IMPROVING THE PERFORMANCE OF NONLINEAR STABILIZATION OF MULTIPLE INTEGRATORS WITH BOUNDED CONTROLS

Nicolas Marchand * **Ahmad Hably** *,[1]

* *Laboratoire d'Automatique de Grenoble, CNRS-INPG-UJF, ENSIEG BP 46, 38402 Saint Martin d'Hères Cedex, France*

Abstract: This paper deals with the global stabilization of integrators chain by means of a bounded feedback. Two nonlinear control laws made of summation of saturation of the state are proposed. The first one extends [Sussmann et al., 1994] and the second proposes variable level of the saturation to improve the performance. Both remains very simple and improve significantly the efficiency of the approach that is very competitive with respect to the other existing methods, especially for its performance/complexity ratio.

Keywords: Integrators, linear system, bounded control, nonlinear control

1. INTRODUCTION

Bounded control is inherent to practical stabilization problem: valves can only be operated between closed and open, cars have limited steering angles, tanks can only contain a finite volume, etc. Therefore, design of controllers for systems with bounds is an active area of research (see for instance recent books Hu and Lin [2001], Saberi et al. [2000] or special issues Bernstein and Michel [1995] and the chronological bibliography therein).

Model predictive control is based on an online computation of an open-loop optimal input defined on a certain number of instant called "horizon" (that theoretically can be infinite) and recalculated at each new instant. This method is known to enable the stabilization of linear systems with constrained control inputs Athans and Falb [1966], Lee and Markus [1967], Vincent and Grantham [1997] - with some possible restrictions. However, due to the intensive computations led by the saturation function, the method is not always applicable to real problems. Moreover, the optimal solution may be discontinuous as for the time-optimal problem. We shall mention the work of Gauthier and Bornard [1983] where the problem is reduced to a classical quadratic programming problem that ensures the convergence and a relative efficiency of the algorithm. However, the stability result can only be obtained in an infinite horizon scheme whereas the algorithm works with a finite horizon.

Solutions derived from the linear control techniques have also been proposed. The linear anti-windup compensation consist in designing a linear feedback, ignoring the input nonlinearity, and then adding a compensation feedback to minimize its effects [see Åstrom and Rundquist, 1989, Campo and Morari, 1990, Kothare et al., 1994, etc.]. Unfortunately, as mentioned by Megretski [1996], a rigorous stability and robustness analysis is rarely done because of its complexity.

Low gain control laws also gave rise to an important literature [Lin and Saberi, 1993, Teel, 1995,

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Saberi et al., 1996, Megretski, 1996, etc.]. This consist in saturating a linear controller usually obtained by solving a Riccati equation. It is known that one can not achieve global stabilization by this mean for system of dimension $n \geq 3$ Sussmann and Yang [1991] and to disguise this drawback, it is proposed to tune the Riccati equation with some parameter ε that can be adapted online Lin and Saberi [1993], Megretski [1996], Grognard et al. [2002]. This enables global stability and better performances in terms of convergence. Unfortunately, this requires to solve at each time a convex optimization problem that requires at each step to solve a linear Riccati equation that turns out to be very expensive. Moreover, to obtain good performances, the control law must to be stiff (even discontinuous in Grognard et al. [2002]) due to a necessarily fast adaptation of the parameter ε. However, it should be mentioned that an important work was carried out for these method to handle perturbations Saberi et al. [1996], Sepulchre [2000].

Fully nonlinear approaches where also developed. This axis was initiated by Teel [1992] who proposed a fully nonlinear control law based on nested saturation functions for the stabilization of integrator chain. It followed various works extending the result to general controllable linear systems Sussmann et al. [1994] (with a slightly different class of feedbacks) or handling measurements bounds Lin [1995]. A common property of these control laws is their extreme simplicity compared with the other existing approaches. Furthermore, as mentioned in Rao and Bernstein [2001] in a comparison paper for the double integrator case, the nonlinear approach shows good performances in terms of robustness and performance degradation. Unfortunately, as mentioned by Megretski [1996], for larger system the performance of the closed loop system are degraded. The efficiency and the simplicity of the nonlinear approaches encourage to further explore this field. The purpose of this paper is to improve the performances of the nonlinear control law proposed in Sussmann et al. [1994] for the integrator case while trying to keep as possible its extreme simplicity.

Notations: Let $\mathbb{R}^\star := \mathbb{R}\setminus\{0\}$ and $\mathbb{R}^{\star+} := \mathbb{R}^+\setminus\{0\}$. For any matrix P, p_{ij} will stand for the element situated at the i^{th} row and j^{th} column, P_j for its j^{th} column. For any $y \in \mathbb{R}$, $\text{sat}_M(y) = y$ if $|y| \leq M$ and $\text{sat}_M(y) = \text{sign}(y)M$ otherwise.

2. PROBLEM STATEMENT AND PRELIMINARY DEFINITIONS

An integrator chain is defined by:

$$\dot{x} = Ax + Bu \tag{1}$$

where A is such that $a_{ij} = 1$ if $j = i+1$ and $a_{ij} = 0$ otherwise, B is such that $b_i = 0$ for $i = 1, \ldots, n-1$ and $b_n = 1$, n being the dimension of the system. The topic of this paper is to find stabilizing control laws u for system (1) such that:

$$-\bar{u} \leq u \leq \bar{u} \tag{2}$$

where $\bar{u}$ is some a priori known strictly positive real number. After an appropriate coordinates change in order to fall within the generic case (1-2), the control law proposed in Sussmann et al. [1994] is:

$$u = -\sigma \sum_{i=1}^{n} \varepsilon^{n-i+1}\,\text{sat}_1(y_i) \tag{3}$$

where $0 < \varepsilon \leq \frac{1}{4}$, $\sigma := \bar{u}/\sum_{i=1}^{n}\varepsilon^i$ is a scaling factor and $y := \frac{1}{\sigma}Tx$ is a linear transformation of x given by:

$$\begin{array}{ll} \prod_{i=1}^{n}(\lambda+\varepsilon^i) = p_0 + p_1\lambda + \cdots + p_{n-1}\lambda^{n-1} + \lambda^n & \\ T_n \quad = B & \\ T_{n-1} = (A + p_{n-1}I)B & \\ T_{n-2} = (A^2 + p_{n-1}A + p_{n-2}I)B & \\ \quad\vdots & \\ T_1 \quad = (A^{n-1} + p_{n-1}A^{n-2} + \cdots + p_1 I)B & \end{array} \tag{4}$$

The drawback of this control law shows through (3). Indeed, if the y_i's are small for $i \in \{1, \ldots, n-1\}$, then for any large y_1 the control law will be close to $\sigma\varepsilon^n$ when a range of $\bar{u}$ could be used to drive faster the system to the origin. Hence, the higher the dimension of the system is, worse the performance is. Moreover, for a given dimension, the choice of ε directly influences the performances. The result of this paper extends the range of possible ε that guarantees the global stability of the closed loop and proposes a modification of the control law (3) that improves the performances.

3. MAIN RESULTS

For any $n > 2$, the polynomial equation $\varepsilon^n - 2\varepsilon + 1 = 0$ has only one real solution in the interval $]0,1[$; let $\bar{\varepsilon}(n)$ denote this solution. For $n = 2$ let us define $\bar{\varepsilon}(2) := 1$, the case $n = 1$ is not treated since it has the trivial $u = \text{sat}_{\bar{u}}x$. One can easily see that for all $n > 1$, $\bar{\varepsilon}(n) > \frac{1}{2}$ and $\lim_{n\to\infty}\bar{\varepsilon}(n) = \frac{1}{2}$. With the above definitions, one has:

Theorem 1. For all ε with $0 < \varepsilon < \bar{\varepsilon}(n)$, the control law

$$u = -\sigma \sum_{i=1}^{n} \varepsilon^{n-i+1} \operatorname{sat}_1(y_i) \qquad (5)$$

with y as in (3), globally asymptotically stabilizes (1).

Corollary 2. The feedback law (5) globally asymptotically stabilizes (1) for any $\varepsilon < \frac{1}{2}$.

Theorem 1 is proved in appendix A.1.

Corollary 2 directly follows from theorem 1 since for all $n > 1$, $\bar{\varepsilon}(n) > \frac{1}{2}$. This result is a generalisation of Sussmann et al. [1994] that assumes $\varepsilon < \frac{1}{4}$. This extension sensibly improves the behavior of the closed loop as shown on figure 1.

To improve the above mentioned bad performance of control laws made of addition of saturation functions due to a loss of possible *"control energy"* if the states are badly scaled, we propose:

Theorem 3. Let ε be such that $0 < \varepsilon < \bar{\varepsilon}(n)$ and let the M_i's be defined by:

$$\begin{cases} M_n = 1 \\ M_j = 1 + \dfrac{1}{\varepsilon}\left[M_{j+1} - \left|\operatorname{sat}_{M_{j+1}}(y_{j+1})\right|\right] \\ \qquad \text{for } j = 1, \ldots, n-1 \end{cases} \qquad (6)$$

then, the control law

$$u = -\sigma \sum_{i=1}^{n} \varepsilon^{n-i+1} \operatorname{sat}_{M_i}(y_i) \qquad (7)$$

with y as in (3), globally asymptotically stabilizes (1).

Theorem 3 is proved in appendix A.2.

Thanks to this modification, the state y_i will inherit of the possible control left by the states $\{y_j\}_{j=i+1,\ldots,n}$ in (7).

The next section is dedicated to a comparison on the triple integrator case of the proposed feedbacks and control laws of the literature: the time optimal control, the control law proposed by Megretski [1996] improved as recommended in Lin [1998] based on solutions of a parametrized Riccati equation, the original control law of Sussmann et al. [1994] and the first nonlinear feedback proposed by Teel [1992].

4. THE TRIPLE INTEGRATOR CASE

In this section, we consider a chain of three integrators:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

with u such that $|u| \leq 1$.

We first apply the time optimal control. This control law is known to minimize the time needed by the state to join the origin. It consist in a bang-bang control with at most $n-1$ switch instant (two switch in the present case) that is $n-1$ instants where the control can switch from +1 to -1 or from -1 to +1. This control is the best result that can be obtained in term of convergence speed, however, it is hard to compute especially for large systems.

Beside optimal control, the control law proposed in Megretski [1996] was tested. This control laws belongs to the class of feedbacks of the form:

$$u = -\operatorname{sat}(kB^T P(\varepsilon)x) \qquad (8)$$

where k and ε are two real parameters and P is the solution of the Riccati equation:

$$P(\varepsilon)A + A^T P(\varepsilon) - P(\varepsilon)BB^T P(\varepsilon) = -\varepsilon I$$

Megretski proposed to adapt ε according to the following rule:

$$\varepsilon(x) := \max\{\eta \in (0,1]; \; (x^T P(\eta)x)(B^T P(\eta)B) \leq 1\} \qquad (9)$$

with $\varepsilon(x) = 0$ if the set over which the maximum is taken is empty. This maximization can be solved very efficiently with the Newton's method since the function $(x, \eta) \to x^T P(\eta)x - (B^T P(\eta)B)^{-1}$ can be proved to be monotonically increasing and concave for $\eta \in [0,1]$. The adaptation is issued from the observation that the gain ε is required to be large close to the origin for good performances and small far from the origin to ensure the stability. As in Grognard et al. [2002], we combined with the online adaptation of ε, a gain $k = \frac{1}{\varepsilon}$ as proposed by Lin [1998] to increase the performances near the origin. This control law ensures global stability but requires to optimize at each step a Riccati equation. A lighter version that does not require to solve online an algebraic Riccati equation is also proposed in Megretski [1996] but only semi-global stability can then be insured. Based on the same approach of adapting a control law of the form (8), Grognard et al. [2002] proposed a control law in three phases. The performances are slightly improved but the method is harder to implement and the obtained control law may be discontinuous which is often critical in case unknown delays Rao and Bernstein [2001].

More in the spirit of the control law proposed in this paper, we applied the control law proposed in Teel [1992]. Taking $M_i = L_{i+1}$ for all $i \in \{1, \dots, n-1\}$ and $M_i = \frac{1}{2.00001} L_{i+1}$ for $i = 1, \dots, n-1$ in order to fulfill the stability condition $M_i < \frac{1}{2} L_{i+1}$, the control law is given by:

$$u = -\,\text{sat}_{M_n}(y_n + \text{sat}_{M_{n-1}}(y_{n-1} + \cdots))$$

with $M_n = 1$ and $y_{n-i} = \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} x_{n-j}$.

The above control laws are compared with theorem 1 and 3 with $\varepsilon = 0.618$ so to insure the stability. The results are shown on figure 1 and 2. The control law proposed in theorem 3 clearly appears to be the best in term of convergence performance of the nonlinear approaches tested in this paper. For instance, the time needed to join the ball of radius $0.5\,\|x(0)\|$ is reduced by a factor 10 from Sussmann et al. [1994] to theorem 3 for large initial condition. Corollary 2, that simply increases the range of possible values for ε initially proposed in Sussmann et al. [1994] brings a significant performance contribution. Simulations showed that further increasing ε reduces the amplitude of $\|x(t)\|$ but is detrimental to the convergence speed in the neighbourhood of the origin (and to the global stability of the closed-loop). Hence, further improving the possible range for ε would not bring much benefits. Time optimal and Megretski [1996]/Lin [1998] have slightly better performances in terms of convergence speed. However, this must be paid with intensive computations that prevent using these methods on large systems.

The robustness of the proposed approach is tested using the achieved settling time (AST) as indicator. AST is the time needed to join and remain in a ball centered at the origin of radius 0.05% of the initial state norm. For the robustness to measurement delay, the application of the theorem 3 stabilizes the system for a delay $\tau < 1.6$ s, which is greater than the delay allowed by Teel [1992] ($\tau < 1.1$ s) and more than the double of the delay allowed by the control law of Megretski [1996]. Model uncertainty is also tested. First the real poles of the system are moved to get $\frac{1}{(s+a)^3}$. Theorem 3 succeeded to settle the system till $a = -0.065$, Teel [1992] failed to stabilize the system beyond $a = -0.01$ while Megretski [1996] continued to stabilize the system for values of a down to -0.07 on the real axis. Second the poles are moved along the imaginary axis and the plant becomes $\frac{1}{s(s^2+w^2)}$. The nonlinear control laws of Teel [1992] and of Theorems 1 and 3 ceased to stabilize the system for $w > 2$. The control law proposed by Megretski [1996] performs better but this is paid with high frequency chattering phenomena of the control which may damage practical experiments.

5. CONCLUSION

In this paper, a generalization of the nonlinear control law proposed by Sussmann et al. [1994]

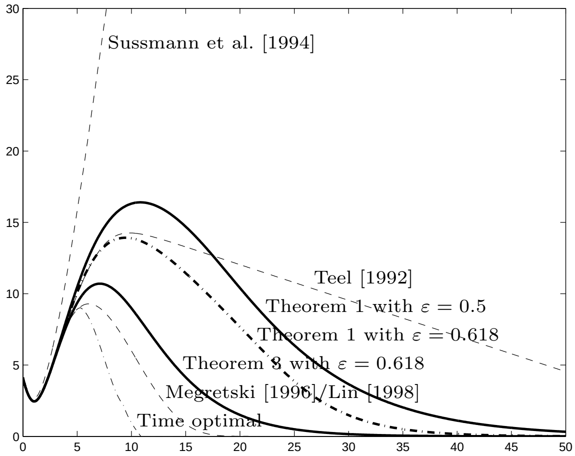

Fig. 1. Control of the third order integrator - Evolution of $\|x(t)\|$ for an initial condition $x_0 = (2\ -2\ 3)^T$

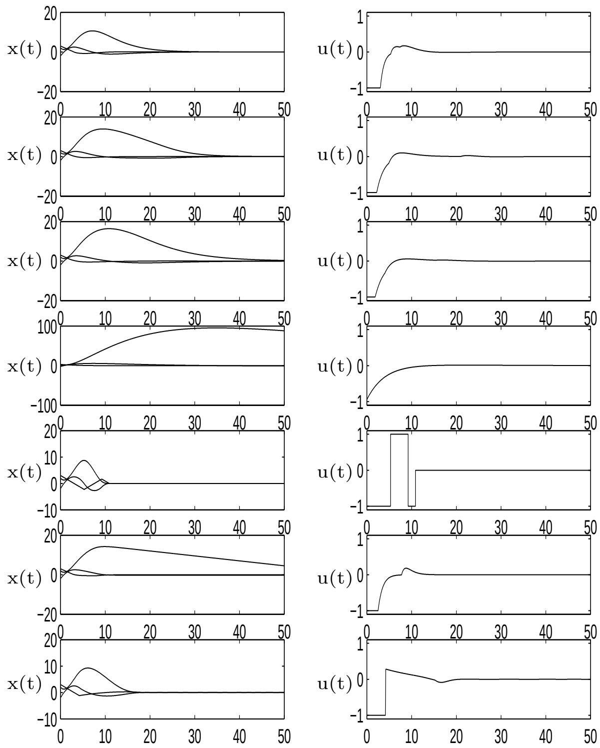

Fig. 2. Control of the third order integrator - Evolution of $x(t)$ (left) and $u(t)$ (right) with the approaches (from top to bottom) of theorem 3 with $\varepsilon = 0.618$, theorem 1 with $\varepsilon = 0.618$, theorem 1 with $\varepsilon = 0.5$, Sussmann et al. [1994], time optimal control, Teel [1992] and Megretski [1996] with the improving factor proposed by Lin [1998]

is proposed as well as a possible improvements. The convergence of the closed loop trajectory is so deeply speeded. The proposed extensions render nonlinear feedback very competitive compared to the other existing methods particularly for the simplicity of the feedback or the ability thanks to the Lyapunov theory to quantify the acceptable measurement or model errors. The next step is clearly to extend the result to general linear systems as in [Sussmann et al., 1994].

Appendix A. PROOFS

Let us start the proof with a preliminary lemma that avoids to use a global lipschitz argument as in Teel [1992] or in Lin [1995] to justify that the trajectory of the closed loop system can not diverge in finite time.

Lemma 4. Any closed loop trajectory of any linear system can not diverge in finite time under bounded control.

PROOF. Let $\dot{x} = Ax + Bu$ where u is such that $\|u\| \leq M$. Then, one has:

$$\begin{aligned}\frac{d\|x\|^2}{dt} &= 2x^T Ax + 2x^T Bu \\ &\leq 2\lambda_{\max}(A)\|x\|^2 + 2\lambda_{\max}(B)M\|x\|\end{aligned}$$

Recognizing a Bernoulli ordinary differential equation, it follows:

$$\|x(t)\|^2 \leq \Big[\frac{\lambda_{\max}(B)M}{\lambda_{\max}(A)}\left(e^{\lambda_{\max}(A)t} - 1\right) + \|x(0)\|\, e^{\lambda_{\max}(A)t}\Big]^2$$

which ends the proof, the system can not blow up in finite time. Note that the result does only assumes the existence of a closed loop trajectory.

Now, if one takes $z := \frac{1}{\sigma}x$ and $v := \frac{1}{\sigma}u$, system (1) becomes:

$$\dot{z} = Az + Bv$$

with the constraint $|v| \leq \sum_{i=1}^{n} \varepsilon^i$ and after the coordinate change $y = Tz$, with T as in (4) one gets:

$$\dot{y} = \begin{pmatrix} 0 & \varepsilon^{n-1} & \varepsilon^{n-2} & \dots & \varepsilon \\ 0 & 0 & \varepsilon^{n-2} & \dots & \varepsilon \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \varepsilon \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} y + \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} v \quad \text{(A.1)}$$

A.1 Proof of Theorem 1

Assume that $|y_n| > 1$ and let us prove that y_n joins $[-1, 1]$ in finite time. Let $V_n := \frac{1}{2}y_n^2$.

$$\begin{aligned}\dot{V}_n = -y_n\varepsilon \operatorname{sat}_1(y_n) - y_n[\varepsilon^2 \operatorname{sat}_1(y_{n-1}) + \\ \cdots + \varepsilon^n \operatorname{sat}_1(y_1)]\end{aligned}$$

Clearly, the decrease of the Lyapunov function V_n holds if $\varepsilon > \sum_{i=2}^{n} \varepsilon^i$. But one has:

$$\begin{aligned}\varepsilon > \sum_{i=2}^{n} \varepsilon^i \quad &\Leftrightarrow \quad \varepsilon - 2\varepsilon^2 + \varepsilon^{n+1} > 0 \\ &\Leftrightarrow \quad 1 - 2\varepsilon + \varepsilon^n > 0\end{aligned}$$

and $p : \varepsilon \to \varepsilon^n - 2\varepsilon + 1$ has its unique extremum for $\varepsilon = \sqrt[n-1]{\frac{2}{n}}$. It is also easy to see that for $n > 2$, this extremum necessarily lies in $]0, 1[$ since $p(0) = 1$, $\frac{dp}{d\varepsilon}(0) = -2 < 0$, $p(1) = 0$ and $\frac{dp}{d\varepsilon}(1) = -2 + n > 0$. Hence, for all $n > 2$, p necessarily has one and only one root $\bar{\varepsilon}(n)$ in the open interval $]0, 1[$ and $p(\varepsilon) > 0$ for all $\varepsilon < \bar{\varepsilon}(n)$ and $p(\varepsilon) < 0$ if $\varepsilon > \bar{\varepsilon}(n)$. One can trivially notice that this also holds for $n = 2$. Hence, y_n necessarily joins $[-1, 1]$ in finite time. During that time, the other states can not blow up thanks to Lemma 4.

Once y_n lies in $[-1, 1]$, the evolution of $V_{n-1} := y_{n-1}^2$ satisfies:

$$\begin{aligned}\dot{V}_{n-1} = &-y_{n-1}\varepsilon^2 \operatorname{sat}_1(y_{n-1}) \\ &-y_{n-1}[\varepsilon^3 \operatorname{sat}_1(y_{n-2}) + \cdots + \varepsilon^n \operatorname{sat}_1(y_1)]\end{aligned}$$

With the same argumentation as above, y_{n-1} joins $[-1, 1]$ in finite time.

Hence, after some finite time, all the y_i's are in the interval $[-1, 1]$ where the system is linear with strictly negative eigenvalues:

$$\dot{y} = \begin{pmatrix} -\varepsilon^n & 0 & 0 & \dots & 0 \\ -\varepsilon^n & -\varepsilon^{n-1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\varepsilon^n & -\varepsilon^{n-1} & \dots & -\varepsilon^2 & 0 \\ -\varepsilon^n & -\varepsilon^{n-1} & \dots & -\varepsilon^2 & -\varepsilon \end{pmatrix} y \quad \text{(A.2)}$$

which ends the proof.

A.2 Proof of Theorem 3

Let here again assume that $|y_n| > 1$ and take $V_n := \frac{1}{2}y_n^2$. It follows:

$$\begin{aligned}\dot{V}_n = -y_n\varepsilon \operatorname{sat}_{M_n}(y_n) - y_n[\varepsilon^2 \operatorname{sat}_{M_{n-1}}(y_{n-1}) + \\ \cdots + \varepsilon^n \operatorname{sat}_{M_1}(y_1)]\end{aligned}$$

The decrease of the Lyapunov function holds if

$$|\operatorname{sat}_{M_n}(y_n)| > \left| \sum_{i=1}^{n-1} \varepsilon^{n-i} \operatorname{sat}_{M_i}(y_i) \right| \quad \text{(A.3)}$$

But, using (6), one has:

$$\begin{aligned} \left| \sum_{i=1}^{n-1} \varepsilon^{n-i} \operatorname{sat}_{M_i}(y_i) \right| &\leq \sum_{i=1}^{n-1} \varepsilon^{n-i} \, |\operatorname{sat}_{M_i}(y_i)| \\ &\leq \sum_{i=2}^{n-1} \varepsilon^{n-i} \, |\operatorname{sat}_{M_i}(y_i)| + \varepsilon^{n-1} M_1 \\ &\leq \sum_{i=2}^{n-1} \varepsilon^{n-i} \, |\operatorname{sat}_{M_i}(y_i)| + \varepsilon^{n-1} \\ &\qquad + \varepsilon^{n-2} [M_2 - |\operatorname{sat}_{M_2}(y_2)|] \\ &\leq \sum_{i=3}^{n-1} \varepsilon^{n-i} \, |\operatorname{sat}_{M_i}(y_i)| + \varepsilon^{n-1} + \varepsilon^{n-2} M_2 \\ &\vdots \\ &\leq \varepsilon^{n-1} + \varepsilon^{n-2} + \cdots + \varepsilon^2 + \varepsilon M_{n-1} \end{aligned}$$

Since $|y_n| > 1$, $M_{n-1} = 1$ and it follows:

$$\left| \sum_{i=1}^{n-1} \varepsilon^{n-i} \operatorname{sat}_{M_i}(y_i) \right| \leq \sum_{i=1}^{n-1} \varepsilon^i$$

A choice of ε strictly lower than $\bar{\varepsilon}(n)$ ensuring that $\varepsilon > \sum_{i=2}^{n} \varepsilon^i$ will also ensure (A.3). Hence, here again, y_n necessarily joins $[-1, 1]$ in finite time. During that time, the other states can not blow up thanks to Lemma 4.

Repeating the same reasoning for all the states, all the y_i's rejoin $[-1, 1]$ in finite time where the system takes the form (A.2). This ends the proof.

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