DECENTRALISED SLIDING MODE CONTROL FOR NONMINIMUM PHASE NONLINEAR INTERCONNECTED SYSTEMS

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Abstract: A class of interconnected systems with nonlinear interconnections and nonlinear disturbances is considered. A continuous nonlinear reduced-order compensator is established by exploiting the structure of the uncertainties. A sliding surface is proposed in an augmented space formed by the system output and the compensator variables, and the stability of the corresponding sliding mode is analysed. Then, a robust decentralised dynamical output feedback sliding mode controller is designed to drive the system to the composite sliding surface and maintain a sliding motion on it thereafter.

Keywords: nonlinear interconnected system, sliding modes, nonminimum phase

1. INTRODUCTION

Sliding mode techniques are employed to study the stabilization of a class of nonlinear interconnected systems. Mismatched uncertainties and nonlinear interconnections are considered, and the bounds on the uncertainties take more general forms as in (Yan *et al.* 2004),(Yan and Xie 2003). By using the structure of the uncertainties, a continuous reduced-order compensator is proposed based on constrained Lyapunov equations. Then, a sliding surface is proposed in the augmented space formed by the compensator and system output. Using an equivalent control approach and a local coordinate transformation, the sliding mode dynamics are established and the stability is analysed. A robust decentralised output feedback sliding mode control scheme is synthesized such that the interconnected system can be driven to the pre-designed sliding surface. This approach allows both the nominal isolated subsystem and the whole nominal system to be nonminimum phase. It should be emphasised that methods to deal with nonlinear interconnections are a key issue in the control of interconnected systems. So far nearly all associated work treats such interconnections as disturbances and then uses an extra stability margin to reject the effect of the interconnections. By dealing with uncertain interconnections and known interconnections separately, the conservatism is reduced to some extent as claimed in (Yan and Xie 2003). However, the interconnections are still treated as a disturbance in the sense that the interconnections are not used explicitly in the control design. In this paper, it is shown that by employing sliding mode techniques, the interconnections are directly used in the control design, which together with the fact that the sliding mode dynamics are reduced-order systems, reduces conservatism and enhances robustness.

Notation: For a square matrix A, $\underline{\lambda}(A)$ and $\overline{\lambda}(A)$ denote the minimum and maximum eigenvalues

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respectively. $A > 0$ means that A is positive definite. I_n denotes the unit matrix with dimension n. The set of $n\times m$ matrices with elements defined in $\mathbb{R}$ will be denoted by $\mathbb{R}^{n\times m}$. For a function/vector $f(x)$, L_f denote its Lipschitz constant in an associated domain. $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

2. SYSTEM DESCRIPTION

Consider a nonlinear interconnected system composed of N subsystems as follows

$$\dot{x}_i = A_i x_i + B_i u_i + \Delta f_i(x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N}\Big(H_{ij}(x_j) + \Delta H_{ij}(x_j,t)\Big), \qquad (1)$$

$$y_i = C_i x_i, \qquad i = 1,2,\ldots,N, \qquad (2)$$

where $x_i \in \Omega_i \subset \mathbb{R}^{n_i}$ $(0 \in \Omega_i)$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the i-th subsystem respectively with $m_i < n_i$; (A_i, B_i, C_i) are constant matrices of appropriate dimensions with B_i and C_i of full rank; Δf_i is the mismatched uncertainty of the i-th isolated subsystem, $\sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}$ and $\sum_{\substack{j=1\\ j\neq i}}^{N} \Delta H_{ij}$ are respectively the known and the uncertain interconnections of the i-th subsystem with $H_{ij}(0) = 0$. The functions are all assumed to be continuous in their arguments.

Without loss of generality, suppose that the nonlinear functions $H_{ij}(\cdot)$ have decompositions

$$H_{ij}(x_j) = \Phi_{ij}(x_j)x_j, \qquad i\neq j, \quad i,j = 1,2\ldots,N \qquad (3)$$

where $\Phi_{ij}(\cdot)$ are continuous. The decomposition (3) is always true for $H_{ij}(\cdot)$ smooth enough in their domain of definition satisfying $H_{ij}(0) = 0$.

In order to facilitate the analysis,

- All equations and inequalities involving the indexes i and/or j are satisfied for all $i,j = 1,2,\ldots,N$ $(i\neq j)$;
- The considered domain is

  $$x = \mathrm{col}(x_1, x_2, \ldots, x_N) \in \Omega \equiv: \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$$

  with $x_i \in \Omega_i \subset \mathbb{R}^{n_i}$;
- Output matrices

  $$C_i = [\, I_{p_i} \quad 0 \,]$$

Assumption 1. The matrix pairs (A_i, B_i) and (A_i, C_i) are controllable and detectable respectively, and the function $H_{ij}(x_j)$ $(i \neq j)$ satisfies Lipschitz conditions in the considered domain.

In view of the detectability of (A_i, C_i), there exists a matrix L_i such that $(A_i - L_iC_i)$ is stable and thus for any $Q_i > 0$ the following Lyapunov equation has a unique solution $P_i > 0$

$$(A_i - L_iC_i)^\tau P_i + P_i(A_i - L_iC_i) = -Q_i \qquad (4)$$

Assumption 2. The uncertainties have structural decompositions of the following form

$$\Delta f_i(x_i,t) = D_i \Delta \widetilde{f}_i(x_i,t), \qquad \Delta H_{ij}(x_j,t) = E_{ij}\Delta\widetilde{H}_{ij}(x_j,t) \qquad (5)$$

where D_i, E_{ij} $(i\neq j)$ are constant matrices, and

$$\|\Delta\widetilde{f}_i(x_i,t)\| \le \rho_i(y_i,t)\gamma_i(x_i,t), \qquad \|\Delta\widetilde{H}_{ij}(x_j,t)\| \le \vartheta_{ij}(y_j,t)\zeta_{ij}(x_j,t) \qquad (6)$$

where $\gamma_i \le \widetilde{\gamma}_i(x_i,t)\|x_i\|$ and $\zeta_{ij} \le \widetilde{\zeta}_{ij}(x_j,t)\|x_j\|$ $(i\neq j)$ are Lipschitz with $\widetilde{\gamma}_i$ and $\widetilde{\zeta}_{ij}$ continuous.

Assumption 3. There exist matrices G_i and F_{ij} $(i\neq j)$ such that

$$D_i^\tau P_i = G_iC_i, \qquad E_{ij}^\tau P_i = F_{ij}C_i \qquad (7)$$

where P_i satisfies (4) and the matrices D_i, E_{ij} $(i\neq j)$ satisfy (5).

It should be noted that Assumption 3 implies that

$$\mathrm{rank}(D_i^\tau P_i) = \mathrm{rank}([D_i^\tau P_i \quad C_i])$$
$$\mathrm{rank}(E_{ij}^\tau P_i) = \mathrm{rank}([E_{ij}^\tau P_i \quad C_i])$$

with $i,j = 1,2,\ldots,N \quad (i\neq j)$.

The objective of this paper is to use sliding mode techniques to develop an output feedback control scheme based on a continuous reduced-order compensator such that the corresponding closed-loop system is asymptotically stable.

3. COMPENSATOR DESIGN

Consider system (1)–(2). Following the partition of $C_i = [\, I_{p_i} \quad 0\,]$, the system can be rewritten

$$\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\end{bmatrix} = \begin{bmatrix}A_{i1} & A_{i2}\\ A_{i3} & A_{i4}\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\end{bmatrix} + \begin{bmatrix}B_{i1}\\ B_{i2}\end{bmatrix}u_i + \begin{bmatrix}D_{i1}\\ D_{i2}\end{bmatrix}\Delta\widetilde{f}_i + \begin{bmatrix}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(H_{ij1} + E_{ij1}\Delta\widetilde{H}_{ij}\Big)\\ \sum_{\substack{j=1\\ j\neq i}}^{N}\Big(H_{ij2} + E_{ij2}\Delta\widetilde{H}_{ij}\Big)\end{bmatrix} \qquad (8)$$

$$y_i = x_{i1} \qquad (9)$$

where $x_i = \mathrm{col}(x_{i1}, x_{i2})$ with $x_{i1} \in \mathbb{R}^{p_i}$, $A_{i1} \in \mathbb{R}^{p_i\times p_i}$, $B_{i1} \in \mathbb{R}^{p_i\times m_i}$; D_{i1}, E_{ij1} and H_{ij1} are the first p_i rows of D_i, E_{ij} and $H_{ij}(x_j)$ respectively.

Partition P_i, Q_i and L_i conformably with the decomposition (8)–(9) as

$$P_i = \begin{bmatrix}P_{i1} & P_{i2}\\ P_{i2}^\tau & P_{i3}\end{bmatrix}, Q_i = \begin{bmatrix}Q_{i1} & Q_{i2}\\ Q_{i2}^\tau & Q_{i3}\end{bmatrix}$$

$$L_i = \begin{bmatrix}L_{i1}\\ L_{i2}\end{bmatrix} \qquad (10)$$

Then, construct a dynamical system

$$\dot{\hat z}_{i2} = (A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2})\hat z_{i2} + (P_{i3}^{-1}P_{i2}^{\tau}(A_{i1} - A_{i2}P_{i3}^{-1}P_{i2}^{\tau}) + A_{i3} - A_{i4}P_{i3}^{-1}P_{i2}^{\tau})y_i + \left(P_{i3}^{-1}P_{i2}^{\tau}B_{i1} + B_{i2}\right)u_i + \sum_{\substack{j=1\\ j\neq i}}^{N}[P_{i3}^{-1}P_{i2}^{\tau}H_{ij1}(y_j,\hat\nu_j) + H_{ij2}(y_j,\hat\nu_j)]$$

where $\hat\nu_j = \hat z_{j2} - P_{j3}^{-1}P_{j2}y_j$ and $\hat z_{i2} \in \mathbb{R}^{n_i-p_i}$. The following conclusion can be drawn:

Theorem 1. Let $\hat x_{i2} = -P_{i3}^{-1}P_{i2}^{\tau}y_i + \hat z_{i2}$ with $\hat z_{i2}$ as given above. Then, under Assumptions 1–3 there exist positive constants α_1 and α_2 such that

$$\|x_{i2}(t) - \hat x_{i2}(t)\| \le \alpha_1 \exp\{-\alpha_2 t\} \tag{11}$$

if $W^T + W$ is positive definite with $W = (w_{ij})_{N\times N}$ defined by

$$w_{ij} = \begin{cases} \underline{\lambda}(Q_{i3}), & i = j \\ -2\left(\|P_{i2}\|L_{H_{ij1}} + \|P_{i3}\|L_{H_{ij2}}\right), & i \neq j \end{cases}$$

where H_{ij1} and H_{ij2} are, respectively, the first p_i and the last $n_i - p_i$ components of $H_{ij}(x_j)$, and P_{i2}, P_{i3} and Q_{i3} are defined by (10).

Proof: From Assumption 3, $C_i = [I_{p_i} \quad 0]$ and the partition (10) of P_i, it follows that

$$P_{i2}^{\tau}D_{i1} + P_{i3}D_{i2} = 0, \tag{12}$$

$$P_{i2}^{\tau}E_{ij1} + P_{i3}E_{ij2} = 0 \quad (i \neq j) \tag{13}$$

Introduce a nonsingular coordinate transformation $z_i = \hat T_i x_i$ defined by

$$\hat T_i : \begin{cases} z_{i1} = x_{i1} \\ z_{i2} = P_{i3}^{-1}P_{i2}^{\tau}x_{i1} + x_{i2} \end{cases} \tag{14}$$

Since (12)-(13) implies $P_{i3}^{-1}P_{i2}^{\tau}D_{i1} + D_{i2} = 0$ and $P_{i3}^{-1}P_{i2}^{\tau}E_{ij1} + E_{ij2} = 0$, it follows from (8)–(9) that in the new coordinates $z = \mathrm{col}(z_{i1},\dots,z_{iN})$, system (1)–(2) is described by

$$\dot z_{i1} = \left(A_{i1} - A_{i2}P_{i3}^{-1}P_{i2}^{\tau}\right)z_{i1} + A_{i2}z_{i2} + B_{i1}u_i + D_{i1}\Delta\widetilde{f}_i + \sum_{\substack{j=1\\ j\neq i}}^{N}(H_{ij1} + E_{ij1}\Delta\widetilde{H}_{ij})$$

$$\dot z_{i2} = \Big(P_{i3}^{-1}P_{i2}^{\tau}(A_{i1} - A_{i2}P_{i3}^{-1}P_{i2}^{\tau}) + A_{i3} - A_{i4}P_{i3}^{-1}P_{i2}^{\tau}\Big)z_{i1} + (A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2})z_{i2} + \left(P_{i3}^{-1}P_{i2}^{\tau}B_{i1} + B_{i2}\right)u_i + \sum_{\substack{j=1\\ j\neq i}}^{N}\Big[P_{i3}^{-1}P_{i2}^{\tau}H_{ij1}(y_j,\nu_j) + H_{ij2}(y_j,\nu_j)\Big]$$

$$y_i = z_{i1}$$

where $\nu_j = z_{j2} - P_{j3}^{-1}P_{j2}^{\tau}z_{j1}$. From the above, (14), and $\hat x_{i2} = -P_{i3}^{-1}P_{i2}^{\tau}y_i + \hat z_{i2}$, it follows that

$$x_{i2} - \hat x_{i2} = x_{i2} + P_{i3}^{-1}P_{i2}^{\tau}y_i - \hat z_{i2} = z_{i2} - \hat z_{i2}$$

It is only required to prove that $\|z_{i2} - \hat z_{i2}\| \le \alpha_1\exp\{-\alpha_2 t\}$ for positive constants α_1 and α_2.

Let $e_i = z_{i2} - \hat z_{i2}$. Substituting from the dynamics,

$$\dot e_i = \left(A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2}\right)e_i + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\Big\{P_{i3}^{-1}P_{i2}^{\tau}\left[H_{ij1}(y_j,\nu_j) - H_{ij1}(y_j,\hat\nu_j)\right] + H_{ij2}(y_j,\nu_j) - H_{ij2}(y_j,\hat\nu_j)\Big\} \tag{15}$$

where $\nu_j = z_{j2} - P_{j3}^{-1}P_{j2}^{\tau}y_j$ and $\hat\nu_j = \hat z_{j2} - P_{j3}^{-1}P_{j2}^{\tau}y_j$. For system (15), consider a Lyapunov function candidate $V_1 = \sum_{i=1}^{N} e_i^{\tau}P_{i3}e_i$. Then, the time derivative of V_1 along the trajectories of system (15) is described by

$$\dot V_1 = \sum_{i=1}^{N} e_i^{\tau}\Big(\left[A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2}\right]^{\tau}P_{i3} + P_{i3}\left[A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2}\right]\Big)e_i + 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} e_i^{\tau}\Big(P_{i2}^{\tau}\left[H_{ij1}(y_j,\nu_j) - H_{ij1}(y_j,\hat\nu_j)\right] + P_{i3}\left[H_{ij2}(y_j,\nu_j) - H_{ij2}(y_j,\hat\nu_j)\right]\Big) \tag{16}$$

From (4), (10) and $C_i = [I_{p_i} \quad 0]$, it follows that

$$\left(P_{i3}^{-1}P_{i2}^{\tau}A_{i2} + A_{i4}\right)^{\tau}P_{i3} + P_{i3}\left(P_{i3}^{-1}P_{i2}^{\tau}A_{i2} + A_{i4}\right) = -Q_{i3} \tag{17}$$

Since Assumption 2 implies that both H_{ij1} and H_{ij2} are Lipschitz in their domain of definition, $L_{H_{ij1}}$ and $L_{H_{ij2}}$ are well defined. Then, substituting (17) into (16), it is observed from $\nu_i - \hat\nu_i = e_i$ that

$$\begin{aligned}\dot V_1 &\le -\sum_{i=1}^{N} e_i^{\tau}Q_{i3}e_i + 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\|P_{i2}\|L_{H_{ij1}} + \|P_{i3}\|L_{H_{ij2}}\Big)\|e_i\|\,\|e_j\| \\ &\le -\frac{1}{2}[\|e_1\|\cdots\|e_N\|](W + W^{\tau})[\|e_1\|\cdots\|e_N\|]^{\tau} \\ &\le -\frac{\underline{\lambda}(W + W^{\tau})}{2\max_i\left\{\overline{\lambda}(P_{i3})\right\}}V_1\end{aligned}$$

This implies $V_1 \le (V_1\ |_{t=0})\exp\{-\frac{\underline{\lambda}(W+W^{\tau})}{2\max_i\left\{\overline{\lambda}(P_{i3})\right\}}t\}$. Then, from

$$\min_i\{\underline{\lambda}(P_{i3})\}\|e_i\|^2 \le e_i^{\tau}P_{i3}e_i \le \sum_{i=1}^{N} e_i^{\tau}P_{i3}e_i = V_1$$

the conclusion follows if $\alpha_1 > \sqrt{\frac{V_1|_{t=0}}{\max_i\{\underline{\lambda}(P_{i3})\}}}$ and $\alpha_2 \ge \frac{\underline{\lambda}(W+W^{\tau})}{2\max_i\left\{\overline{\lambda}(P_{i3})\right\}}$. □

4. SLIDING MODE ANALYSIS

For system (1)–(2), consider the sliding surface

$$\sigma \equiv: \mathrm{col}(\sigma_1, \sigma_2, \dots, \sigma_N) = 0 \tag{18}$$

$$\sigma_i(y_i, \hat x_{i2}) = S_{i1}y_i + S_{i2}\hat x_{i2} \tag{19}$$

where $\hat x_{i2}$ is the compensator state in Theorem 1, and $S_{i1} \in \mathbb{R}^{m_i\times p_i}$ and $S_{i2} \in \mathbb{R}^{m_i\times(n_i-p_i)}$ are the design parameters.

As in the proof of Theorem 1, let $e_i = x_{i2} - \hat{x}_{i2}$ and define $S_i = [S_{i1} \quad S_{i2}]$. In the new coordinate system (x_i, e_i), the sliding function matrices

$$\sigma_i = [\, S_{i1} \quad S_{i2} \,]\, x_i - S_{i2} e_i = S_i x_i - S_{i2} e_i. \quad (20)$$

The matrices S_i can be chosen using any existing state feedback sliding mode design approach on the pairs (A_i, B_i) such that:

i) the matrices $S_i B_i$ are nonsingular;
ii) the matrices $A_{eqi} \equiv: A_i - B_i \left(S_i B_i\right)^{-1} S_i A_i$ have $n_i - m_i$ eigenvalues which lie in the open left-half plane.

During a sliding motion, both $\sigma_i = 0$ and $\dot{\sigma}_i = 0$. From (1), (15), (20) and $\dot{\sigma}_i = 0$, the equivalent control (Utkin 1978) necessary to maintain a sliding motion is given by

$$u_{ieq} = -(S_i B_i)^{-1} \Big\{ S_i A_i x_i - S_{i2} \left(A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \right) e_i + S_i \Delta f_i + \sum_{\substack{j=1\\ j \neq i}}^{N} S_i \Big(H_{ij}(x_j) + \Delta H_{ij} \Big) - S_{i2} \sum_{\substack{j=1\\ j \neq i}}^{N} \Big(P_{i3}^{-1} P_{i2}^{\tau} \Big(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \Big) + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \Big) \Big\}$$

When system (1)–(2) is restricted to the sliding surface (18), it follows by applying the control above to system (1) that the associated dynamics are given by

$$\begin{bmatrix} \dot{x}_i \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} A_{eqi} & B_i (S_i B_i)^{-1} S_{i2} \left(A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \right) \\ 0 & A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \end{bmatrix} \begin{bmatrix} x_i \\ e_i \end{bmatrix} + \begin{bmatrix} \left(I_{n_i} - B_i (S_i B_i)^{-1} S_i \right) \Big(\Delta f_i + \sum_{\substack{j=1\\ j \neq i}}^{N} (H_{ij} + \Delta H_{ij}) \Big) \\ 0 \end{bmatrix} + \begin{bmatrix} B_i (S_i B_i)^{-1} S_{i2} \\ I_{n_i - p_i} \end{bmatrix} \times \sum_{\substack{j=1\\ j \neq i}}^{N} \Big\{ P_{i3}^{-1} P_{i2}^{\tau} \Big(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \Big) + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \Big\}$$

where $A_{eqi} = A_i - B_i (S_i B_i)^{-1} S_i A_i$. Since $S_i B_i$ is nonsingular, matrix S_i is full row rank and thus there exist nonsingular matrices $T_{i1} \in \mathbb{R}^{n_i \times n_i}$ and $T_{i2} \in \mathbb{R}^{m_i \times m_i}$ such that

$$T_{i2} S_i T_{i1} = [\, I_{m_i} \quad 0 \,] \quad (21)$$

In order to further analyse the stability of the sliding mode, it is required to derive a reduced order representation. The coordinate transformation $\mathrm{col}(\xi_i, \eta_i) = T_{i1}^{-1} x_i$ is introduced, where $\xi_i \in \mathbb{R}^{m_i}$ and T_{i1} is determined by (21). Then, noticing the condition ii), it follows that in the new coordinates (ξ_i, η_i, e_i), the system is described by the equations

$$\begin{bmatrix} \dot{\xi}_i \\ \dot{\eta}_i \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & * \\ \widetilde{A}_{i1} & \widetilde{A}_{i2} & \widetilde{A}_{i3} \\ 0 & 0 & A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \\ e_i \end{bmatrix} + \begin{bmatrix} * \\ \Delta f_{i1} \\ 0 \end{bmatrix} + \sum_{\substack{j=1\\ j \neq i}}^{N} \begin{bmatrix} * \\ \Pi_{ij} \\ 0 \end{bmatrix} + \sum_{\substack{j=1\\ j \neq i}}^{N} \begin{bmatrix} * \\ \Theta_{ij1} \\ \Theta_{ij2} \end{bmatrix} \quad (22)$$

where $\widetilde{A}_{i2} \in \mathbb{R}^{(n_i - m_i) \times (n_i - m_i)}$, $\widetilde{A}_{i3} \in \mathbb{R}^{(n_i - m_i) \times (n_i - p_i)}$, and

$$\begin{bmatrix} 0 & 0 \\ \widetilde{A}_{i1} & \widetilde{A}_{i2} \end{bmatrix} = T_{i1}^{-1} A_{eqi} T_{i1} \quad (23)$$

The notation $*$ denotes items which do not play a role in the subsequent analysis; Δf_{i1}, Π_{ij} and Θ_{ij1} are the last $n_i - m_i$ components of

$$T_{i1}^{-1} \left(I_{n_i} - B_i (S_i B_i)^{-1} S_i \right) \Delta f_i,$$
$$T_{i1}^{-1} \left(I_{n_i} - B_i (S_i B_i)^{-1} S_i \right) (H_{ij} + \Delta H_{ij})$$

and

$$T_{i1}^{-1} B_i (S_i B_i)^{-1} S_{i2} \{ P_{i3}^{-1} P_{i2}^{\tau} \Big(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \Big) + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \}$$

respectively, and

$$\Theta_{ij2} = P_{i3}^{-1} P_{i2}^{\tau} \left(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \right) + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \quad (24)$$

From (21), it follows that

$$\sigma_i = S_i x_i - S_{i2} e_i = T_{i2}^{-1} [\, I_{m_i} \quad 0 \,] \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} - S_{i2} e_i = T_{i2}^{-1} \xi_i - S_{i2} e_i \quad (25)$$

This implies that in the new coordinate system (ξ_i, η_i, e_i), $\sigma_i = 0$ can be depicted by $\xi_i = T_{i2} S_{i2} e_i$. Consequently, when the system is restricted to the sliding surface (18), it can be described in coordinate system (ξ_i, η_i, e_i) by

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{i2} & \widetilde{A}_{i3} + \widetilde{A}_{i1} T_{i2} S_{i2} \\ 0 & A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \end{bmatrix} \begin{bmatrix} \eta_i \\ e_i \end{bmatrix} + \begin{bmatrix} \Delta f_{i1} \\ 0 \end{bmatrix} + \sum_{\substack{j=1\\ j \neq i}}^{N} \begin{bmatrix} \Pi_{ij} \\ 0 \end{bmatrix} + \sum_{\substack{j=1\\ j \neq i}}^{N} \begin{bmatrix} \Theta_{ij1} \\ \Theta_{ij2} \end{bmatrix}$$

From condition ii) and (23), the matrix $\widetilde{A}_{i2}$ has $n_i - m_i$ negative eigenvalues. This implies that for any $\widetilde{Q}_i > 0$, the Lyapunov equations

$$\widetilde{A}_{i2}^{\tau} \widetilde{P}_i + \widetilde{P}_i \widetilde{A}_{i2} = -\widetilde{Q}_i \quad (26)$$

have unique solutions $\widetilde{P}_i > 0$.

From (3) and Assumption 2, there exist continuous functions φ_{i1}, φ_{i2}, ψ_{ij} and χ_{ij} such that

$$\|\widetilde{P}_i \Delta f_{i1}\| \leq \varphi_{i1}(\eta_i, e_i) \|\eta_i\| + \varphi_{i2}(\eta_i, e_i) \|e_i\|$$
$$\|\widetilde{P}_i \left(\Pi_{ij} + \Theta_{ij1} \right)\| \leq \psi_{ij}(\eta_j, e_j) \|\eta_j\| + \chi_{ij}(\eta_j, e_j) \|e_j\|$$

where $\widetilde{P}_i$ satisfies (26).

Theorem 2. Under Assumptions 1–3, the sliding mode dynamics are asymptotically stable if there exists a domain of the origin $\mathcal{O}_i \subset \mathbb{R}^{2n_i - m_i - p_i}$ such that for $col(\eta_1, e_1, \ldots, \eta_N, e_N) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_N$, the matrix $M^{\tau} + M$ is positive definite with $M \in \mathbb{R}^{2N \times 2N}$ defined by

$$
\left[\begin{array}{cccc|cccc}
\underline{\lambda}(\widetilde{Q}_1)-2\varphi_{11} & -2\psi_{12} & \cdots & -2\psi_{1N} & -2(\varphi_{12}+\varpi_1) & -2\chi_{12} & \cdots & -2\chi_{1N} \\
-2\psi_{21} & \underline{\lambda}(\widetilde{Q}_2)-2\varphi_{21} & \ddots & \vdots & -2\chi_{21} & -2(\varphi_{22}+\varpi_2) & \ddots & \vdots \\
\vdots & \ddots & \ddots & -2\psi_{(N-1)N} & \vdots & \ddots & \ddots & -2\chi_{(N-1)N} \\
-2\psi_{N1} & \cdots\cdots & -2\psi_{N(N-1)} & \underline{\lambda}(\widetilde{Q}_N)-2\varphi_{N1} & -2\chi_{N1} & \cdots\cdots & -2\chi_{N(N-1)} & -2(\phi_{N2}+\varpi_N) \\
\hline
-2(\varphi_{12}+\varpi_1) & -2\chi_{12} & \cdots & -2\chi_{1N} & \underline{\lambda}(Q_{13}) & -2\kappa_{12} & \cdots & -2\kappa_{1N} \\
-2\chi_{21} & -2(\varphi_{22}+\varpi_2) & \ddots & \vdots & -2\kappa_{21} & \underline{\lambda}(Q_{23}) & \ddots & \vdots \\
\vdots & \ddots & \ddots & -2\chi_{(N-1)N} & \vdots & \ddots & \ddots & -2\kappa_{(N-1)N} \\
-2\chi_{N1} & \cdots\cdots & -2\chi_{N(N-1)} & -2(\varphi_{N2}+\varpi_N) & -2\kappa_{N1} & \cdots\cdots & -2\kappa_{N(N-1)} & \underline{\lambda}(Q_{N3})
\end{array}\right]
$$

where φ_{i1}, φ_{i2}, ψ_{ij} and χ_{ij} are determined by the equations above, $\kappa_{ij} :\equiv (\|P_{i2}\|L_{H_{ij1}} + \|P_{i3}\|L_{H_{ij2}})$ and $\varpi_i :\equiv \|\widetilde{P}_i(\widetilde{A}_{i3} + \widetilde{A}_{i1}T_{i2}S_{i2})\|$.

Proof: Consider a Lyapunov function $V = \sum_{i=1}^{N}\left(\eta_i^\tau \widetilde{P}_i\eta_i + e_i^\tau P_{i3}e_i\right)$. Then, the time derivative of V along the trajectories of the system is given by

$$
\begin{aligned}
\dot{V} = & -\sum_{i=1}^{N}\eta_i^\tau \widetilde{Q}_i\eta_i + 2\sum_{i=1}^{N}\eta_i^\tau \widetilde{P}_i\left(\widetilde{A}_{i3} + \widetilde{A}_{i1}T_{i2}S_{i2}\right)e_i \\
& +2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\eta_i^\tau \widetilde{P}_i\left(\Pi_{ij} + \Theta_{ij1}\right) + 2\sum_{i=1}^{N}\eta_i^\tau \widetilde{P}_i\Delta f_{i1} \\
& -\sum_{i=1}^{N}e_i^\tau Q_{i3}e_i + 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}e_i^\tau P_{i3}\Theta_{ij2} \\
\leq & -\sum_{i=1}^{N}\left(\underline{\lambda}(\widetilde{Q}_i)\|\eta_i\|^2 + \underline{\lambda}(Q_{i3})\|e_i\|^2\right) \\
& +2\sum_{i=1}^{N}\left\|\widetilde{P}_i\left(\widetilde{A}_{i3} + \widetilde{A}_{i1}T_{i2}S_{i2}\right)\right\|\|e_i\|\,\|\eta_i\| \\
& +2\sum_{i=1}^{N}\left\|\widetilde{P}_i\Delta f_{i1}\right\|\,\|\eta_i\| + 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big\{ \\
& \left\|\widetilde{P}_i\left(\Pi_{ij} + \Theta_{ij1}\right)\right\|\,\|\eta_i\| + \|P_{i3}\Theta_{ij2}\|\,\|e_i\|\Big\} \quad (27)
\end{aligned}
$$

where (17) and (26) are used above. From (24),

$$
\begin{aligned}
\|P_{i3}\Theta_{ij2}\| = & \Big\|P_{i2}^\tau\Big(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2})\Big) + \\
& P_{i3}\Big(H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2})\Big)\Big\| \\
\leq & (\|P_{i2}\|_{H_{ij1}} + \|P_{i3}\|_{H_{ij2}})\|e_j\| = \kappa_{ij}\|e_j\|
\end{aligned}
$$

Then, from the above, and the definitions of the functions φ_{i1}, φ_{i2}, ψ_{ij} and χ_{ij}:

$$
\begin{aligned}
\dot{V} \leq & -\sum_{i=1}^{N}(\underline{\lambda}(\widetilde{Q}_i) - 2\varphi_{i1})\|\eta_i\|^2 - \sum_{i=1}^{N}\underline{\lambda}(Q_{i3})\|e_i\|^2 \\
& +2\sum_{i=1}^{N}(\varphi_{i2} + \varpi_i)\|\eta_i\|\,\|e_i\| \\
& +2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big\{\psi_{ij}\|\eta_i\|\,\|\eta_j\| + \chi_{ij}\|\eta_i\|\,\|e_j\| \\
& +\kappa_{ij}\|e_i\|\,\|e_j\|\Big\} \\
= & \frac{1}{2}\left[\|\eta_1\| \cdots \|\eta_N\|\; \|e_1\| \cdots \|e_N\|\right] \\
& (M^\tau + M)\left[\|\eta_1\| \cdots \|\eta_N\|\; \|e_1\| \cdots \|e_N\|\right]^\tau
\end{aligned}
$$

Hence, the conclusion follows by the positive definiteness of $M^\tau + M$. □

5. SLIDING MODE CONTROLLER

For the system (1)–(2) with the designed composite sliding surface (18), construct the following sliding mode control

$$
\begin{aligned}
u_i = & -(S_iB_i)^{-1}\Big\{(S_{i1}A_{i1} + S_{i2}A_{i3})\,y_i + \Big(S_{i1}A_{i2} + \\
& S_{i2}A_{i4}\Big)\hat{x}_{i2} + \Big(\|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t) \\
& +K_i(y_i,t) + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\|S_jE_{ji}\|\vartheta_{ji}(y_i,t)\zeta_{ji}(y_i,\hat{x}_{i2},t) + \\
& \|S_jH_{ji}(y_i,\hat{x}_{i2})\|\Big)\Big)\frac{\sigma_i}{\|\sigma_i\|}\Big\}
\end{aligned}
$$

where σ_i is defined by (19), and $K_i(y_i,t)$ is the control gain to be determined later. The control law is decentralised and only depends on the $\hat{x}_{i2}$ and the system output y_i. It is necessary to show that the above control can drive the system (1)–(2) to the sliding surface (18) and maintain sliding. It is required to prove that the composite reachability condition (see (Hsu 1997))

$$
\sum_{i=1}^{N}\frac{\sigma_i^\tau(y_i,\hat{x}_{i2})\dot{\sigma}_i(y_i,\hat{x}_{i2})}{\|\sigma_i(y_i,\hat{x}_{i2})\|} < 0. \qquad (28)
$$

is satisfied, where $\sigma_i(y_i,\hat{x}_{i2})$ defined by (19) is the sliding function for the i-th subsystem.

Theorem 3. Under Assumptions 1–3 with (11) satisfied, the controller drives the system (1)–(2) to the composite sliding surface (18) and maintain a sliding motion thereafter if the control gains K_i are chosen such that

$$
\begin{aligned}
K_i(y_i,t) > & \alpha_1\exp\{-\alpha_2 t\}\Big\{\|S_{i1}A_{i2} + S_{i2}A_{i4}\| + \\
& L_{\gamma_i}\|S_iD_i\|\rho_i(y_i,t) + \|S_{i2}P_{i3}^{-1}(P_{i2}^\tau A_{i2} + P_{i3}A_{i4})\| \\
& +\textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\|S_j\|L_{H_{ji}} + \|S_jE_{ji}\vartheta_{ji}\|L_{\zeta_{ji}} + \\
& \|S_{i2}P_{i3}^{-1}\|\left(\|P_{i2}\|L_{H_{ij1}} + \|P_{i3}\|L_{H_{ij2}}\right)\Big)\Big\}
\end{aligned}
$$

with the constants α_1 and α_2 determined by (11).

Proof: From the proof of Theorem 1 the error dynamics in (15) can be rewritten

$$
\begin{aligned}
\dot{e}_i = & P_{i3}^{-1}\left(P_{i2}^\tau A_{i2} + P_{i3}A_{i4}\right)e_i + \sum_{\substack{j=1\\ j\neq i}}^{N}P_{i3}^{-1}\times \\
& \Big\{P_{i2}^\tau\left(H_{ij1}(y_j,x_{j2}) - H_{ij1}(y_j,\hat{x}_{j2})\right) \\
& +P_{i3}\left(H_{ij2}(y_j,x_{j2}) - H_{ij2}(y_j,\hat{x}_{j2})\right)\Big\} \quad (29)
\end{aligned}
$$

From (20), (1) and (29)

$$\dot{\sigma}_i = S_iA_ix_i + S_iB_iu_i + S_i\Delta f_i + \sum_{\substack{j=1\\ j\neq i}}^{N} S_i\Big(H_{ij}(x_j) + \Delta H_{ij}(x_j,t)\Big) - S_{i2}P_{i3}^{-1}\left(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4}\right)e_i - \sum_{\substack{j=1\\ j\neq i}}^{N} S_{i2}P_{i3}^{-1}\Big\{P_{i2}^{\tau}\left(H_{ij1}(y_j,x_{j2}) - H_{ij1}(y_j,\hat{x}_{j2})\right) + P_{i3}\left(H_{ij2}(y_j,x_{j2}) - H_{ij2}(y_j,\hat{x}_{j2})\right)\Big\} \tag{30}$$

Then, substituting the proposed control u_i into the above equation,

$$\sum_{i=1}^{N}\frac{\sigma_i^{\tau}\dot{\sigma}_i}{\|\sigma_i\|} = \sum_{i=1}^{N}\frac{\sigma_i^{\tau}}{\|\sigma_i\|}\Big\{\Big(S_iA_ix_i - (S_{i1}A_{i1} + S_{i2}A_{i3})y_i - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2}\Big) - S_{i2}P_{i3}^{-1}(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4})e_i\Big\} + \sum_{i=1}^{N}\Big(\frac{\sigma_i^{\tau}}{\|\sigma_i\|}S_i\Delta f_i - \|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t)\Big) - \sum_{i=1}^{N}K_i + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\sigma_i^{\tau}}{\|\sigma_i\|}\Big(S_i\left[H_{ij} + \Delta H_{ij}\right] - \frac{\sigma_i}{\|\sigma_i\|}\times \left[\|S_jH_{ji}(y_i,\hat{x}_{i2})\| + \|S_jE_{ji}\|\vartheta_{ji}(y_i,t)\zeta_{ji}(y_i,\hat{x}_{i2},t)\right]\Big) - \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} S_{i2}P_{i3}^{-1}\Big\{P_{i2}^{\tau}[H_{ij1}(y_j,x_{j2}) - H_{ij1}(y_j,\hat{x}_{j2})] + P_{i3}\left[H_{ij2}(y_j,x_{j2}) - H_{ij2}(y_j,\hat{x}_{j2})\right]\Big\}$$

Using the previous partition of A_i in (8) and $S_i = [S_{i1} \quad S_{i2}]$, it follows that

$$\begin{aligned} &S_iA_ix_i - (S_{i1}A_{i1} + S_{i2}A_{i3})y_i - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2} \\ &= [S_{i1} \quad S_{i2}]\begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4}\end{bmatrix}\begin{bmatrix} x_{i1} \\ x_{i2}\end{bmatrix} - (S_{i1}A_{i1} + S_{i2}A_{i3})x_{i1} \\ &\quad - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2} \\ &= (S_{i1}A_{i2} + S_{i2}A_{i4})e_i \end{aligned}$$

From Assumption 2

$$\begin{aligned} &\frac{\sigma_i^{\tau}}{\|\sigma_i\|}S_i\Delta f_i - \|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t) \le \|S_iD_i\| \times \\ &\|\Delta\widetilde{f}_i\| - \|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t) \\ &\le \rho_i(y_i,t)_{\gamma_i}\|S_iD_i\|\,\|e_i\| \end{aligned}$$

and as $\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ji}$:

$$\begin{aligned} &\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\sigma_i^{\tau}}{\|\sigma_i\|}\Big(S_i\left[H_{ij}(x_j) + \Delta H_{ij}(x_j,t)\right] - \frac{\sigma_i}{\|\sigma_i\|}\times \\ &\left[\|S_jH_{ji}(y_i,\hat{x}_{i2})\| + \|S_jE_{ji}\|\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)\right]\Big) \\ &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big\{\frac{\sigma_i^{\tau}}{\|\sigma_i\|}S_iH_{ij}(x_j) - \|S_jH_{ji}(y_i,\hat{x}_{i2})\| + \\ &\frac{\sigma_i^{\tau}}{\|\sigma_i\|}S_iE_{ij}\Delta\widetilde{H}_{ij} - \|S_jE_{ji}\|\,\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)\Big\} \\ &\le \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\Big\{L_{H_{ij}}\|S_i\|\,\|e_j\| + \|S_jE_{ji}\|\,\vartheta_{ji}(y_i,t)\times \\ &\zeta_{ji}(x_i,t) - \|S_jE_{ji}\|\,\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)\Big\} \\ &\le \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left(\|S_j\|L_{H_{ji}} + \|S_jE_{ji}\|\vartheta_{ji}(y_i,t)L_{\zeta_{ji}}\right)\|e_i\| \end{aligned} \tag{31}$$

Using the above, it follows from (11) that

$$\begin{aligned} &\sum_{i=1}^{N}\frac{\sigma_i^{\tau}\dot{\sigma}_i}{\|\sigma_i\|} \\ &\le \sum_{i=1}^{N}\Big\{\alpha_1\exp\{-\alpha_2t\}\Big(\|S_{i1}A_{i2} + S_{i2}A_{i4}\| + \\ &L_{\gamma_i}\rho_i\|S_iD_i\| + \|S_{i2}P_{i3}^{-1}(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4})\| \\ &+ \sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\|S_j\|L_{H_{ji}} + \|S_jE_{ji}\|\vartheta_{ji}L_{\zeta_{ji}} + \\ &\|S_{i2}P_{i3}^{-1}\|\left(\|P_{i2}\|L_{H_{ij1}} + \|P_{i3}\|L_{H_{ij2}}\right)\Big)\Big) \\ &- K_i(y_i,t)\Big)\Big\} \end{aligned} \tag{32}$$

Hence, from the conclusion follows. □

6. CONCLUSION

A dynamical decentralised output feedback control has been presented using sliding mode techniques. Equivalent control theory and a local coordinate transformation are exploited to establish the stability of the reduced-order sliding mode. Known interconnections are used in the control design which insures the composite reachability condition can be satisfied by the control law. The approach allows both nominal isolated subsystems and the overall nominal interconnected system to be nonminimum phase. The uncertainties are mismatched and have nonlinear bounds.

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