period portfolio selection analysis in order to describe and compare the sample path of the final wealth processes.

Keywords: Performance ratios, heavy tails, portfolio choice, risk measure, expected utility, efficient frontier.

1. INTRODUCTION[1]

More than thirty five years ago Sharpe (1966) introduced the so called Sharpe Ratio, a performance measure for mutual funds that is justified by the classic Markowitz mean-variance analysis. Leaving behind the assumption of normality in return distributions, the classic Sharpe performance measure has become a questionable tool for ranking portfolio choices. As a matter of fact, the many shortcomings and ambiguous results of the empirical and theoretical mean-variance analysis represent the main reason and justification for the creation of alternative performance measures, such as those proposed in the last decade (see Goetzmann, Spiegel, Ingersoll, Welch (2003), Farinelli, Tibiletti (2003), Dowd (2001), Sortino (2000), Pedersen and Satchell (2002)).

In the spirit of these recent researches, we want to consider more general risk-reward ratios best suited to compare skewed and heavy tailed return distributions with respect to a benchmark. In the spirit of these recent researches, we want to consider more general risk-reward ratios best suited to compare skewed and heavy tailed return distributions with respect to a benchmark. In view of this consideration we introduce and discuss several performance measures. In particular, we compare the classic Sharpe ratio with other ratios proposed in literature: minimax ratio (Young (1998)), Stable ratio (Ortobelli, Rachev Schwartz (2003)), MAD ratio (Konno and Yamazaki (1991)), Farinelli-Tibiletti ratio (Farinelli,Tibiletti (2003)) Sortino-Satchell ratio (Sortino (2000), Pedersen, Satchell (2002)), Colog-type ratios (Giacometti, Ortobelli (2001)), VaR and CVaR ratios (Favre and Galeano (2002) and Martin, Rachev, Siboulet (2003)) Rachev-type ratios (see

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Biglova, Ortobelli, Rachev, Stoyanov (2004) and Ortobelli, Rachev, Biglova, Stoyanov and Fabozzi (2004)).

First, we propose an ex-ante static comparison among portfolio selection models based on different performance measures. In particular, we compare the expected utility of investors when the market portfolio is computed by maximizing a given performance measure. We analyze two allocation problems for investors with different risk aversion coefficients. We determine the efficient frontiers generated by linear combinations of the market portfolio and the riskless asset. Each investor, characterized by his/her utility function, will prefer the portfolio which maximizes his/her expected utility on the efficient frontier. Hence the portfolios obtained with this methodology represent the optimal investors' choices in each distinct case. Second, we propose an ex-post dynamic analysis as another approach to performance comparison. Here we compare the final wealth of investors who maximize the expected logarithmic utility function under several portfolio selection models.

Section 2 introduces the performance ratios. Section 3 proposes a comparison among the different models. In the last section, we briefly summarize the results.

2. PERFORMANCE RATIOS

This section describes the different performance ratios examined in portfolio selection problems. Particularly, we analyze the problem of optimal allocation among n+1 assets: n of those assets are risky with returns (continuously compounded) $z=[z_1,...,z_n]'$, and the (n+1)th asset is risk-free with return z_0. No short selling is allowed, i.e., the portfolio risky weights $x_i \in [0,1]$ for every i=1,...,n and the riskless weight $\lambda = 1-\sum_{i=1}^{n} x_i$ are equal to or greater than zero. Assume that all portfolios are uniquely determined by the mean and by a risk measure consistent with some stochastic dominance order. The investor will choose an optimal portfolio which is the linear combination of the riskless asset and an optimal risky portfolio. The optimal risky portfolio is given by the portfolio that maximizes the performance ratio. Thus, for any performance measure ρ we compute a "market portfolio" $x_M' z$ that is the solution of the following optimization problem

$$\max_x \rho(x'z)$$
$$\sum_{i=1}^{n} x_i = 1 \qquad (1)$$
$$x_i \geq 0$$

For different performance measures we obtain different optimal portfolios. Therefore, the market portfolio composition $x_M' = [x_{M,1},....,x_{M,n}]'$ found for each performance measure ρ is based on a diverse risk perception. In particular, we consider the following performance measures

a) **Sharpe ratio** (see Sharpe (1966), (1994))

$$\rho(x'z) = \frac{E(x'z)-z_0}{STD(x'z)},$$

where STD(x′z) is the standard deviation of the portfolio x′z.

b) **Minimax ratio** (see Young (1998))

$$\rho(x'z) = \frac{E(x'z)-z_0}{MM(x'z)}$$

where $MM(x'z) = z_0 - \min_t x'z_t$ and z_t is the vector of returns at time t.

c) **Stable ratio** (see Ortobelli, Rachev, Schwartz (2003))

$$\rho(x'z) = \frac{E(x'z)-z_0}{\sigma_{x'z}},$$

where $\sigma_{x'z} = \sqrt{x'Qx}$, $Q=[\sigma_{i,j}^2]$ is the dispersion matrix of the vector z that we assume to be α-stable sub-Gaussian distributed. The elements of Q are defined for every p∈[1,α)

$$\sigma_{i,j}^2 = \left(A(\alpha,p)\right)^{2/p} f(p,\tilde{z}_i,\tilde{z}_j)$$

where $f(p,\tilde{z}_i,\tilde{z}_j) = E\left(\tilde{z}_i \tilde{z}_j^{<p-1>}\right)\left(E\left(\left|\tilde{z}_j\right|^p\right)\right)^{\frac{2-p}{p}}$;

$$A(\alpha,p) = \frac{\sqrt{\pi}\,\Gamma\left(1-\frac{p}{2}\right)}{2^p\,\Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{1+p}{2}\right)};\quad \tilde{z}_i = z_i - \mu_i;$$

$\tilde{z}_j^{<p-1>} = \mathrm{sgn}(\tilde{z}_j)\left|\tilde{z}_j\right|^{p-1}$; the optimal p is determined as in Lamantia, Ortobelli, Rachev (2004) and α is the index of stability that we assume being the mean of indexes of stability.

d) **MAD ratio** (see Konno and Yamazaki (1991))

$$\rho(x'z) = \frac{E(x'z)-z_0}{\sigma_{x'z}}$$

where $\sigma_{x'z} = \frac{1}{N}\sum_{k=1}^{N}\left|x'z_{(k)} - E(x'z)\right|$ and $z_{(k)}$ points out the k-th observation of vector z.

e) **Gini ratio** (see Yitzhaki (1982)):

$$\rho(x'z) = \frac{E(x'z)-z_0}{\sigma_{x'z}}$$

where $\sigma_{x'z} = \frac{1}{T(T-1)}\sum_{k=1}^{T}\sum_{t>k}^{T}\left|\sum_{i=1}^{n} x_i(z_{i,t}-z_{i,k})\right|$

f) **Farinelli-Tibiletti ratio** (See Farinelli-Tibiletti 2003)

$$\rho(x'z) = \frac{\sqrt[p]{E\left((x'z-t_1)_+{}^p\right)}}{\sqrt[q]{E\left((x'z-t_2)_-{}^q\right)}}$$

where $\sqrt[p]{E\left((x'z-t)_+{}^p\right)} = \left(\frac{1}{T}\sum_{k=1}^{T}\left(x'z_{(k)}-t\right)_+^{\,p}\right)^{1/p}$, $\left(x'z_{(k)}-t\right)_+^{\,p} = \left(\max(x'z_{(k)}-t,0)\right)^p$ and

$$\sqrt[q]{E\left((x'z-t)_-^{\,q}\right)}=\left(\frac{1}{T}\sum_{k=1}^{T}\left(x'z_{(k)}-t\right)_-^{\,q}\right)^{1/q},$$

$\left(x'z_{(k)}-t\right)_-^{\,q}=\left(\max(t-x'z_{(k)},0)\right)^q$. We use $t_1=z_0$, $t_2=z_0/2$, p=$\alpha=\frac{1}{n}\sum_{i=1}^{n}\alpha_i$ and $q=\alpha/2$ and also we use $p=\alpha/2$ and q=$\alpha=\frac{1}{n}\sum_{i=1}^{n}\alpha_i$.

g) **Sortino-Satchell ratio** (see Sortino F. (2000), Pedersen C., Satchell S. E. (2002))

$$\rho(x'z)=\frac{E(x'z)-z_0}{\sigma_{x'z}(t)}$$

where $\sigma_{x'z}(t)=\left(\frac{1}{T}\sum_{k=1}^{T}\left(t-x'z_{(k)}\right)_+\right)$ either with $\sigma_{x'z}(t)=\sqrt{E\left((t-x'z)_+^{\,2}\right)}=\left(\frac{1}{T}\sum_{k=1}^{T}\left(t-x'z_{(k)}\right)_+^{\,2}\right)^{1/2}$.

We suppose $t=z_0/2$.

h) **Colog ratio** (see Giacometti Ortobelli (2001))

$$\rho(x'z)=\frac{E(x'z)-z_0}{VAR_{x'z}}$$

where $VAR_{x'z}$ is the variance of *x'z*.

i) **Cologdsr ratio** (see Giacometti Ortobelli (2001))

$$\rho(x'z)=\frac{E(x'z)-z_0}{\sigma_{x'z}(t)},$$

where $\sigma_{x'z}(t)=\left(\frac{1}{T}\sum_{k=1}^{T}\left(x'z_{(k)}-t\right)_-^2\right)$ and $t=z_0/2$.

j) **VaR ratio** (see Favre and Galeano (2002))

$$\rho(x'z)=\frac{E(x'z)-z_0}{VaR_{99\%}(x'z)+z_0},$$

where $VaR_{99\%}(x'z)$ is the opposite of 1% quantile, implicitly defined by $P(x'z<-VaR_{99\%}(x'z))=0.01$.

k) **CVaR ratio** (Martin, Rachev, Siboulet (2003), Favre and Galeano (2002))

$$\rho(x'z)=\frac{E(x'z)-z_0}{CVaR_{99\%}(x'z)+z_0}$$

where $CVaR_{99\%}(x'z)=-E(x'z/x'z\le -VaR_{99\%}(x'z))$.

l) **Rachev ratio** (Biglova, Ortobelli, Rachev, Stoyanov (2004) and Ortobelli, Rachev, Biglova, Stoyanov and Fabozzi (2004))

$$\rho(x'z)=\frac{CVaR_\beta(z_0-x'z)}{CVaR_\alpha(x'z-z_0)}$$

for some opportune α and β belonging to [0,1].

m) **Rachev generalized ratio** (Biglova, Ortobelli, Rachev, Stoyanov (2004) and Ortobelli, Rachev, Biglova, Stoyanov and Fabozzi (2004))

$$\rho(x'z)=\frac{ETL_{\beta,\lambda}(z_0-x'z)}{ETL_{\alpha,\gamma}(x'z-z_0)}$$

where $ETL_{\beta,\lambda}(X)=E\left(\left(\max(-X,0)\right)^\lambda / X\le -VaR_\beta\right)$.

3. AN EMPIRICAL COMPARISON

In this section we propose two distinct types of comparison: an ex-ante comparison where we consider several ex-ante utility maximizers and an ex-post comparison where we analyze the sample paths of final wealth obtained with the different approaches.

3.1 *An ex-ante comparison*

Once we determine the optimal market portfolios, we can compare the efficiency of alternative performance measures from the point of view of different decision making processes. In particular, assuming that no short sales are allowed, we examine the issue of optimal allocation among the riskless return LIBOR and *n* asset returns. In particular, we consider daily returns in the period 1999-2003. The first analysis approximates the expected utility of the following utility functions

$$u(r)=\frac{(1+r)^\gamma}{\gamma}\qquad with\ \gamma<1$$

$$u(r)=\log(1+r)$$

$$u(r)=-e^{-cr}\qquad with\ c>0$$

where $r=\lambda z_0+(1-\lambda)x'_M z$, and λ is the allocation in the riskless asset. In order to compare the various models, we use the algorithm proposed by Giacometti and Ortobelli (2004). Considering *i*=1,…,T i.i.d. observations $z_{(i)}$ of the vector $z_{(i)}=\left[z_{1,(i)},z_{2,(i)},\dots,z_{n,(i)}\right]'$, the main steps of our comparison are the following

Step 1. Fit the efficient frontiers corresponding to the different market portfolio $x_M=[x_{1,M},\dots,x_{n,M}]'$.

Step. 2 Choose a utility function *u* with a given coefficient of risk aversion.

Step. 3 Calculate for every efficient frontier

$$\max_\lambda \sum_{i=1}^{T}u(\lambda z_0+(1-\lambda)x_M{}'z_{(i)})$$

Step. 4 Repeat steps 2 and 3 for every utility function and for every risk aversion coefficient c and γ.

Finally, we obtain tables which approximate the maximum expected utility for the multiplicative factor. We implicitly assume the approximation

$$\frac{1}{T}\sum_{i=1}^{T}u(\lambda z_0+(1-\lambda)x_M{}'z_{(i)})\approx \approx E(u(\lambda z_0+(1-\lambda)x_M{}'z_{(i)})).$$

We know that too large or too small risk aversion coefficients imply that investors choose respectively either the riskless or the market portfolio. Therefore, in order to obtain significant results, we calibrate risk aversion coefficients so that the portfolios which maximize the expected utility are optimal portfolios in the segment of the efficient frontier considered. As a consequence of this analysis, it follows that the Rachev-type ratios, Farinelli-Tibiletti ratio and the

Cologdsr ratio models often show a superior performance with respect to the classic Sharpe ratio. In contrast, the other approaches do not seem to diverge significantly from the mean-variance one, even though we observe that the optimal portfolio weights are significantly different. This result implicitly supports the hypothesis that Rachev-type ratios, Farinelli-Tibiletti ratio and the Cologdsr ratio capture the distributional behavior of the data (typically the component of risk due to heavy tails) better than the classic mean-variance model.

3.2 *An ex-post multi-period comparison*

Let us suppose that investors with logarithmic utility function invest their wealth in the market assuming that the market portfolio is determined by maximizing one of the above performance measures. Then the investors will choose a convex combination between the riskless and the market portfolio. We want to compare the sample path and the final wealth obtained from the several approaches. Thus, everyday and for each performance measure ρ we have to solve two optimization problems: the first in order to determine the market portfolio and the second to determine the optimal expected utility on the efficient frontier. In particular, everyday we calibrate the portfolio using the last 250 observations. Therefore, without considering transaction costs and taxes, we first determine the market portfolio solving the optimization problem (1). Thus, after k periods, we get the market portfolio composition $x_M^{(k)}$ and the investors will choose the portfolio that maximizes their expected utility given by $\max_{\lambda} \frac{1}{250} \sum_{i=k}^{249+k} \log(1+\lambda z_0^{(i)} + (1-\lambda) x_M^{(k)}{}' z_{(i)})$, where $z_0^{(i)}$ is the corresponding i-th observation of the LIBOR. In this way we get the optimal investment $\lambda \in [0,1]$ in the riskless and the vector composition of risk assets $(1-\lambda)x_M^{(k)}$ after k periods. Therefore, the final wealth at the k-th step is given by

$$W_k = W_{k-1}(1+\lambda z_0^{(250+k)} + (1-\lambda) x_M^{(k)}{}' z_{(250+k)}).$$

This ex-post multi-period analysis generally confirms the results of the previous analysis. In particular, we observe that Rachev-type ratios present the best performances during all the period considered.

4. CONCLUSIONS

This paper proposes and compares alternative portfolio selection models. In the first part we describe several performance measures. Specifically, we justify the importance of some new portfolio choice models because they consider the fundamental financial impact of the tail distribution. As it follows from the previous considerations, the performance ratios introduced here can be theoretically improved and empirically tested. However, a more general theoretical and empirical analysis with further discussion and comparisons will be the subject of future research.

The empirical comparison confirms that the classic Sharpe ratio presents less forecast abilities than other performance measures proposed in literature. The Rachev-type ratios present better performance for most decision makers among the alternative models proposed. In addition, these performance measures reveal a high degree of efficiency for large portfolios. Thus, it is reasonable to believe that implementing these portfolio selection models for online calculation is a realistic issue.

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