# STOCHASTIC SUBSPACE IDENTIFICATION GUARANTEEING STABILITY AND MINIMUM PHASE

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Abstract: This paper presents a stochastic subspace identification algorithm to compute stable, minimum phase models from a stationary time-series data. The algorithm is based on spectral factorization techniques and a stochastic subspace identification method via a block LQ decomposition (Tanaka and Katayama, 2003*c*). Two Riccati equations are solved to ensure both stability and minimum phase property of resulting Markov models.


Keywords: Subspace identification, Stochastic realization, Canonical Correlation Analysis, Balanced realization, LQ decomposition, Spectral factorization technique

## 1. INTRODUCTION

Stochastic subspace identification algorithms compute stochastic state space systems from a finite string of a time-series data (Van Overschee and De Moor (1993; 1996)), where the numerical operations include not only the singular value decomposition (SVD) and QR decomposition, but also computation of a stabilizing solution of an associated Riccati equation.

Lindquist and Picci pointed out that stochastic subspace identification algorithms (Aoki, 1990; Van Overschee and De Moor, 1993) may fail in solving the Riccati equation, since the failure is related to a non-trivial problem of positivity in the stochastic realization theory, where an essential part of the problem is equivalent to the covariance extension problem (Lindquist and Picci, 1996*a*). Some stochastic subspace identification methods therefore have been developed taking positive realness into account (Van Overschee and De Moor, 1996; Mari *et al.*, 2000; Goethals *et al.*, 2003).

In order to review the use of Riccati equation in the context of the stochastic realization theory, we have re-visited stochastic realization theory (Tanaka and Katayama, 2003*a*), and obtained a finite-interval realization method based on a block LQ decomposition (Tanaka and Katayama, 2003*b*). Furthermore, we have proved that an approximate innovation representation due to (Maciejowski, 1996) is of minimum phase under the idealized assumption that a finite complete covariance data is given (Tanaka and Katayama, 2004*b*). It should be noted that this fact implies that a minimum phase model is obtained without solving Riccati equations in the idealized case.

Adapting the finite-interval realization method via a block LQ decomposition (Tanaka and Katayama, 2003*b*) to a finite time-series data, we have presented a stochastic subspace identification algorithm (Tanaka and Katayama, 2003*c*). The algorithm, however, does not guarantee that the identified forward innovation representation is stable and of minimum phase. Based on a spectral factorization technique (SFT), we have developed a prototype algorithm to obtain a minimum phase model (Tanaka and Katayama, 2004*a*).

In this paper, using Riccati equations for Kalman filters, we give an explicit algorithm to obtain a stable, minimum phase model. A numerical simulation result is also shown.

## 2. PROBLEM SETTING

### 2.1 Problem statement

Consider a second-order stationary process $\{y_t,\ t = 0,\ \pm 1,\ \pm 2,\ \ldots\}$, where $y_t$ is a $p$-dimensional non-deterministic process with mean zero and covariance matrices

$$\Lambda_k = \mathrm{E}\left\{y_{t+k} y_t^T\right\}, \quad k = 0, \pm 1, \pm 2, \ldots \quad (1)$$

where a set of covariance matrices $\{\Lambda_k \,|\, k = 0,\ \pm 1,\ \ldots\}$ is a positive real sequence: $\sum_{i,j} u_i^T \Lambda_{i-j} u_j > 0,\ u_i \not\equiv 0$. We assume that there exists a finite dimensional realization for $y_t$, so that the covariance matrix has a decomposition $\Lambda_k = H F^{k-1} \Gamma,\ k = 1,\ 2,\ \ldots$, where $(F,\ \Gamma,\ H)$ is a minimal realization with $F \in \mathbb{R}^{n\times n}$ stable. The spectral density of the stationary time series $y_t$ is given by $\Upsilon(z) = \sum_{j=-\infty}^{\infty} \Lambda_j z^{-j}$, and it has a canonical spectral factorization,

$$\Upsilon(z) = \hat{W}(z)\hat{W}^T(z^{-1}),$$

where $\hat{W}(z)$ is stable and of minimum phase.

Given a finite time series data $\{y_0,\ y_1,\ \ldots,\ y_{\nu+2\tau-2}\}$ ($\tau > n$), our problem is to estimate a forward innovation representation of $y_t$ or estimate $\hat{W}(z)$, where the model must be stable and of minimum phase.

### 2.2 Innovation representation

Define a vector space as

$$\mathcal{Y} := \left\{ \sum a_k^T y_{t_k} \mid a_k \in \mathbb{R}^p,\ k = 0,\ \pm 1,\ \pm 2,\ \ldots \right\},$$

which is a linear space spanned by all finite linear combinations of row vector of $y_t$. Define a bilinear form (inner product) as

$$\langle a^T y_i, b^T y_j \rangle := a^T \mathrm{E}\{y_i y_j^T\} b = a^T \Lambda_{i-j} b. \quad (2)$$

By completing the vector space $\mathcal{Y}$ with the norm induced by the inner product (2), we get a Hilbert space (Lindquist and Picci (1996*a*; 1996*b*)), which is also written as $\mathcal{Y}$. For $\mathcal{U} \subseteq \mathcal{Y}$ and $z \in \mathcal{Y}$, $\hat{\mathrm{E}}(z \,|\, \mathcal{U})$ expresses an orthogonal projection of $z$ onto $\mathcal{U}$. The notation $\hat{\mathrm{E}}(z \,|\, \mathcal{U})$ is also used for a vector $z = \begin{bmatrix} z_1 & z_2 & \cdots \end{bmatrix}^T$, the symbol will then just denote the vector with components $\hat{\mathrm{E}}(z_j \,|\, \mathcal{U})$.

Define the past and future matrices, $Y_t^-$ and $Y_t^+$, respectively as

$$Y_t^- = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \end{bmatrix}, \quad Y_t^+ = \begin{bmatrix} y_t \\ y_{t+1} \\ y_{t+2} \\ \vdots \end{bmatrix}.$$

We also define matrices $\Phi := \mathrm{E}\{(Y_t^-)(Y_t^-)^T\}$, $\Psi := \mathrm{E}\{(Y_t^+)(Y_t^+)^T\}$ and $\mathcal{H} := \mathrm{E}\{(Y_t^+)(Y_t^-)^T\}$. The block Hankel matrix $\mathcal{H}$ has a decomposition $\mathcal{H} = \mathcal{O}\mathcal{C}$ such that $\mathcal{C}\Phi^{-1}\mathcal{C}^T = \mathcal{O}^T\Psi^{-1}\mathcal{O}$ (Desai *et al.*, 1985).

The matrices $\mathcal{O} \in \mathbb{R}^{\infty\times n}$ and $\mathcal{C} \in \mathbb{R}^{\infty\times n}$ are extended observability and reachability matrices, respectively, which are described as

$$\mathcal{O} = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots \end{bmatrix}^T,$$
$$\mathcal{C} = \begin{bmatrix} G & AG & A^2G & A^3G & \cdots \end{bmatrix},$$

using stochastically balanced matrices $A \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$ (Desai *et al.*, 1985), where $\Lambda_k = CA^{k-1}G$ holds.

Consider the following Riccati equation (Faurre, 1976)

$$P = APA^T + (G - APC^T) \times (\Lambda_0 - CPC^T)^{-1}(G - APC^T)^T. \quad (3)$$

Using the stabilizing solution of (3), define

$$\hat{K} = (G - APC^T)(\Lambda_0 - CPC^T)^{-1}, \quad (4)$$
$$\hat{R} = \Lambda_0 - CPC^T. \quad (5)$$

Defining $\hat{x}_t = \mathcal{C}\Phi^{-1}Y_t^-$, we have $\hat{\mathrm{E}}(Y_t^+ \,|\, Y_t^-) = \mathcal{O}\hat{x}_t$, where $\hat{\mathrm{E}}(Y_t^+ \,|\, Y_t^-) := \hat{\mathrm{E}}(Y_t^+ \,|\, \mathrm{span}(Y_t^-))$, and $\mathrm{span}(Y_t^-)$ is a subspace of $\mathcal{Y}$ spanned by row vectors of $Y_t^-$.

*Proposition 1.* (Desai *et al.*, 1985) The forward innovation representation of $y_t$ is given by

$$\hat{x}_{t+1} = A\hat{x}_t + \hat{K}\hat{v}_t, \quad (6a)$$
$$y_t = C\hat{x}_t + \hat{v}_t, \quad (6b)$$

where $\hat{v}_t$ is stationary Gaussian defined as $\hat{v}_t := y_t - C\hat{x}_t$, and its variance $\hat{R} = \mathrm{E}\{\hat{v}_t\hat{v}_t^T\}$.

From (6), we have a whitening filter of $y_t$,

$$\hat{x}_{t+1} = (A - \hat{K}C)\hat{x}_t + \hat{K}y_t, \quad (7)$$
$$\hat{v}_t = -C\hat{x}_t + y_t. \quad (8)$$

It should be noted that $\hat{v}_t$ is obtained from $y_t$ via a stable whitening filter, since $A - \hat{K}C$ is stable.

## 3. STOCHASTIC SUBSPACE IDENTIFICATION METHOD

We review a subspace identification method (Tanaka and Katayama (2003*c*; 2004*a*)).

### 3.1 Assumptions

Define a vector space spanned by all finite linear combinations of vectors $\boldsymbol{\eta} \in \mathbb{R}^{1\times\nu}$ as $\mathcal{Y}^\nu$. For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \mathcal{Y}^\nu$, define an inner product as $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_{\frac{L}{\nu}} := \frac{1}{\nu}\boldsymbol{\alpha}\boldsymbol{\beta}^T$. The vector space equipped with the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\frac{L}{\nu}}$ is an inner product space, which is also written as $\mathcal{Y}^\nu$. We extend $\mathcal{Y}^\nu$ to $\mathcal{Y}^{\bullet\times\nu}$ so that matrices are included as its elements.

Define a matrix

$$\tilde{\boldsymbol{y}}_t := \left[\, y_t \;\, y_{t+1} \;\, \cdots \;\, y_{t+\nu-1} \,\right] \in \mathcal{Y}^{p\times\nu}$$

for $t = 0, 1, \ldots, 2\tau - 1$, and define matrices as

$$\tilde{Y}_t^- = \begin{bmatrix} \tilde{\boldsymbol{y}}_{t-1} \\ \tilde{\boldsymbol{y}}_{t-2} \\ \vdots \\ \tilde{\boldsymbol{y}}_1 \\ \tilde{\boldsymbol{y}}_0 \end{bmatrix}, \quad \tilde{Y}_s^+ = \begin{bmatrix} \tilde{\boldsymbol{y}}_s \\ \tilde{\boldsymbol{y}}_{s+1} \\ \vdots \\ \tilde{\boldsymbol{y}}_{2\tau-2} \\ \tilde{\boldsymbol{y}}_{2\tau-1} \end{bmatrix} \qquad (9)$$

for $t = 1, \ldots, 2\tau$, and for $s = 0, \ldots, 2\tau - 1$. Define also incomplete covariance matrices as

$$\tilde{\Phi}_\tau := \langle \tilde{Y}_\tau^-, \tilde{Y}_\tau^- \rangle_{\frac{I}{\nu}}, \quad \tilde{\Psi}_{-\tau} := \langle \tilde{Y}_\tau^+, \tilde{Y}_\tau^+ \rangle_{\frac{I}{\nu}}$$

and also define $\tilde{\mathcal{H}}_\tau := \langle \tilde{Y}_\tau^+, \tilde{Y}_\tau^- \rangle_{\frac{I}{\nu}}$. We assume $\langle \tilde{Y}_0^+, \tilde{Y}_0^+ \rangle_{\frac{I}{\nu}} > 0$ and $\operatorname{rank} \tilde{\mathcal{H}}_\tau = \tilde{n} < \tau$.

### 3.2 *Identification algorithm*

Compute the standard LQ decomposition

$$\frac{1}{\sqrt{\nu}} \tilde{Y}_0^+ = LQ^T, \qquad (10)$$

where $\tilde{Y}_0^+$ is defined in (9). Partition $L$ as

$$L = \begin{bmatrix} L_{0,0} & & 0 \\ \vdots & \ddots & \\ L_{2\tau-1,0} & \cdots & L_{2\tau-1,2\tau-1} \end{bmatrix} = \begin{bmatrix} L_{pp} & 0 \\ L_{fp} & L_{ff} \end{bmatrix},$$

where $L_{i,j} \in \mathbb{R}^{p\times p}$ and $L_{pp}$, $L_{fp}$, $L_{ff} \in \mathbb{R}^{p\tau\times p\tau}$. Define a matrix as

$$D_L := \text{block-diag}(L_{0,0}, \ldots, L_{2\tau-1,2\tau-1}),$$

where $D_L$ is non-singular from assumptions.

Define matrices as

$$\acute{\mathcal{L}}_0^+ := L D_L^{-1}, \qquad (11)$$

$$\acute{\mathcal{R}}_0^+ := D_L D_L^T. \qquad (12)$$

The following equations are then obtained

$$\tilde{\Psi}_{-\tau} = L_{fp} L_{fp}^T + L_{ff} L_{ff}^T, \qquad (13)$$

$$\acute{\mathcal{R}}_0^+ = \text{block-diag}(\acute{R}_0, \ldots, \acute{R}_{2\tau-1}), \qquad (14)$$

$$\acute{\mathcal{L}}_0^+ = \begin{bmatrix} \acute{L}_{0,0} & & 0 \\ \vdots & \ddots & \\ \acute{L}_{2\tau-1,0} & \cdots & \acute{L}_{2\tau-1,2\tau-1} \end{bmatrix}, \qquad (15)$$

where $\acute{R}_t \in \mathbb{R}^{p\times p}$, $\acute{L}_{i,j} = L_{i,j} L_{j,j}^{-1}$ and $\acute{L}_{i,i} = I_p$.

We summarize a stochastic subspace identification algorithm (Tanaka and Katayama, 2004*a*).

**A stochastic subspace identification algorithm**

**Step 1:** Compute the standard LQ decomposition (10) and define $\acute{\mathcal{L}}_0^+$, $\acute{\mathcal{R}}_0^+$ and $\tilde{\Psi}_{-\tau}$ as (11), (12) and (13), respectively.

**Step 2:** Calculate the SVD

$$(\tilde{\Psi}_{-\tau})^{-\frac{1}{2}} L_{fp} = \tilde{U}_\tau \tilde{\Sigma}_\tau \tilde{V}_J^T, \quad \tilde{\Sigma}_\tau \in \mathbb{R}^{\tilde{n}\times\tilde{n}}, \quad (16)$$

where $\tilde{U}_\tau^T \tilde{U}_\tau = I$, $\tilde{V}_J^T \tilde{V}_J = I$ and $\operatorname{rank} \tilde{\Sigma}_\tau = \tilde{n}$. Based on the SVD (16), define $\tilde{\mathcal{O}}_\tau$ as

$$\tilde{\mathcal{O}}_\tau = \left(\tilde{\Psi}_{-\tau}\right)^{\frac{1}{2}} \tilde{U}_\tau \tilde{\Sigma}_\tau^{\frac{1}{2}}. \qquad (17)$$

**Step 3:** Compute $\tilde{C}$ and $\acute{A}$ by

$$\tilde{C} = \tilde{\mathcal{O}}_\tau(1:p,:),$$

$$\acute{A} = \tilde{\mathcal{O}}_{\tau-1}^\dagger \tilde{\mathcal{O}}_\tau(p+1:p\tau,:),$$

where $\tilde{\mathcal{O}}_{\tau-1} := \tilde{\mathcal{O}}_\tau(1:p(\tau-1),:)$, and $(\cdot)^\dagger$ denotes the Moore-Penrose pseudo-inverse.

**Step 4:** Define $\acute{R}_\tau$ from (14), and compute $\acute{K}_\tau$ from

$$\acute{K}_\tau = \tilde{\mathcal{O}}_{\tau-1}^\dagger \left[\, \acute{L}_{\tau+1,\tau}^T \;\, \acute{L}_{\tau+2,\tau}^T \;\, \cdots \;\, \acute{L}_{2\tau-1,\tau}^T \,\right]^T,$$

where $\acute{L}_{\tau+1,\tau}$, $\acute{L}_{\tau+2,\tau}, \ldots, \acute{L}_{2\tau-1,\tau}$ are found in the matrix $\acute{\mathcal{L}}_0^+$ in (15).

Define transfer functions

$$\acute{\Upsilon}_\tau(z) := \acute{W}_\tau(z) \acute{W}_\tau^T(z^{-1}), \qquad (18)$$

$$\acute{W}_\tau(z) := (\tilde{C}(zI - \acute{A})^{-1} \acute{K}_\tau + I) \acute{R}_\tau^{\frac{1}{2}}. \qquad (19)$$

It can be shown that the transfer function $\acute{\Upsilon}_\tau(z)$ is positive real, and is a good approximation to the true spectral density $\Upsilon(z)$ for large $\nu$ and $\tau$ (Tanaka and Katayama (2003*c*; 2004*a*)). It is however not guaranteed that $\acute{W}_\tau(z)$ is stable and of minimum phase.

## 4. SPECTRAL FACTORIZATION TECHNIQUE

In this section, we summarize an SFT based on Riccati equations for Kalman filters.

Consider the following linear stochastic system

$$x_{t+1} = Ax_t + w_t, \qquad (20a)$$

$$y_t = Cx_t + v_t, \qquad (20b)$$

where $A \in \mathbb{R}^{n\times n}$, and variables $w_t$ and $v_t$ are stationary Gaussian with zero mean and variance

$$\mathrm{E}\left\{ \begin{bmatrix} w_s \\ v_s \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}^T \right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{st}$$

with $R > 0$. We assume that covariance matrices of the system (20) is also given by $\mathrm{E}\{y_{t+k} y_t^T\} = \Lambda_k$, and this assumption implies that $\Lambda_k = CA^{k-1}G$ holds, where $G = \mathrm{E}\{x_{t+1} y_t^T\}$, and that the spectral density function of $y_t$ is given by $\Upsilon(z) = \sum_{j=-\infty}^{\infty} \Lambda_j z^{-j}$, which is positive real.

Assume that we can observe $y_t$, and consider the problem of estimating $x_t$ which minimize $\mathrm{E}\{\|x_t - \hat{x}_t\|^2\}$, where $\hat{x}_t$ is an estimate of $x_t$ based on $\{y_{t-1}, y_{t-2}, y_{t-3}, \ldots\}$. It is well known that such a "$\hat{x}_t$" is given by the Kalman filter. Associated with this problem, consider the following Riccati equation for the Kalman filter

$$\Xi = A\Xi A^T - (A\Xi C^T + S) \times (C\Xi C^T + R)^{-1} (A\Xi C^T + S)^T + Q. \quad (21)$$

Assume here that $Q$, $S$ and $R$ satisfies $Q = SR^{-1}S^T$. This implies that $\Xi = 0$ is a solution of (21), and that there exists $K$ such that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} K \\ I \end{bmatrix} R \begin{bmatrix} K \\ I \end{bmatrix}^T.$$

We also assume that there are no eigenvalues $\lambda_i$ of $A - KC$, such that $\lambda_i \lambda_j = 1$ $(i \neq j)$ or $\lambda_i = 0$.

In order to solve the Riccati equation (21), define $M$ and $N$ as

$$M := \begin{bmatrix} (A - SR^{-1}C)^T & 0 \\ SR^{-1}S^T - Q & I_n \end{bmatrix},$$
$$N := \begin{bmatrix} I_n & C^T R^{-1} C \\ 0 & A - SR^{-1}C \end{bmatrix}.$$

*Proposition 2.* Consider an eigenvalue problem

$$\lambda N x = M x. \tag{22}$$

Assume that $(C, A)$ is observable, and that $(A, Q)$ is stabilizable. If $\lambda$ satisfies (22), then there exists $z$ such that

$$\lambda M z = N z \tag{23}$$

holds. This implies that $1/\lambda$ is also an eigenvalue of (22).

*Proposition 3.* (Arnold and Laub, 1984) Consider a generalized eigenvalue problem:

$$M \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = N \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \Lambda, \tag{24}$$

where $\Lambda \in \mathbb{R}^{n \times n}$ has a Jordan form. Then, the solution of Riccati equation (21) is given by $W_2 W_1^{-1}$.

From Proposition 2 and assumptions, there exists $\Lambda$ whose every diagonal element satisfies $|\lambda_i| < 1$, and we define it as $\hat{\Lambda}$. Using $\hat{\Lambda}$ in (24), we define a solution $\hat{\Xi}$, from Proposition 3.

*Proposition 4.* Define matrices $\hat{K}$ and $\hat{R}$ as

$$\hat{K} = (A\hat{\Xi}C^T + S)(C\hat{\Xi}C^T + R)^{-1}, \tag{25}$$
$$\hat{R} = C\hat{\Xi}C^T + R. \tag{26}$$

Then, a set of every eigenvalue of $A - \hat{K}C$ coincides with the set of diagonal element of $\hat{\Lambda}$.

Proposition 4 implies that $A - \hat{K}C$ is stable, and the matrix $\hat{\Xi}$ is a stabilizing solution of (21).

*Proposition 5.* Suppose that $\hat{\Xi}$ is a stabilizing solution, and define $\hat{K}$ as (25). Then, $A - \hat{K}C$ is stable.

*Proposition 6.* The transfer function $\Upsilon(z)$ satisfies

$$\Upsilon(z) = W(z)W^T(z^{-1}) = \hat{W}(z)\hat{W}^T(z^{-1}), \tag{27}$$

where $W(z)$ and $\hat{W}(z)$ are given by

$$W(z) := \left(C(zI - A)^{-1}K + I\right)R^{\frac{1}{2}}, \tag{28}$$
$$\hat{W}(z) := \left(C(zI - A)^{-1}\hat{K} + I\right)\hat{R}^{\frac{1}{2}}, \tag{29}$$

where $\hat{K}$ and $\hat{R}$ are given by (25) and (26), respectively.

Proposition 6 implies that a minimum phase factor $\hat{W}(z)$ is obtained from $W(z)$ satisfying (27) based on the stabilizing solution of the Riccati equation (21).

## 5. STABLE, MINIMUM PHASE MODEL

We obtain a stable, minimum phase model from $\acute{W}_\tau(z)$ in (19) based on the SFT.

### 5.1 *Enforcing stability*

Assume that $\acute{A}$ is unstable, $(\acute{A}^T, \acute{K}_\tau^T)$ is detectable, and that no eigenvalues of $\acute{A}$ are on the unit circle or on the origin in the complex plane. We derive a stable spectral factor, using the inverse of $\acute{\Upsilon}_\tau(z)$,

$$\acute{\Upsilon}_\tau^{-1}(z) = \acute{W}_\tau^{-T}(z^{-1})\acute{W}_\tau^{-1}(z),$$

where $\acute{W}_\tau^{-T}(z^{-1})$ is given by

$$\acute{W}_\tau^{-T}(z^{-1}) = \left(-\acute{K}_\tau^T(zI - \acute{F}^T)^{-1}\tilde{C}^T + I\right)\acute{R}_\tau^{-\frac{T}{2}}, \tag{30}$$
$$\acute{F} := \acute{A} - \acute{K}_\tau \tilde{C}.$$

Since zeros of $\acute{W}_\tau^{-T}(z^{-1})$ are eigenvalues of $\acute{A}^T$, we can find a stable factor, applying Proposition 6 to $\acute{W}_\tau^{-T}(z^{-1})$ as shown below.

Define matrices as

$$\begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} := \begin{bmatrix} \tilde{C}^T \\ I \end{bmatrix} \acute{R}_\tau^{-1} \begin{bmatrix} \tilde{C}^T \\ I \end{bmatrix}^T. \tag{31}$$

Then, the Riccati equation associated with (30) is obtained by

$$\tilde{\Xi} = \acute{F}^T\tilde{\Xi}\acute{F} - (-\acute{F}^T\tilde{\Xi}\acute{K}_\tau + \tilde{S})$$
$$\times (\acute{K}_\tau^T\tilde{\Xi}\acute{K}_\tau + \tilde{R})^{-1}(-\acute{F}^T\tilde{\Xi}\acute{K}_\tau + \tilde{S})^T + \tilde{Q}. \tag{32}$$

*Lemma 1.* Define matrices $\breve{C}$ and $\breve{A}$ as

$$\breve{C} := (\acute{K}_\tau^T\tilde{\Xi}\acute{K}_\tau + \tilde{R})^{-1}(-\acute{F}^T\tilde{\Xi}\acute{K}_\tau + \tilde{S})^T, \tag{33}$$
$$\breve{A} := \acute{F} + \acute{K}_\tau\breve{C}, \tag{34}$$

using a stabilizing solution of Riccati equation (32). Then, $\breve{A}$ is stable.

*Theorem 1.* A spectral factorization of $\acute{\Upsilon}_\tau(z)$ is given by

$$\acute{\Upsilon}_\tau(z) = \acute{\tilde{W}}_\tau(z)\acute{\tilde{W}}_\tau^T(z^{-1}),$$

where $\acute{\tilde{W}}_\tau(z)$ is stable and given by

$$\acute{\tilde{W}}_\tau(z) := (\breve{C}(zI - \breve{A})^{-1}\acute{K}_\tau + I)\acute{\tilde{R}}_\tau^{\frac{1}{2}}, \tag{35}$$
$$\acute{\tilde{R}}_\tau := \acute{K}_\tau^T\tilde{\Xi}\acute{K}_\tau + \tilde{R}. \tag{36}$$

If $\acute{A}$ is stable, a stabilizing solution of (32) is given by $\tilde{\Xi} = 0$ from $\tilde{C} = \acute{R}_\tau\tilde{S}^T$, and hence we have

$$\breve{A} := \acute{A}, \quad \breve{C} := \tilde{C}, \quad \acute{\tilde{R}}_\tau := \acute{R}_\tau. \tag{37}$$

### 5.2 *Enforcing minimum phase*

Assume that $\breve{A} - \acute{K}_\tau\breve{C}$ is unstable, and that no eigenvalue of $\breve{A} - \acute{K}_\tau\breve{C}$ is on the unit circle or on the origin.

It should be noted that $(\breve{A}, \breve{C})$ is detectable, since $\breve{A}$ is stable. Define matrices as

$$\begin{bmatrix} \acute{\acute{Q}} & \acute{\acute{S}} \\ \acute{\acute{S}}^T & \acute{\acute{R}}_\tau \end{bmatrix} := \begin{bmatrix} \acute{K}_\tau \\ I \end{bmatrix} \acute{\acute{R}}_\tau \begin{bmatrix} \acute{K}_\tau \\ I \end{bmatrix}^T . \tag{38}$$

Consider the following Riccati equation

$$\begin{aligned} \breve{\Xi} &= \breve{A}\breve{\Xi}\breve{A}^T - (\breve{A}\breve{\Xi}\breve{C}^T + \acute{\acute{S}}) \\ &\quad \times (\breve{C}\breve{\Xi}\breve{C}^T + \acute{\acute{R}}_\tau)^{-1}(\breve{A}\breve{\Xi}\breve{C}^T + \acute{\acute{S}})^T + \acute{\acute{Q}}. \end{aligned} \tag{39}$$

*Lemma 2.* Define $\breve{K}_\tau$ as

$$\breve{K}_\tau := (\breve{A}\breve{\Xi}\breve{C}^T + \acute{\acute{S}})(\breve{C}\breve{\Xi}\breve{C}^T + \acute{\acute{R}}_\tau)^{-1}, \tag{40}$$

in terms of a stabilizing solution of Riccati equation (39). Then, $\breve{A} - \breve{K}_\tau\breve{C}$ is stable.

*Theorem 2.* A spectral factorization of $\acute{\Upsilon}_\tau(z)$ is given by

$$\acute{\Upsilon}_\tau(z) = \breve{W}_\tau(z)\breve{W}_\tau^T(z^{-1}),$$

where $\breve{W}_\tau(z)$ is defined as

$$\breve{W}_\tau(z) := (\breve{C}(zI - \breve{A})^{-1}\breve{K}_\tau + I)\breve{R}_\tau^{\frac{1}{2}}, \tag{41}$$

$$\breve{R}_\tau := \breve{C}\breve{\Xi}\breve{C}^T + \acute{\acute{R}}_\tau. \tag{42}$$

The transfer function $\breve{W}_\tau(z)$ is stable and of minimum phase.

If $\breve{A} - \acute{K}_\tau\breve{C}$ is stable, a stabilizing solution of (39) is given by $\acute{\acute{\Xi}} = 0$ from $\acute{K}_\tau = \acute{\acute{S}}\acute{\acute{R}}_\tau^{-1}$, and we therefore have

$$\breve{K}_\tau := \acute{K}_\tau, \quad \breve{R}_\tau := \acute{\acute{R}}_\tau. \tag{43}$$

### 5.3 *Stochastic subspace identification algorithm*

We summarize a stochastic subspace identification algorithm which provides a stable, minimum phase model based on Theorems 1 and 2.

**A new subspace identification algorithm**

**Steps 1-4:** Compute Steps 1-4 in the stochastic subspace identification algorithm in Section 3.

**Step 5:** If $\acute{A}$ is unstable, find a stabilizing solution of Riccati equation (32) to define $\breve{C}$, $\breve{A}$ and $\acute{\acute{R}}_\tau$ as (33), (34) and (36), respectively. If $\acute{A}$ is stable, define $\breve{C}$, $\breve{A}$ and $\acute{\acute{R}}_\tau$ as (37).

**Step 6:** If $\breve{A} - \acute{K}_\tau\breve{C}$ is unstable, find a stabilizing solution of Riccati equation (39) to define $\breve{K}_\tau$ and $\breve{R}_\tau$ as (40) and (42), respectively. If $\breve{A} - \acute{K}_\tau\breve{C}$ is stable, define $\breve{K}_\tau$ and $\breve{R}_\tau$ as (43).

Steps 5 and 6 give a stable, minimum phase model $\breve{W}_\tau(z)$, which can be used as an approximation to $\hat{W}(z)$, by solving two Riccati equations (32) and (39) [1]. It should be noted that Steps 5 and 6 can be calculated by means of a Matlab function whose inputs are $(\acute{F}, \tilde{C}^T, -\acute{K}_\tau, \acute{R}_\tau)$ and $(\breve{A}, \acute{K}_\tau, \breve{C}, \acute{\acute{R}}_\tau)$, respectively

The computational time of the proposed algorithm compares favorably with former stochastic subspace identification algorithms (Mari *et al.*, 2000; Goethals *et al.*, 2003); in fact, the computation tasks needed to guarantee stability and minimum phase property in Steps 5 and 6 are only solving Riccati equations, while the former algorithms (Mari *et al.*, 2000; Goethals *et al.*, 2003) use numerical optimization methods in order to take positive realness into account.

## 6. NUMERICAL SIMULATION

We present a simulation result to explain feasibility of the subspace identification method proposed in this paper. Simulated data is generated by a system $y_t = \hat{W}(z)e_t$, where $e_t$ is a white noise with zero mean and unit variance, and $\hat{W}(z)$ is given by

$$\begin{aligned} \hat{W}(z) =& W_N(z)/W_D(z), \\ W_N(z) =& 1.0 \times 10^{-3} + 0.0090z^{-1} + 0.0081z^{-2} \\ &+ 0.0073z^{-3} + 0.0066z^{-4} + 0.0059z^{-5}, \\ W_D(z) =& 1 - 2.6908z^{-1} + 4.3502z^{-2} - 4.2269z^{-3} \\ &+ 2.5542z^{-4} - 0.8714z^{-5}. \end{aligned}$$

We estimated the system for 30 simulation runs carried out with different noise realizations where $\tau = 12$, $\nu = 3,000$ and $\tilde{n} = 7$. We confirmed that $\breve{W}_\tau(z)$ computed through Steps 1-6 in Section 5 is stable and of minimum phase in every simulation, while we obtain only 6 stable, minimum phase models for $\acute{W}_\tau(z)$ based on Steps 1-4 in Section 3.

Figure 1 shows Bode plots of the systems $\breve{W}_\tau(z)$ estimated by the present method for 30 simulations. Bode plots of $\breve{W}_\tau(z)$ are clustered around $\hat{W}(z)$.

Figure 2 shows plots of poles and zeros of $\acute{W}_\tau(z)$ and $\breve{W}_\tau(z)$ in a sample of 30 simulations. We observe that

[1] We can derive an alternative method for Step 6 based on the Riccati equation (3).

**Step 6':** Solve Lyapunov equation

$$\breve{X} = \breve{A}\breve{X}\breve{A}^T + \acute{\acute{Q}}.$$

Define $\breve{\Lambda}_0$ and $\breve{G}$ as

$$\breve{\Lambda}_0 = \breve{C}\breve{X}\breve{C}^T + \acute{\acute{R}}_\tau, \quad \breve{G} = \breve{A}\breve{X}\breve{C}^T + \acute{\acute{S}}.$$

Find a stabilizing solution $\breve{P}$ of the following Riccati equation:

$$\begin{aligned} \breve{P} &= \breve{A}\breve{P}\breve{A}^T + (\breve{G} - \breve{A}\breve{P}\breve{C}^T) \\ &\quad \times (\breve{\Lambda}_0 - \breve{C}\breve{P}\breve{C}^T)^{-1}(\breve{G} - \breve{A}\breve{P}\breve{C}^T)^T \end{aligned} \tag{44}$$

Define $\breve{K}_\tau$ and $\breve{R}_\tau$ as

$$\breve{K}_\tau = (\breve{G} - \breve{A}\breve{P}\breve{C}^T)(\breve{\Lambda}_0 - \breve{C}\breve{P}\breve{C}^T)^{-1}, \tag{45}$$

$$\breve{R}_\tau = \breve{\Lambda}_0 - \breve{C}\breve{P}\breve{C}^T. \tag{46}$$

Riccati equation (44) is always solvable from (38). Moreover, $\breve{K}_\tau$ and $\breve{R}_\tau$ in (45) and (46) coincide with the ones in (40) and (42), respectively, since stabilizing solutions for (39) and (44), $\breve{P}$ and $\breve{\Xi}$, satisfy $\breve{P} = \breve{X} - \breve{\Xi}$.

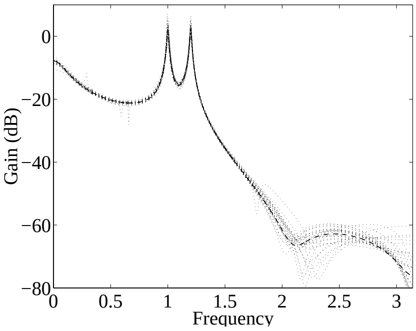
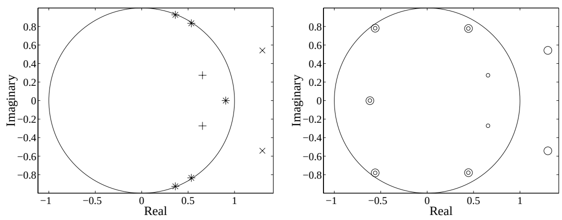



Fig. 1. Bode plots of $W(z)$ and $\breve{W}_\tau(z)$ where the dotted line expresses the plots of $\acute{W}_j$ for 30 simulation runs.

only unstable poles and zeros of $\acute{W}_\tau(z)$ are reflected into the unit circle.




(a) Poles

(b) Zeros

Fig. 2. Poles and zeros of $\acute{W}_\tau(z)$ and $\breve{W}_\tau(z)$, where "$\times$" and "$+$" in (a) express poles of $\acute{W}_\tau(z)$ and $\breve{W}_\tau(z)$, respectively, and where "$\bigcirc$" and "$\circ$" in (b) express zeros of $\acute{W}_\tau(z)$ and $\breve{W}_\tau(z)$, respectively.

## 7. CONCLUSIONS

We developed a stochastic subspace identification method which guarantees stability and minimum phase property. Two Riccati equations are solved to find a stable, minimum phase model $\breve{W}_\tau(z)$. A model $\acute{W}_\tau(z)$ is obtained without solving Riccati equations, though it is not always stable and of minimum phase.

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