CONSTRUCTION OF ROBUST ROOT LOCI FOR LINEAR SYSTEMS WITH ELLIPSOIDAL UNCERTAINTY OF PARAMETERS

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Abstract: In this paper we consider the construction of the robust root locus (RRL) for the systems with ellipsoidally parametric uncertainties. By characterizing the principal points of ellipsoidal parameter set $\mathbf{Q}$ associated with the root mapping $s(\mathbf{q}) : \mathbf{R}^m \to \mathbf{C}$, we present a necessary condition for the point $(s, \mathbf{q}) \in \mathbf{C} \times \mathbf{Q}$ to satisfy $p(s; \mathbf{q}) = 0$ and $s \in \partial Z(p, \mathbf{Q})$, the boundary of the RRL $Z(p, \mathbf{Q})$. This condition renders analytic manifolds of dimension one in the domain $\mathbf{C} \times \mathbf{Q}$. Hence, the boundary of each section of the RLL $Z(p, \mathbf{Q})$ can be accurately constructed via tracing the manifolds by a path-following algorithm. This approach to constructing the RRL provides an alternative way of verifying the robust stability of uncertain systems with ellipsoidal perturbations.

Keywords: Root locus, uncertainty, uncertain linear systems

1. INTRODUCTION

Since the inception of seminal Kharitonov's stability theorem (Kharitonov, 1979) on interval polynomials, robust stability analysis for linear systems subject to parametric uncertainties has been an active area of research in the last two decades. Given a polynomial $p(s; \mathbf{q})$ whose coefficients depend smoothly on parameter vector $\mathbf{q} \in \mathbf{R}^m$ taking values in a bounded and closed domain $\mathbf{Q}$ in $\mathbf{R}^m$, the problem of robust stability analysis is to answer if all the roots of the polynomial family $p(s; \mathbf{Q}) \triangleq \{p(s; \mathbf{q}) : \mathbf{q} \in \mathbf{Q}\}$ lie inside a certain simply-connected region $\mathbf{D}$ in the complex plane. In general, there are two research directions toward the solution of this robust D-stability problem. The first research direction focuses on, without actually finding part or all the roots of the polynomial family $p(s; \mathbf{Q})$, exploring a testable subset of the polynomial family $p(s; \mathbf{Q})$ whose D-stability ensures the D-stability of the whole polynomial family (Bartlett *et al.*, 1988; Bhattacharyya *et al.*, 1995).

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Instead of giving 'yes' or 'no' answer to whether all the roots of the polynomial family $p(s;\mathbf{Q})$ are in $\mathbf{D}$, the second research direction in the robust stability analysis attempts to find the root distribution of $p(s;\mathbf{Q})$. This led to the so-called robust root locus problem (Barmish and Tempo, 1990; Barmish and Tempo, 1991; Cerone, 1997; Hwang and Chen, 1999) or multi-parameter zero set problem (Fruchter *et al.*, 1987; Walach and Zeheb, 1982). Since the knowledge of root distribution of a parametric polynomial family allows one to analyze not only the robust stability but also the robust performance of an uncertain system, the robust root locus problem has recently gained considerable attention (Barmish and Tempo, 1990; Barmish and Tempo, 1991; Cerone, 1997; Hwang and Chen, 1999; Hwang and Yang, 2003; Yang and Hwang, 2001). The methods recently proposed in (Barmish and Tempo, 1991; Cerone, 1997; Hwang and Chen, 1999) to solve the robust root locus problem with linear or multilinear parameter dependency are essentially based on performing zero inclusion test for the value sets $p(s_i;\mathbf{Q})$ with a finite set of specially selected points s_i. All these methods are applicable to the case where the unertain parameter set $\mathbf{Q}$ is an m-D box.

In this paper we solve the problem of constructing robust root locus for systems with ellipsoidal parametric uncertianties because ellipsoidal uncertainty descriptions arise naturally in the context of set-membership parameter identification using successive an ellipsoid-bounding algorithm (Belforte *et al.*, 1990). Such a problem has recently been approached in (Hwang and Yang, 2003; Yang and Hwang, 2001) with a method combining zero inclusion/exclusion test, parameter domain subdivision, and integer-labelled pivoting procedure. Actually, a bisection-based box domain subdivision algorithm is used, the method presented in (Hwang and Yang, 2003; Yang and Hwang, 2001) is not efficient in the case when the parameter set corresponding to an RLL boundary lies on the ellipsoid. To avoid this disadvantage, we propose in this paper an analytical characterization of the boundary of the robust root locus. This characterization is based on the notions of principal points (Kiselev *et al.*, 1997) for characterizing the boundary of the image of a convex compact set under a smooth mapping. It allows one to construct the boundaries of the RLL of an ellipsoidally uncertain system by using a path-following algorithm. Moreover, it provides a direct link between the roots lying on the RLL boundaries and the uncertain parameters in the ellipsoidal parameter set. This link allows one to identify the worst or critical case of the parametrically uncertain plant set.

2. ANALYTICAL CHARACTERIZATION OF VALUE-SET BOUNDARY

Let $\mathbf{Q}$ be a convex compact set in $\mathbf{R}^m$ and $\mathbf{q} \triangleq (q_0, q_1, \cdots, q_{m-1})^T \in \mathbf{Q}$, where the superscript T denotes transpose. For a smooth mapping $f \triangleq (f_1, f_2, \cdots, f_l)^T : \mathbf{R}^m \to \mathbf{R}^l$, the image

$$f(\mathbf{Q}) = \{f(\mathbf{q}) : \forall \mathbf{q} \in \mathbf{Q}\} \subset \mathbf{R}^l \qquad (1)$$

is often referred to as a value set. The construction of the value set $f(\mathbf{Q})$ plays a central role in the robust stability and performance analysis for control systems in the presence of parametric uncertainties. In this section, we shall review some existing results on the characterization of the smallest testable set of points in $\mathbf{Q}$ whose image under f covers the boundary of the value set $f(\mathbf{Q})$, denoted by $\partial f(\mathbf{Q})$. By definition, a point $\mathbf{z}$ lies on the boundary $\partial f(\mathbf{Q})$ if every neighborhood of $\mathbf{z}$ contains points from the value set $f(\mathbf{Q})$ and its set-theoretic complement.

We first give some definitions. A vector $\mathbf{n}$ is called a *normal vector* to $\mathbf{Q}$ at the point $\mathbf{q}^0$ if the inner product of the vectors $\mathbf{n}$ and $\mathbf{q}^0 - \mathbf{q}$ is nonnegative. All normal vectors to $\mathbf{Q}$ at $\mathbf{q}^0$ constitute a *normal cone* $\mathbf{N}(\mathbf{q}^0)$. It can be shown that $\mathbf{N}(\mathbf{q}^0) = \emptyset$ for an interior point $\mathbf{q}^0$ of $\mathbf{Q}$. If $\mathbf{q}^0 \in \partial\mathbf{Q}$, where $\partial\mathbf{Q}$ denotes the boundary of $\mathbf{Q}$, the normal cone $\mathbf{N}(\mathbf{q}^0)$ is nonempty. In particular, if the convex set $\mathbf{Q}$ is defined by $\{\mathbf{q} : \phi_1(\mathbf{q}) \le 0, \cdots, \phi_\nu(\mathbf{q}) \le 0\}$, where $\phi_i, i = 1, 2, \cdots, \nu$, are convex differentiable functions such that $\phi_k(\mathbf{q}^0) = 0, \mathbf{q}^0 \in \partial\mathbf{Q}$ and $\phi_k(\mathbf{q}^1) < 0$ for an interior point $\mathbf{q}^1 \in \mathbf{Q}$ and a $k \in \{1, 2, \cdots, \nu\}$, then

$$\mathbf{N}(\mathbf{q}^0) = \left\{\lambda \nabla\phi_k(\mathbf{q}^0),\ 0 \le \lambda < \infty\right\} \qquad (2)$$

where

$$\nabla\phi_k(\mathbf{q}^0) = \Big(\frac{\partial\phi_k(\mathbf{q}^0)}{\partial q_0}, \frac{\partial\phi_k(\mathbf{q}^0)}{\partial q_1}, \cdots, \frac{\partial\phi_k(\mathbf{q}^0)}{\partial q_{m-1}}\Big)^T \qquad (3)$$

Now we quote the necessary condition given by Kiselev *et al.* (1997) for characterizing the the points in $\mathbf{Q}$ whose image under the mapping f lie on value-set boundary $\partial f(\mathbf{Q})$.

Theorem 1 (Kiselev *et al.*, 1997). Assume that $\mathbf{q}^0 \in \mathbf{Q}$ and $f(\mathbf{q}^0) \in \partial f(\mathbf{Q})$. Then there exists a nonzero $\mathbf{y} \in \mathbf{R}^l$ such that $[J_f(\mathbf{q}^0)]^T \mathbf{y} \in \mathbf{N}(\mathbf{q}^0)$, where

$$J_f(\mathbf{q}^0) = \begin{pmatrix} \frac{\partial f_1(\mathbf{q}^0)}{\partial q_0} & \frac{\partial f_1(\mathbf{q}^0)}{\partial q_1} & \cdots & \frac{\partial f_1(\mathbf{q}^0)}{\partial q_{m-1}} \\ \frac{\partial f_2(\mathbf{q}^0)}{\partial q_0} & \frac{\partial f_2(\mathbf{q}^0)}{\partial q_1} & \cdots & \frac{\partial f_2(\mathbf{q}^0)}{\partial q_{m-1}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_l(\mathbf{q}^0)}{\partial q_0} & \frac{\partial f_l(\mathbf{q}^0)}{\partial q_1} & \cdots & \frac{\partial f_l(\mathbf{q}^0)}{\partial q_{m-1}} \end{pmatrix} \qquad (4)$$

We outline in the following the consequences of Theorem 1 for the case where f is a complex-valued smooth function and the domain $\mathbf{Q}$ is an m-ellipsoid, which is of particular interest to us in this paper. In the sequel, $f(\mathbf{q})$ is assumed to be a complex-valued smooth function of $\mathbf{q} \in \mathbf{R}^m$ and $g_k(\mathbf{q})$ denotes the partial derivative of $f(\mathbf{q})$ with respect to q_k, i.e.,

$$\begin{aligned} g_k(\mathbf{q}) &:= g_{k,\mathrm{r}}(\mathbf{q}) + j g_{k,\mathrm{i}}(\mathbf{q}) \\ &= \frac{\partial f(\mathbf{q})}{\partial q_k},\ k = 0, 1, \cdots, m-1 \end{aligned} \quad (5)$$

where $g_{k,\mathrm{r}}(\mathbf{q})$ and $g_{k,\mathrm{i}}(\mathbf{q})$ are the real and imaginary parts of $g_k(\mathbf{q})$, respectively. For an ellipsoidal uncertainty set $\mathbf{E}$, the following corollary characterizes the value-set boundary $\partial f(\mathbf{E})$.

Corollary 1. Let $\mathbf{E}$ be an m-ellipsoid in $\mathbf{R}^m$ with the center $\mathbf{q}^c = (q_0^c, q_1^c, \cdots, q_{m-1}^c)^T$ and the positive axis lengths $\mathbf{w} = (w_0, w_1, \cdots, w_{m-1})^T$:

$$\mathbf{E} = \{\mathbf{q} : \phi(\mathbf{q}) = \sum_{k=0}^{m-1} (\frac{q_k - q_k^c}{w_k})^2 - 1 \le 0\} \quad (6)$$

If $f(\mathbf{q}^0) \in \partial f(\mathbf{E})$ for a point $\mathbf{q}^0 \in \mathbf{E}$, then there exists a nonzero $g \in \mathbf{C}$ such that for $k = 0, 1, \cdots, m-1$

$$|g|^2 \Im\{\frac{g_k(\mathbf{q}^0)}{g}\} = \begin{cases} 0, & \mathbf{q}^0 \in \mathbf{E}\backslash\partial\mathbf{E} \\ -\dfrac{\partial\phi(\mathbf{q}^0)}{\partial q_k}, & \mathbf{q}^0 \in \partial\mathbf{E} \end{cases} \quad (7)$$

A point $\mathbf{q}^0$ belonging to the m-ellipsoid $\mathbf{E}$ while satisfying condition (7) will be referred to as a principal point associated with the mapping f. The notion of principal points was introduced in (Polyak and Kogan, 1995) to characterize the value-set boundary and further generalized in (Hwang and Chen, 1999) to facilitate the boundary construction. Let the set of principal points in $\mathbf{E}$ associated with the mapping f be denoted by $\mathbf{P}$. Since condition (7) is only a necessary condition for the image of a point $\mathbf{q}^0 \in \mathbf{E}$ under the mapping f to lie on the boundary of the value set $f(\mathbf{E})$, the image of the set of principal points $\mathbf{P} \subset \mathbf{E}$ thus covers the boundary of the value set $f(\mathbf{E})$, i.e.,

$$\partial f(\mathbf{E}) \subset f(\mathbf{P}) \quad (8)$$

For a principal point $\mathbf{q}$ lying in the interior of the m-ellipsoid $\mathbf{E}$, it follows from (7) that there exists a nonzero complex number g such that

$$\Im\{\frac{g_k(\mathbf{q})}{g}\} = 0, \quad \forall k \in \{0, 1, \cdots, m-1\} \quad (9)$$

Hence the set of principal points in the interior of the m-ellipsoid $\mathbf{E}$ are the one-dimensional manifolds defined by the following $m-1$ equations:

$$\begin{vmatrix} g_{l,\mathrm{r}}(\mathbf{q}) & g_{l,\mathrm{i}}(\mathbf{q}) \\ g_{k,\mathrm{r}}(\mathbf{q}) & g_{k,\mathrm{i}}(\mathbf{q}) \end{vmatrix} = 0,\ \forall k \in \{0, 1, \cdots, m-1\}\backslash\{l\} \quad (10)$$

where $l \in \{0, 1, \cdots, m-1\}$. On the other hand, if a principal point $\mathbf{q}$ is on the surface

$$\phi(\mathbf{q}) = 0 \quad (11)$$

of the m-ellipsoid $\mathbf{E}$, then it satisfies the second equation in (7), which is equivalent to

$$\begin{aligned} g_{\nu,\mathrm{r}}(\mathbf{q}) y_{\mathrm{r}}(\mathbf{q}) + g_{\nu,\mathrm{i}}(\mathbf{q}) y_{\mathrm{i}}(\mathbf{q}) = \Delta_{l,k}(\mathbf{q}) \frac{\partial\phi(\mathbf{q})}{\partial q_\nu}, \\ \forall \nu \in \{0, 1, \cdots, m-1\}\backslash\{l, k\} \end{aligned} \quad (12)$$

where $l, k \in \{0, 1, \cdots, m-1\}$, and

$$\Delta_{l,k}(\mathbf{q}) = \begin{vmatrix} g_{l,\mathrm{r}}(\mathbf{q}) & g_{l,\mathrm{i}}(\mathbf{q}) \\ g_{k,\mathrm{r}}(\mathbf{q}) & g_{k,\mathrm{i}}(\mathbf{q}) \end{vmatrix} \ne 0 \quad (13a)$$

$$y_{\mathrm{r}}(\mathbf{q}) = \begin{vmatrix} \dfrac{\partial\phi(\mathbf{q})}{\partial q_l} & g_{l,\mathrm{i}}(\mathbf{q}) \\ \dfrac{\partial\phi(\mathbf{q})}{\partial q_k} & g_{k,\mathrm{i}}(\mathbf{q}) \end{vmatrix} \quad (13b)$$

$$y_{\mathrm{i}}(\mathbf{q}) = \begin{vmatrix} g_{l,\mathrm{r}}(\mathbf{q}) & \dfrac{\partial\phi(\mathbf{q})}{\partial q_l} \\ g_{k,\mathrm{r}}(\mathbf{q}) & \dfrac{\partial\phi(\mathbf{q})}{\partial q_k} \end{vmatrix} \quad (13c)$$

Hence the set of principal points lying on the ellipsoidal surface are also one-dimensional manifolds described by the $m-1$ equations in (11) and (12).

3. CHARACTERIZATION OF RLL FOR SYSTEMS WITH ELLIPSOID UNCERTAINTY DESCRIPTIONS

Consider a parametric polynomial

$$p(s; \mathbf{q}) = \sum_{k=0}^{n} a_k(\mathbf{q}) s^k \quad (14)$$

where the real coefficients $a_k(\mathbf{q})$ depend smoothly on the parameters $\mathbf{q}$. The robust root locus (RRL) of the polynomial family $p(s; \mathbf{Q})$ with $\mathbf{Q}$ a compact convex set in $\mathbf{R}^m$ is defined as

$$\begin{aligned} Z(p, \mathbf{Q}) &= \{s \in \mathbf{C} : p(s; \mathbf{q}) = 0 \text{ for some } \mathbf{q} \in \mathbf{Q}) \\ &= \mathbf{C}_1 \cup \mathbf{C}_2 \cup \cdots \cup \mathbf{C}_\mu,\ \mu \le n \end{aligned} \quad (15)$$

It represents the smallest set of regions $\mathbf{C}_k \subset \mathbf{C}, k = 1, 2, \cdots, \mu \le n$ in the complex plane within which the roots of the polynomial members in the set $p(s; \mathbf{Q})$ lie. In the sequel, we shall call a root region $\mathbf{C}_k$ a cross section of the robust root locus of the polynomial family $p(s; \mathbf{Q})$. Since the roots of a polynomial are continuous functions of its coefficients, a cross section of the robust root locus is simply-connected. Moreover, under the assumption that the polynomial family $p(s; \mathbf{Q})$

contains no members which have different degrees, each cross section of the robust root locus is a bounded region in the complex plane.

In this section, we shall characterize the smallest testable subset $\mathbf{M}$ of the parameter domain $\mathbf{Q}$ such that the robust root locus of the parametric polynomial family $p(s;\mathbf{M})$, $Z(p,\mathbf{M})$, covers the boundary of the robust root locus $Z(p,\mathbf{Q})$, denoted by $\partial Z(p,\mathbf{Q})$. The following theorem provides the desired characterization of the boundary of the robust root locus.

Theorem 2. If z is a point on the boundary of the robust root locus $Z(p,\mathbf{Q})$, then there exists a $\mathbf{q} \in \mathbf{Q}$ such that the following two conditions hold

(i) $p(z;\mathbf{q}) = 0$.
(ii) $p(z;\mathbf{q}) \in \partial p(z;\mathbf{Q})$.

Proof. Condition (i) follows from the definition of RRL given in (15). In the following we prove condition (ii).

Let z^* be a point in the complex plane which does not belong to the RRL of the polynomial family $p(s;\mathbf{Q})$, i.e., $0 \notin p(z^*;\mathbf{Q})$. Since $p(s;\mathbf{q})$ is a continuous function of s, the distance between the origin and the value set $p(s;\mathbf{Q})$ varies continuously with respect to s. Therefore, when z^* approaches z, the origin approaches the boundary of the value set $p(z;\mathbf{Q})$. Consequently, we have $0 \in \partial p(z;\mathbf{Q})$ and from condition (i), $p(z;\mathbf{q}) = 0 \in \partial p(z;\mathbf{Q})$. This completes the proof. ∎

Note that Theorem 2 provides a necessary condition for a point $\mathbf{q}^0 \in \mathbf{Q}$ with that some roots of the polynomial $p(s;\mathbf{q}^0)$ lie on the boundary of the robust root locus $Z(p,\mathbf{E})$. Following the condition (ii) of Theorem 2 and the characterization of principal points for the boundary of the value set $p(z;\mathbf{E})$, we see that the necessary condition for a root z of the polynomial $p(s;\mathbf{q}^0), \mathbf{q}^0 \in \mathbf{E}$ to lie on the boundary of the robust root locus $Z(p,\mathbf{E})$ is that the point $\mathbf{q}^0$ belongs to the set of principal points $\mathbf{P}$ and satisfies the equation $p(z;\mathbf{q}^0) = 0$. Hence, the smallest testable subset $\mathbf{M}$ of $\mathbf{E}$ such that the robust root locus $Z(p,\mathbf{M})$ covers the RLL boundary $\partial Z(p,\mathbf{E})$ is exactly the set of principal points $\mathbf{P}$ associated with the complex-valued mapping $p(s;\mathbf{E})$. As a result, the branches of the root locus set $\partial Z(p,\mathbf{E})$ can be constructed by tracing the one-dimensional manifold defined by equation $p(z,\mathbf{q}) = 0$ and those for defining the principal points in $\mathbf{E}$.

4. AN ILLUSTRATIVE EXAMPLE

In this example, we apply the proposed analytical characterization of value set boundary to generate the robust root loci for the parametric polynomial family

$$p(s;\mathbf{E}) = \{p(s;\mathbf{q}) = s^4 + q_0^2 q_2 s^3 + q_1 q_2^2 s^2 + q_1^2 s + q_0 q_2 : \mathbf{q} = (q_0,q_1,q_2)^T \in \mathbf{Q}\} \quad (16)$$

where the parameter domain $\mathbf{E}$ is a 3-ellipsoid define by

$$\mathbf{E} = \{(q_0,q_1,q_2)^T : \phi(q_0,q_1,q_2) \le 0\} \quad (17)$$

where

$$\phi(q_0,q_1,q_2) = (\frac{q_0-0.8}{0.1})^2 + (\frac{q_1-0.3}{0.5})^2 + (\frac{q_2-0.7}{0.3})^2 - 1 \quad (18)$$

For a given complex number $s = \sigma + j\omega$, the set of principal points associated with the mapping $p(\sigma + j\omega;\mathbf{q})$ includes the principal point (PP) manifolds lying in the interior of the 3-ellipsoid $\mathbf{E}$ and those on the ellipsoidal surface defined by $\phi(q_0,q_1,q_2) = 0$. In the interior of the 3-ellipsoid $\mathbf{E}$, the PP manifolds in the domain $\mathbf{C} \times \mathbf{E}$ are defined by

$$\Re\{p(\sigma + j\omega;\mathbf{q})\} = 0 \quad (19a)$$

$$\Im\{p(\sigma + j\omega;\mathbf{q})\} = 0 \quad (19b)$$

$$\begin{vmatrix} g_{0,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{0,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \\ g_{1,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{1,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \end{vmatrix} = 0 \quad (19c)$$

$$\begin{vmatrix} g_{0,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{0,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \\ g_{2,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{2,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \end{vmatrix} = 0 \quad (19d)$$

$$\phi(q_0,q_1,q_2) < 0 \quad (19e)$$

On the elliptic surface, the PP manifolds are defined by the following equations:

$$\Re\{p(\sigma + j\omega;\mathbf{q})\} = 0 \quad (20a)$$

$$\Im\{p(\sigma + j\omega;\mathbf{q})\} = 0 \quad (20b)$$

$$\phi(\mathbf{q}) = 0 \quad (20c)$$

$$g_{2,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) y_\mathrm{r}(\sigma + j\omega;\mathbf{q}) + g_{2,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \times y_\mathrm{i}(\sigma + j\omega;\mathbf{q}) = \Delta_{0,1}(\sigma + j\omega;\mathbf{q}) \frac{\partial \phi(\mathbf{q})}{\partial q_2} \quad (20d)$$

where

$$\Delta_{0,1}(\sigma + j\omega;\mathbf{q}) = \begin{vmatrix} g_{0,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{0,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \\ g_{1,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & g_{1,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \end{vmatrix} \quad (21a)$$

$$y_\mathrm{r}(\sigma + j\omega;\mathbf{q}) = \begin{vmatrix} \frac{\partial \phi(\mathbf{q})}{\partial q_0} & g_{0,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \\ \frac{\partial \phi(\mathbf{q})}{\partial q_1} & g_{1,\mathrm{i}}(\sigma + j\omega;\mathbf{q}) \end{vmatrix} \quad (21b)$$

$$y_\mathrm{i}(\sigma + j\omega;\mathbf{q}) = \begin{vmatrix} g_{0,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & \frac{\partial \phi(\mathbf{q})}{\partial q_0} \\ g_{1,\mathrm{r}}(\sigma + j\omega;\mathbf{q}) & \frac{\partial \phi(\mathbf{q})}{\partial q_1} \end{vmatrix} \quad (21c)$$

Using a curve-following algorithm, we traced the PP manifolds lying in the interior and on the elliptic surface of the 3-ellipsoid described by (19) and (20), respectively, and their corresponding branches of root locus. Figs. 1 and 2 show the traced PP manifolds and the corresponding two cross sections of the robust root locus lying above the real axis. Note that in Figs. 1a and 2a, points on the ellipse

$$(\frac{q_0 - 0.8}{0.1})^2 + (\frac{q_1 - 0.3}{0.5})^2 = 1 \qquad (22)$$

are also the principal points. To let the Figs. 1 and 2 be more readable, some principal points and their corresponding roots are labelled.

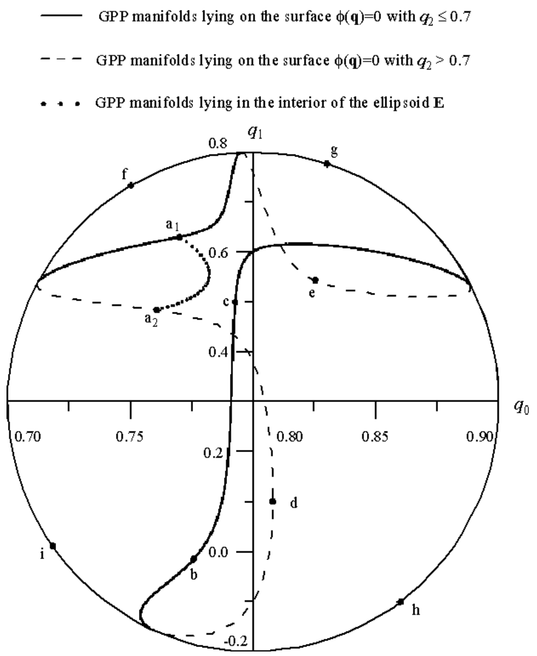

Fig. 1a. The projection of the PP manifolds associated with the cross section of the robust root locus of $p(s;\mathbf{E})$ lying in the first quadrant of the complex plane.

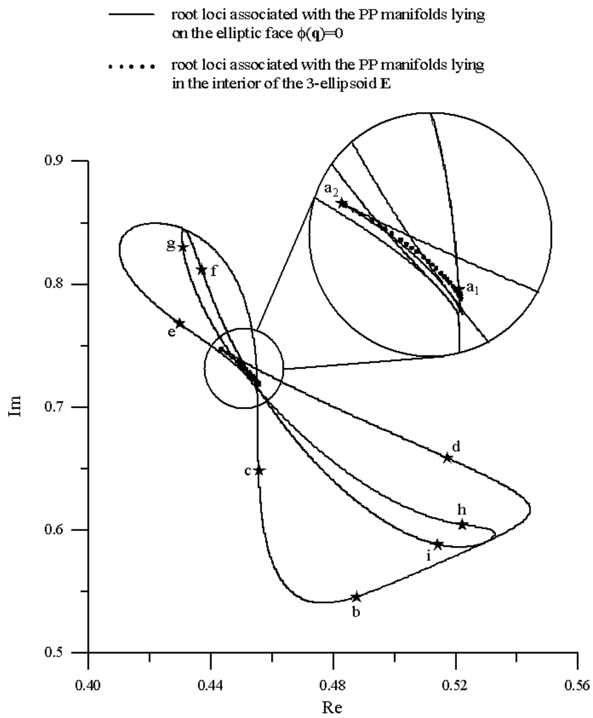

Fig. 1b. The cross section of the robust root locus of $p(s;\mathbf{E})$ lying in the first quadrant of the complex plane.

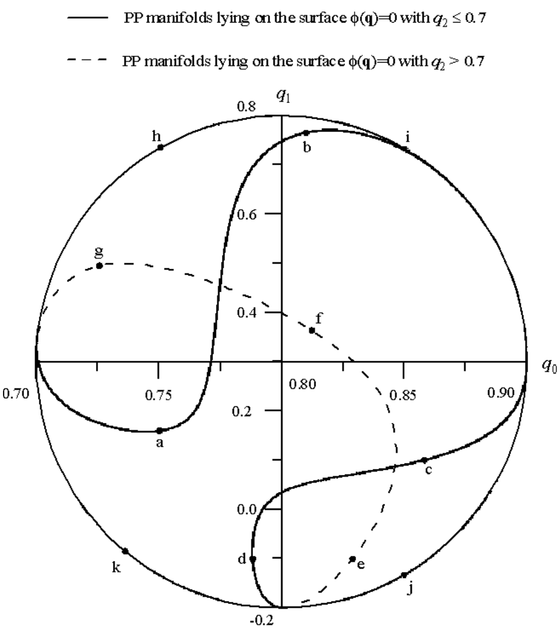

Fig. 2a. The projection of the PP manifolds associated with the cross section of the robust root locus of $p(s;\mathbf{E})$ lying in the second quadrant of the complex plane.

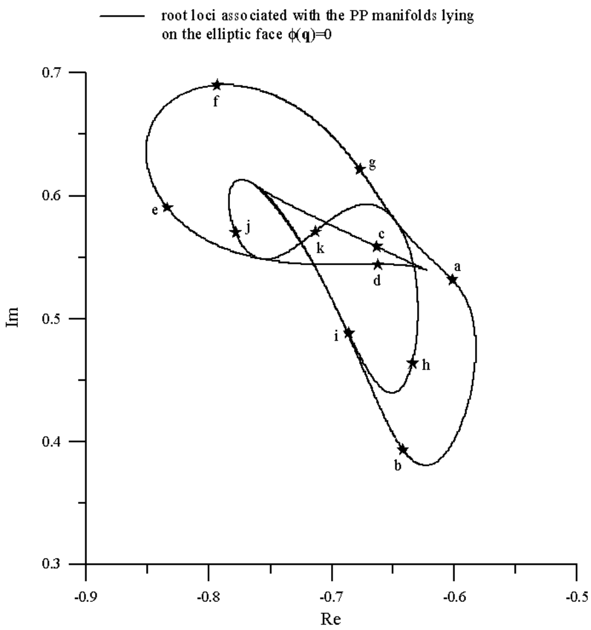

Fig. 2b. The cross section of the robust root locus of $p(s; \mathbf{E})$ lying in the second quadrant of the complex plane.

5. CONCLUSION

In this article, we have presented an analytical characterization of the robust root locus of ellipsoidal parametric polynomial families. Based on exploiting the notion of principal points for characterizing the boundary of the image of a convex domain under a differentiable complex-valued mapping, we have derived necessary conditions for characterizing the boundary of the robust root loci for multi-parameter polynomial families. The derived necessary conditions are given in analytical expressions which describe one-dimensional manifolds in the union of the complex plane and the parameter domain. By tracing the one-dimensional manifolds with an existing path-following algorithm, we can obtain the boundary of each robust root locus section. Although the derivations are given for the case of ellipsoidal uncertainty set, the obtained results can be readily extended to systems with polytopical uncertainty set. Hence, the proposed approach to constructing robust root loci is a highly valuable and useful tool for the robust stability analysis and control design of linear systems having parametric uncertainties.

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