A COMPARATIVE SURVEY IN DETERMINING THE IMAGINARY CHARACTERISTIC ROOTS OF LTI TIME DELAYED SYSTEMS

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Abstract: The aim of this study is to offer a comparison of the numerical procedures for an important problem, the determination of purely imaginary characteristic roots of LTI-Time Delayed Systems (LTI-TDS). This problem, in fact, has a crucial role in assessing the stability of the general class of vector LTI-TDS $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{x}(t-\tau)$. There are many procedures discussed in the literature for this purpose. Those, which are exact, first determine the complete set of imaginary characteristic roots of the dynamics, as they constitute the only points where stability switching can take place. These approaches are, in fact, some variations of the five main methods, which may demand numerical procedures of different complexity and they may result in different precisions in finding the roots. There is, however, no comparative case study known to the authors to demonstrate the strengths and weaknesses of these methods. This document is prepared primarily for this purpose. We first present an overview of each of the five methods and then compare their numerical performances over an example case study.

Keywords: Linear time invariant, time-delay systems, stability, stability tests.

1. INTRODUCTION

The stability of linear time invariant retarded time-delayed systems (LTI-TDS) has been a very active research topic for some time (Cooke and van den Driessche, 1986; Walton and Marshall, 1987; Hale and Verduyn Lunel, 1993; Chen, Gu et al., 1995; Niculescu, 2001). Numerous contributions by renowned investigators can be found in the literature on the subject. It is the authors' belief that LTI-TDS field still remains rich with challenging and unsolved problems. Some existing methods, for instance, present new knowledge, which have not been recognized until recently (Gu and Niculescu, 2001; Olgac and Sipahi, 2002). Some others suggest variations on the earlier techniques to overcome some subtle and hidden impracticalities mainly from numerical deployment point of view (Thowsen, 1981).

The general dynamics in question is

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{x}(t-\tau) \qquad (1)$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A}$ and $\mathbf{B} \in \Re^{n\times n}$ are known matrices with ranks n and p ($\le n$), respectively, $\tau \in \Re^+$ and it is the only free parameter in equation (1). The question is to determine the stability outlook of the system in semi-infinite τ domain. The characteristic equation of the system is

$$CE(s,\tau) = \det(s\,\mathbf{I} - \mathbf{A} - \mathbf{B}\,e^{-\tau s}) = 0 \qquad (2)$$

and it contains time delays of commensurate nature with degree p, i.e. there are $e^{-k\tau s}$, $k = 0, 1, \ldots p$ terms in equation (2) with an assurance that $e^{-p\tau s}$ term is present where $p = rank(\mathbf{B}) \le n$. The system is infinite dimensional and as such it possesses infinitely many characteristic roots. The question of stability translates into some conditions on τ to guarantee that all of these infinitely many characteristic roots lie on the stable left half of the complex plane.

When we study the existing procedures assessing the stability we observe a common thread, which has to do with the determination of purely imaginary roots of Equation (2) (Rekasius, 1980; Thowsen, 1981; Cooke and van den Driessche, 1986; Walton and Marshall, 1987; Hale and Verduyn Lunel, 1993; Chen, Gu et al., 1995; Su, 1995; Niculescu, 2001; Olgac and Sipahi, 2002). It is known for the retarded LTI-TDS, that the imaginary characteristic roots, when and if they exist, are the only possible transition points from stable to unstable behavior, and vice versa (Hale and Verduyn Lunel, 1993). This is known as the “stability switching” in the TDS literature. So, it is of utmost importance to determine all such roots, exhaustively and precisely. It is stated in the literature (Chen, Gu et al., 1995) that an n-dimensional system with *p*-degree of commensuracy like in equation (2) cannot have more than *np* imaginary characteristic roots regardless of the particular composition of **A** and **B** matrices, and for all $\tau \in \Re^+$ values. Additionally it is also a trivial observation that to any one of these crossings, $s = \mp \omega i$, infinitely many periodically distributed time delays are attributed. These time delays display an equidistant distribution given by

$$\tau = \tau_0 + 2\pi k / \omega , \qquad k = 0,1,... \tag{3}$$

where τ_0 is the smallest positive time delay causing a pair of crossings at ωi.

The determination of all the imaginary roots completely and precisely constitutes the common starting point for all the stability methodologies. And we direct the rest of the paper to review and compare these procedures. There are, in fact, five distinguishable approaches in the literature:

a) Schur-Cohn method (Hermite matrix formation) (Barnett, 1983; Chen, Gu et al., 1995)
b) Elimination of transcendental terms (Walton and Marshall, 1987)
c) Matrix pencil, Kronecker sum method (Chen, Gu et al., 1995; Su, 1995)
d) Kronecker multiplication and elementary transformation (Louisell, 2001)
e) Rekasius substitution (Rekasius, 1980)

As a side remark, we wish to state the following for the purpose of motivating the main problem. Once the complete set of purely imaginary roots are found for a given system using any one of the above methods, one can deploy a recent paradigm, **Cluster Treatment of Characteristic Roots (CTCR)** (Olgac and Sipahi, 2002; Sipahi and Olgac, 2003b). This paradigm suggests a crucial property, which was not recognized until recently (Olgac and Sipahi, 2002). It is called the *invariance* of the tendency of the characteristic root crossings. It ultimately results in the determination of the stable delay intervals, exhaustively in $\tau \in \Re^+$ domain.

We briefly review the five procedures leading to the complete set of imaginary roots in Section II. Section III contains a numerical comparison of these procedures from a practical perspective using an example case study. Section IV presents the conclusions.

II. BRIEF REVIEW OF THE METHODOLOGIES

In this section we revisit the five main methodologies mentioned to prepare for the comparative work. Let us take the expanded form of equation (2)

$$CE(s,\tau) = \sum_{k=0}^{p} a_k(s)\, e^{-k\tau s} = 0 \tag{4}$$

where $p = rank(\mathbf{B})$, $p \le n$ is the degree of commensuracy in the dynamics and $a_k(s)$ are polynomials of degree $n-k$.

(a) Schur-Cohn Criterion as per (Barnett, 1983; Chen, Gu et al., 1995). The formation starts with rewriting the equation (4) multiplying it with $e^{k\tau s}$, $k = 0,1,...,p-1$. This generates *p* equations in terms of $e^{k\tau s}$, $k = -p, ..., -1, 0, 1, ..., p-2, p-1$, which are 2*p* linearly independent terms. Next let us consider the companion equation, $CE(-s,\tau) = 0$, which is also satisfied for $s = \omega i$ due to the fact that the imaginary characteristic root ωi always appears as a complex conjugate pair,

$$\overline{CE}(s,\tau) = CE(-s,\tau) = \sum_{k=0}^{p} a_k(-s)\, e^{k\tau s} = \sum_{k=0}^{p} \bar{a}_k(s)\, e^{k\tau s} \tag{5}$$

where $\bar{f}(s) = f(-s)$ is indeed the conjugate operation when $s = \omega i$. We then multiply $\overline{CE}(s,\tau)$ with $e^{-k\tau s}$, $k = 1,2,...,p$, generating another *p* equations in terms of the same 2*p* linearly independent terms $e^{k\tau s}$, $k = -p, -p+1, ..., p-1$. Both of these sets of *p* equations can be combined in a single matrix equation as

$$\begin{bmatrix}
0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & \cdots & a_p \\
0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & & a_p & 0 \\
 & & \iddots & \iddots & \vdots & \vdots & & \iddots & & \vdots \\
 & \iddots & \iddots & & \vdots & \vdots & \iddots & & & \vdots \\
a_0 & a_1 & a_2 & & a_{p-1} & a_p & \cdots & \cdots & \cdots & 0 \\
0 & 0 & & 0 & \bar{a}_p & \bar{a}_{p-1} & \bar{a}_{p-2} & \cdots & \cdots & \bar{a}_0 \\
0 & 0 & \cdots & \bar{a}_p & \bar{a}_{p-1} & \bar{a}_{p-2} & & & \bar{a}_0 & 0 \\
 & & \iddots & \iddots & \vdots & \vdots & & & & \vdots \\
 & \iddots & \iddots & & \vdots & \vdots & & & & \vdots \\
\bar{a}_p & \bar{a}_{p-1} & \cdots & \cdots & \bar{a}_1 & \bar{a}_0 & 0 & \cdots & \cdots & 0
\end{bmatrix}
\begin{bmatrix} e_{p-1} \\ e_{p-2} \\ \vdots \\ \vdots \\ \vdots \\ e_1 \\ e_0 \\ \vdots \\ \\ e_{-n} \end{bmatrix} = \mathbf{0} \tag{6}$$

$$\mathbf{A}_1 \mathbf{E}_1 = 0 \tag{7}$$

where e_k represents $e^{k\tau s}$ as a shorthand notation to prevent cluttering the equation. If one rewrites this equation by rearranging the exponential terms, it can be cast in the form:

$$\begin{bmatrix}
a_0 & 0 & 0 & \cdots & 0 & a_p & a_{p-1} & \cdots & a_2 & a_1 \\
a_1 & a_0 & 0 & \cdots & 0 & 0 & a_p & a_{p-1} & \cdots & a_2 \\
\vdots & & \ddots & & \vdots & \vdots & & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\
a_{p-1} & a_{p-2} & \cdots & \cdots & a_0 & 0 & 0 & \cdots & 0 & a_p \\
\bar{a}_p & 0 & \cdots & \cdots & 0 & \bar{a}_0 & \bar{a}_1 & \cdots & \cdots & \bar{a}_{p-1} \\
\bar{a}_{p-1} & \bar{a}_p & \ddots & & \vdots & 0 & \bar{a}_0 & \cdots & \cdots & \bar{a}_{p-2} \\
\vdots & & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & \vdots & & \ddots & \ddots & \vdots \\
\bar{a}_1 & \bar{a}_2 & \cdots & \cdots & \bar{a}_p & 0 & \cdots & \cdots & 0 & \bar{a}_0
\end{bmatrix}
\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ e_p \\ e_{-p} \\ e_{-p+1} \\ \vdots \\ \vdots \\ e_{-1} \end{bmatrix} = \mathbf{A}_2 \mathbf{E}_2 = \mathbf{0} \tag{8}$$

Obviously for a nontrivial solution of $\mathbf{E}_2$ the $\mathbf{A}_2$ matrix must be singular: $\det \mathbf{A}_2(s) = 0$. This matrix, $\mathbf{A}_2$, is known to be the Schur-Cohn matrix. Notice the favourable fragmentation of $\mathbf{A}_2$ into four $p \times p$ segments in the rearranged form as:

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Lambda}_2^H & \mathbf{\Lambda}_1^H \end{bmatrix} \tag{9}$$

where $\mathbf{\Lambda}^H$ implies the hermitian of $\mathbf{\Lambda}$, and equation (9) presents a compact form adopted by (Barnett, 1983; Chen, Gu et al., 1995)[1]. $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are self evident matrices from equation (7).

This method suggests that if equation (4) has any imaginary root pair $s = \mp \omega i$, it should also satisfy equation (8). Consequently, the question of finding all the imaginary roots of equation (2) reduces to finding the imaginary roots of equation (8), which is a polynomial of *s* with degree *2np*. So the problem is cast into determining the purely imaginary roots of *2np* degree polynomial equation (8), which can produce maximum *np* pairs of imaginary roots. Evaluation of the $\det \mathbf{A}_2(s)$, however, needs a symbolic operation, while *2p* terms are multiplied and added *2p* times for expanding the determinant. Each one of these *2p* multiplications would create some round-off errors eventually resulting a polynomial of *s* with erroneous coefficients. This operation ultimately yields poor precision in determining the desired imaginary roots.

Alternatively to this symbolic evaluation of a determinant one can determine the eigenvalues of a constant matrix (Theorem 2.1 (Chen, Gu et al., 1995)). Namely,

$$\det\,[\omega \mathbf{I} - \mathbf{P}] \equiv \det(\mathbf{P}_n^{-1})\ \det(\mathbf{A}_2)\big|_{s=\omega i} \tag{10}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{I} \\ -\mathbf{P}_n^{-1}\mathbf{P}_0 & -\mathbf{P}_n^{-1}\mathbf{P}_1 & \cdots & \cdots & \cdots & -\mathbf{P}_n^{-1}\mathbf{P}_{n-1} \end{bmatrix},$$

$$\mathbf{P}_k = \begin{bmatrix} i^k \mathbf{T}_k & i^k \mathbf{H}_k \\ (-i)^k \mathbf{H}_k^T & (-i)^k \mathbf{T}_k^T \end{bmatrix},\quad k = 0,1,\dots n \tag{11}$$

$$\mathbf{H}_k = \begin{bmatrix} a_{pk} & a_{p-1,k} & \cdots & a_{1k} \\ 0 & \ddots & \ddots & a_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{pk} \end{bmatrix},$$

$$\mathbf{T}_k = \begin{bmatrix} a_{0k} & 0 & \cdots & 0 \\ a_{1k} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{p-1,k} & \cdots & a_{1k} & a_{0k} \end{bmatrix},$$

$$k = 0,1,\dots n-1 \tag{12}$$

and $a_j(s) = \sum_{k=0}^{p} a_{jk}\, s^k$ which are the terms defined in equation (4). Notice that equation (10) indicates that the imaginary roots of $\det \mathbf{A}_2(s) = 0$ are identical with the real eigenvalues of **P**, which is a constant matrix. So the numerical procedure is now converted into a simpler and more precise "real eigenvalue" determination of a constant matrix.

(b) <u>Elimination of transcendental terms</u> as introduced by (Walton and Marshall, 1987) and utilized in (Arunsawatwong, 1996; Tissir and Hmamed, 1996; Naimark and Zeheb, 1997; Jalili and Olgac, 1999; Filipovic and Olgac, 2002; Tuzcu and Ahmadian, 2002). This procedure follows the similar starting premise as in Schur-Cohn methodology in (a). If $CE(s,\tau)$ of equation (4) has an imaginary root, corresponding $\overline{CE}(s,\tau)$ of equation (5) should also have the same root. Multiplying equation (5) with $e^{-p\tau s}$ we obtain

$$e^{-p\tau s}\,\overline{CE}(s,\tau) = \sum_{k=0}^{p} \overline{a}_k(s)\, e^{(k-p)\tau s} = 0 \tag{13}$$

One can then eliminate the highest commensuracy term (i.e. $e^{-p\tau s}$) between equation (4) and equation (13) yielding a new equation

$$CE_1(s,\tau) = \sum_{k=0}^{p-1} a_k^{(1)}\, e^{-k\tau s} = 0 \tag{14}$$

which is of commensuracy degree of $p-1$. If one repeats this procedure of eliminating the highest degree commensuracy terms *p* times successively, one arrives at

$$CE_p(s) = a_0^{(p)}(s) = 0 \tag{15}$$

an algebraic characteristic equation with no transcendentality left. One can show that $a_0^{(p)}(s)$ is a polynomial of degree $n2^p$, of which purely imaginary roots are in question. Notice that due to the successive substitution of "s" with "-s" during the manipulations, the imaginary roots of the original characteristic equation CE are preserved, although the degree of the *s* terms in polynomials $CE_i(s)$ continuously increases. Ultimately there remains only $n2^p$ finite roots of $CE_p(s)$ instead of the infinitely many roots of the original $CE(s,\tau)$. It is guaranteed that only the imaginary roots of these two equations are identical. Therefore searching for the imaginary roots of $CE_p(s)$ is the sufficient procedure for the mission. The practical usage of this analytically elegant procedure in the literature (Arunsawatwong, 1996; Tissir and Hmamed, 1996; Jalili and Olgac, 1999; Tuzcu and Ahmadian, 2002)

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[1] We wish to make a remark on the formation error of equation (3.79) and (3.80) in (Chen, Gu et al., 1995), which needs to be corrected according to equation (7).

is very limited because of the round-off errors it invites during the successive evaluation of $CE_p(s)$.

Clearly for $n = p = 1,2$ the degrees of the polynomials of equation (8) and equation (15) are identical, and it is equal to 2. For these cases equations are indeed identical. For $n = p > 2$ which implies the case of full rank **B** matrix ($p = n$), $n2^n > 2n^2$ and clearly the procedure in (a) is much favorable proposition for determining the purely imaginary roots. Notice that the $n2^n - 2n^2$ excess roots may also contain some false imaginary roots, which should not appear at all. We suppress the proofs of these statements, but we will revisit them for example case studies later.

(c) <u>Matrix Pencil, Kronecker Sum method</u> introduced in (Chen, Gu et al., 1995; Su, 1995). The procedure departs from equation (2), which is rewritten as

$$\det[s\mathbf{I}-(\mathbf{A}+\mathbf{B}z)] = 0\,,\quad z = e^{-\tau s} \tag{16}$$

Using the argument that if $s = \omega i$ is a root of equation (16) so is $s = -\omega i$ when z is replaced with $1/z$. One can say that the eigenvalues of $\mathbf{A}+\mathbf{B}z$ and $\mathbf{A}+\mathbf{B}z^{-1}$ must be $s = \mp\omega i$, which can also be expressed using the property of the Kronecker sum (see Appendix for definition) of matrices. Commonly known property of this operation is that the eigenvalues of the Kronecker sum of two matrices are equal to the sum of the individual eigenvalues of the matrices (Brewer, 1978; Qiu and Davison, 1991). That is, at least one of the eigenvalues of

$$(\mathbf{A}+\mathbf{B}z)\oplus(\mathbf{A}+\mathbf{B}z^{-1}) \tag{17}$$

has to be zero ($\oplus$ is the Kronecker summation). In other words

$$\det[(\mathbf{A}+\mathbf{B}z)\oplus(\mathbf{A}+\mathbf{B}z^{-1})] = 0 \tag{18}$$

which gives rise to a polynomial in z of degree $2n^2$ for $n = p$. One needs to solve the $2n^2$ roots of equation (18) and determine those, which have the unity magnitude $|z| = 1$. For the roots, which satisfy this condition, one tries to solve next, the imaginary roots $s = \omega i$ from equation (16). Notice that substituting z as a complex number in equation (16) one obtains a polynomial with complex but constant coefficients. Therefore, most of the neat features of ordinary polynomials with constant coefficients disappear. For instance there is no guarantee of the complex conjugate feature of the roots. Therefore to decide whether an imaginary root $s = \omega i + \varepsilon$, $\varepsilon << 1$, is really an imaginary root except that it is displaced infinitesimally due to numerical/computational error, is not a trivial task. This particular point alone brings a weakness from the numerical deployment point of view.

As we explained in methodology (a) one can convert the symbolic determinant evaluation of equation (18) in an equivalent eigenvalue determination of a constant matrix, Theorem 3.1, (Chen, Gu et al., 1995). In this new form, a generalized eigenvalue problem

$$\det[z\mathbf{U}-\mathbf{V}] = z^{n^2}\det[(\mathbf{A}+\mathbf{B}z)\oplus(\mathbf{A}+\mathbf{B}z^{-1})] = 0 \tag{19}$$

where $\mathbf{U} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_2\end{bmatrix}_{2n^2\times 2n^2}$ $\mathbf{V} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{B}_0 & -\mathbf{B}_1\end{bmatrix}_{2n^2\times 2n^2}$

and $\mathbf{B}_0 = \mathbf{I}\otimes\mathbf{B}^T$, $\mathbf{B}_1 = \mathbf{A}\otimes\mathbf{A}^T$, $\mathbf{B}_1 = \mathbf{B}\otimes\mathbf{I}$ all of which are $(n^2\times n^2)$. Again the generalized eigenvalue operation is numerically much more reliable and efficient operation than evaluating the roots of the determinant in equation (18).

(d) <u>Kronecker multiplication /Elementary transformation method</u> (Louisell, 2001) Before we proceed, we wish to define the elementary transformation, $\xi : \mathbf{C}^{n\times n} \to \mathbf{C}^{n^2\times 1}$ (Brewer, 1978; Qiu and Davison, 1991), which is the key step enabling the procedure. ξ converts a matrix

$\mathbf{M}^{(n\times n)} = \begin{bmatrix}\mathbf{m}_1\\ \mathbf{m}_2\\ \vdots\\ \mathbf{m}_n\end{bmatrix}_{(n\times n)}$ into $(\xi\mathbf{M})^{(n^2\times 1)} = \begin{bmatrix}\mathbf{m}_1^T\\ \mathbf{m}_2^T\\ \vdots\\ \mathbf{m}_n^T\end{bmatrix}_{(n^2\times 1)}$ and

the multiplication of three $n\times n$ matrices $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ into a Kronecker product of dimension $n^2\times n^2$ (Brewer, 1978; Qiu and Davison, 1991), which is given as follows

$$\xi\,(\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3) = (\mathbf{P}_1\otimes\mathbf{P}_3^T)\ \xi\,\mathbf{P}_2 \tag{20}$$

where $(\bullet)^T$ denotes the transpose of $(\bullet)$. The aim is to form a $\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3$ product, which will then be mapped into the right hand side of (20). This mapping, as explained below, brings convenience in solving the pure imaginary roots of dynamics (1).

The procedure departs from equation (1) for which a solution of the form $\mathbf{x}(t) = e^{st}\ \mathbf{v}$ is suggested, where (s, $\mathbf{v}^{(n\times 1)}$) is an eigenvalue-eigenvector (both complex in general) pair. Differentiating this expression and substituting in (1), one gets

$$(s\mathbf{I}-\mathbf{A}-\mathbf{B}\ e^{-\tau s})\,\mathbf{v} = 0 \tag{21}$$

which can be rewritten as

$$(s\mathbf{I}-\mathbf{A})\ \mathbf{v} = e^{-\tau s}\ \mathbf{B}\mathbf{v} \tag{22}$$

If $s = \omega i$ is a root of equation (22) so is its conjugate $s = -\omega i$. We can express this by conjugating the complex equation in (22)

$$\mathbf{v}^*\ (-s\mathbf{I}-\mathbf{A}^T)\ = e^{\tau s}\ \mathbf{v}^*\,\mathbf{B}^T \tag{23}$$

where $(\bullet)^*$ denotes the conjugate transpose of $(\bullet)$. One can now multiply (22) and (23) side-by-side to get

$$(s\mathbf{I}-\mathbf{A})\ \mathbf{V}\ (s\mathbf{I}+\mathbf{A}^T)\ = -\mathbf{B}\mathbf{V}\mathbf{B}^T \tag{24}$$

with $\mathbf{V}^{(n\times n)} = \mathbf{v}\mathbf{v}^*$. This equation is exactly in the form to be transformed by using ξ as defined in (20). It returns

$$\begin{aligned}&\{(s\mathbf{I}-\mathbf{A})\otimes(s\mathbf{I}+\mathbf{A})+\mathbf{B}\otimes\mathbf{B}\}\ \xi\mathbf{V} = 0\\ &\Rightarrow\ \boldsymbol{\lambda}(s)\ \xi\mathbf{V} = 0\end{aligned} \tag{25}$$

For non-trivial solutions of $\xi\mathbf{V}\neq\mathbf{0}$, the only way to satisfy (25) is to set

$$\det \boldsymbol{\lambda}(s)=0 \qquad (26)$$

We can conclude that the desired imaginary roots are determined by solving a $2n^2$ degree polynomial (26). However, one should take notice that this root finding algorithm for higher dimensions ($2n^2$>10) becomes numerically unreliable due to repeated round-off errors in the determinant expansion procedure unless the operation is performed using very large number of significant digits. In that case, however, excessive computational cost will appear. In order to circumvent this difficulty using lower precision calculations, one can expand (25) using Kronecker product identities as defined in (Brewer, 1978; Qiu and Davison, 1991). Outcome of this is a matrix polynomial

$$\boldsymbol{\lambda}(s) = \mathbf{G}_0\, s^2 + \mathbf{G}_1\, s + \mathbf{G}_2 \qquad (27)$$

where $\mathbf{G}_0 = \mathbf{I} \otimes \mathbf{I}$, $\mathbf{G}_1 = \mathbf{I} \otimes \mathbf{A} - \mathbf{A} \otimes \mathbf{I}$, $\mathbf{G}_2 = \mathbf{B} \otimes \mathbf{B} - \mathbf{A} \otimes \mathbf{A}$. Then this matrix polynomial can be linearized (Gohberg, Lancaster et al., 1982) on the fact that $\mathbf{G}_0$ is an invertible matrix. The linearized form of (27) is expressed as

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_0 \\ -\mathbf{T}_2 & -\mathbf{T}_1 \end{bmatrix} \qquad (28)$$

with $\mathbf{T}_0 = \mathbf{I} \otimes \mathbf{I}$, $\mathbf{T}_1 = \mathbf{G}_0^{-1}\mathbf{G}_1$, $\mathbf{T}_2 = \mathbf{G}_0^{-1}\mathbf{G}_2$, where the zeros of $\det(\boldsymbol{\lambda}(s))=0$ are the eigenvalues of $\mathbf{F}$, i.e., $\det(\boldsymbol{\lambda}(s)) = \det(s\mathbf{I}-\mathbf{F}) = 0$. The imaginary roots of dynamics (1) have to be among the eigenvalues of $\mathbf{F}$ which can be computed by *eig*(**F**) subroutine of Matlab or *eigenvalues*(**F**) subroutine of Maple. Eventually, one can arrive at numerically more reliable set of imaginary roots of dynamics (1) as demonstrated in the example section.

(e) <u>Rekasius substitution</u> introduced in (Rekasius, 1980) and utilized by (Thowsen, 1981; Hertz, Jury et al., 1984; Olgac and Sipahi, 2002; Sipahi and Olgac, 2003a). The critical procedure is an exact substitution in equation (4)

$$e^{-\tau s} = \frac{1-Ts}{1+Ts}, \text{ when } s=\omega i \text{ \underline{only}}, \ T \in \Re \qquad (29)$$

where

$$\tau = \frac{2}{\omega}[\tan^{-1}(\omega T) \mp \ell\pi], \quad \ell = 0,1,... \qquad (30)$$

This exact substitution creates a new characteristic equation

$$CE(s,T) = \sum_{k=0}^{p} a_k(s)\left(\frac{1-Ts}{1+Ts}\right)^k = 0 \qquad (31)$$

Multiplying (31) with $(1+Ts)^p$, one obtains

$$\sum_{k=0}^{p} a_k(s)(1+Ts)^{p-k}(1-Ts)^k = 0 \qquad (32)$$

Considering that $a_k(s)$ are ordinary polynomials, equation (32) is nothing other than a polynomial in s with parameterized coefficients in (T). Since the system in equation (1) is retarded type, the highest degree term of s is n and it is in $a_0(s)$. Equation (32), therefore, is a polynomial of s in degree $n+p$. The question is to determine <u>all</u> $T \in \Re$ values, which cause imaginary roots of $s=\omega i$. This can be achieved by forming the Routh's array of the equation (32), and setting the only term in the s^1 row to zero (Sipahi and Olgac, 2003b; Olgac and Sipahi, 2004a, 2004b, 2005). It can be shown that this polynomial is of degree np in T, of which only the real roots are searched. Once these roots are determined the corresponding crossing frequencies ($s=\omega i$) can be found using the auxiliary equation, which is formed by the s^2 row of the Routh's array (Sipahi and Olgac, 2003b; Olgac and Sipahi, 2004a, 2004b, 2005). Notice that, the s^2 row has two terms, which are functions of T. They must agree in sign for those T values to yield imaginary roots. Final results are exhaustive in detecting all the imaginary characteristic roots we set out to solve.

In the case of degenerate imaginary roots at the origin, $s=\omega i$ with $\omega=0$, one needs to check in addition, the constant term in equation (32) with no s term; if

$$\sum_{k=0}^{p} a_k(0) = 0 \qquad (33)$$

is satisfied or not. If it does there is at least one root at $s=0$, which remains there for all $\tau \in \Re^+$. It is easy to determine if this root is a multiple root for some τ values.

Notice the search domain of T is an open domain, as such it includes $T=\mp\infty$ as well. Understandably it is easy to check whether these unbounded T values are of interest to us or not. In that case, for $T=\mp\infty$, $e^{-\tau s} \to -1$ leaves $CE(s,\tau)$ as a simple polynomial of s. The roots of this polynomial can be found easily. If any one of these is purely imaginary, that root becomes part of the solution of interest as well.

III. A NUMERICAL CASE STUDY AND COMPARATIVE OBSERVATIONS

We now take an example case study to display a comparison among the five methodologies we discussed above. Consider the numerical example in (Olgac and Sipahi, 2002) which has $p = rank(\mathbf{B}) = 3 = n$.

$$\mathbf{A} = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{pmatrix} \qquad (34)$$

The respective characteristic equation is

$$\begin{aligned} CE(s,\tau) &= s^3 + 6s^2 + 45.5s + 111 \\ &+ (0.9s^2 - 116.8s - 22.1)e^{-\tau s} \\ &+ (90.9s - 185.1)e^{-2\tau s} + 119.4e^{-3\tau s} \end{aligned} \qquad (35)$$

in which $a_k(s)$, $k=0,...,3$ expressions are readily identified as represented in equation (4). For this system the following exhaustive list of (τ_{0i}, ω_i) is given in (Olgac and Sipahi, 2002), using methodology (e).

On equation (35) we now start utilizing the five methodologies as described above. The critical point of comparison is to be able to declare the complete Table 1. In order to conserve space the numerical results are given in truncated forms at the fourth decimal except where necessary for the arguments.

Table 1. The fundamental delays and the resonant frequencies

$\tau_{0i}\,[s]$	$\omega_i\,[rad/s]$
7.208	0.8404
0.8725	2.1109
0.1859	2.9123
0.1624	3.0351
0.2219	15.5032

(a) Schur-Cohn procedure: Λ_1 and Λ_2 matrices of equation (9) are readily formed using the terms of $a_0(s)$, … , $a_3(s)$. Notice that the degree of $\det(\mathbf{A}_2)$ of equation (8) is $2np=18$. Thus the mission is to determine the purely imaginary roots of

$$\begin{aligned}\det(\mathbf{A}_2) = {} & s^{18} + 0.1\ 10^{-7}\ s^{17} + 165.81\ s^{16} \\ & + 0.3\ 10^{-4}\ s^{15} - 18681.53\ s^{14} + 0.022\ s^{13} \\ & - 248693.82\ s^{12} + 4.5\ s^{11} + 35141.36\ s^{10} \\ & + 330.83\ s^{9} + 0.41\ 10^{10}\ s^{8} + 8671.27\ s^{7} \\ & + 0.94\ 10^{11}\ s^{6} + 72932.09\ s^{5} + 0.71\ 10^{12}\ s^{4} \\ & + 148284.13\ s^{3} + 0.19\ 10^{13}\ s^{2} + 29249.72\ s \\ & + 0.1\ 10^{13} = 0\end{aligned} \tag{36}$$

which displays two major obstacles:

(i) Because of the accumulated numerical errors the imaginary roots shape up in the form of $s=\omega i+\varepsilon$, $\varepsilon<<1$. For instance using **20 digits** of precision in MAPLE, equation (36) gives the following, apparently imaginary roots:

$0.21\ 10^{-7} \pm 0.8404\,i$, $0.28\ 10^{-6} \pm 2.1109\,i$,
$-0.55\ 10^{-8} \pm 15.5032\,I$ $-0.13\ 10^{-5} \pm 2.9123\,i$,
$0.69\ 10^{-6} \pm 3.0351\,i$

As can be seen the numerical error in real parts (which are supposed to be zero) are at least in the order of 10 magnitudes larger than the computational precision. Therefore it is problematic to decide whether these roots, which are very close to the imaginary axis, are really imaginary roots or not.

(ii) Equation (36) is expected to have only even powers of s. One cannot achieve this even with 60-digit precision. The trial based determination of the significant digits necessary is an important hindrance. For comparison purposes, we consider the errors at the level of up to *¼ of significant digits* as acceptable accuracy. Such as for 20-digit operation, any error of 10^{-15} and less is considered zero.

An alternative but more reliable procedure on this line was described earlier. Accordingly, the real eigenvalues of the **P** matrix are evaluated. Notice that some of the a_{jk} terms given in equations (11-12) are identified as 0 based on the special formation of Eq.(35). Finally one obtains the five crossing frequencies (i.e., the real eigenvalues of **P**) as $0.8404 - 0.44\ 10^{-17}\,i$, $2.1109 + 0.66\ 10^{-16}\,i$, $2.9123 + 0.63\ 10^{-15}\,i$, $3.0351 + 0.68\ 10^{-15}\,i$, $15.5032 + 0.72\ 10^{-17}\,i$ when 20 digits precision is used. These numbers are all acceptably close to the desired results under the ¼-digit rule expressed earlier.

(b) Elimination of transcendental terms: Starting from the characteristic equation (35) and following the three steps as described in the above section, one can eliminate $e^{-3\tau s}$, $e^{-2\tau s}$ and $e^{-\tau s}$ sequentially. Notice that these steps preserve the purely imaginary roots of the characteristic equation. The final form should contain only even powers of s and it does;

$$\begin{aligned}CE_p(s) = {} & s^{24} + 220.81\,s^{22} - 8823.73\ s^{20} \\ & - 0.12\ 10^{7}\ s^{18} - 0.27\ 10^{8}\ s^{16} + 0.39\ 10^{10}\ s^{14} \\ & + 0.32\ 10^{12}\ s^{12} + 0.89\ 10^{13}\ s^{10} + 0.12\ 10^{15}\ s^{8} \\ & + 0.81\ 10^{15}\ s^{6} + 0.28\ 10^{16}\ s^{4} + 0.44\ 10^{16}\ s^{2} \\ & + 0.2\ 10^{16} = 0\end{aligned} \tag{37}$$

The imaginary roots of the original system have to be among the roots of this polynomial. We again use 20 significant digits of precision in MAPLE during the manipulations. The same level of precision is used in the following steps of root finding. A cautionary note; for the case of up to 18 digits of precision in MAPLE, one can observe odd powered terms appearing also.

Equation (37) results in 24 symmetric roots with respect to the origin, of which only the purely imaginary ones are important. They are

$$\begin{aligned}& \pm 5.9977\,i,\ \pm 3.9460\,i,\ \pm 1.8587\,i, \\ & \pm 0.8404\,i,\ \pm 2.1109\,i, \pm 2.9123\,i, \\ & \pm 3.0351\,i, \pm 15.5032\,i\end{aligned} \tag{38}$$

It is easy to demonstrate that the first three roots on the 1st row are faulty findings and they do not represent true crossings. To see that one can substitute them into the original characteristic equation (35) and show that they do not satisfy this equation. On the other hand the remaining 5 roots do. They also yield some $|e^{-\tau\omega i}|=1$ from which one can determine the respective time delays[2].

One can also observe that the difference between the degrees of equation (36) and equation (37) is 6. In fact the relation between the two polynomials is described as

$$CE_p(s,\tau) = P_1(s)\,\det(\mathbf{A}_2) \tag{39}$$

[2] Incidentally, the procedure described in (Walton and Marshall, 1987), equation (48), yields incorrect time delays due to the accumulation of numerical error. Primarily an ill conditioning occurs due to some coefficients in equation (15) being 15-16 orders of magnitude apart from one another.

where the $\text{degree}(P_1) = 6$, and its roots result in the 3 false pairs of imaginary roots, as explained above.

(c) <u>Matrix pencil, Kronecker sum application</u>

Deploying equation (18) we obtain

$$\begin{aligned}
&\det[(\mathbf{A}+\mathbf{B}z)\oplus(\mathbf{A}+\mathbf{B}z^{-1})] = \frac{1}{z^9}(0.17\,10^7\, z^{18} \\
&-141137.96\, z^{17} + 0.23\,10^8\, z^{16} - 0.17\,10^9\, z^{15} \\
&+ 0.52\,10^9\, z^{14} - 0.81\,10^9\, z^{13} + 0.62\,10^9\, z^{12} \\
&- 0.32\,10^8\, z^{11} - 0.43\,10^9\, z^{10} + 0.56\,10^9\, z^9 \\
&- 0.43\,10^9\, z^8 - 0.32\,10^8\, z^7 + 0.62\,10^9\, z^6 \\
&- 0.81\,10^9\, z^5 + 0.52\,10^9\, z^4 - 0.17\,10^9\, z^3 \\
&+ 0.22\,10^8\, z^2 - 141137.96\, z + 0.17\,10^7)
\end{aligned} \tag{40}$$

We are seeking the roots of equation (40) with magnitude equal to 1. There is a major difficulty, however, specifically in deciding the tolerance level of $|z| = 1$. This difficulty disappears when higher precision (above 20) is used. For example, using 20 digits of precision the errors in magnitudes of the five roots become precisely zero. They are listed together with their corresponding 5 imaginary roots, which are obtained by replacing $z = e^{-\tau\omega i}$, $|z| = 1$, in equation (35) and solving for s

Table 2. The tolerances in evaluationg z's and the corresponding frequencies

$1-\lvert z\rvert$	ω_i
0	$-0.57\ 10^{-19} + 0.8404\,i$
0	$0.47\ 10^{-20} + 2.1109\,i$
0	$-0.73\ 10^{-19} + 2.9123\,i$
0	$0.5\ 10^{-19} + 3.0352\,i$
0	$-10^{-21} + 15.5032\,i$

Notice the cumulative effects of numerical error in ω_i, which are all acceptable at this precision level (in ¼-digit sense). However with up to 13 digits of precision, one observes relatively large accumulated errors (violating the ¼-digit rule) in the computation of imaginary roots. Therefore if a root of z displays $|z|-1 = \varepsilon$ it is quite difficult to assess whether we should take it as an indicator of crossing or not. Similar question arises for $s = \omega i + \varepsilon$. Does it represent a true imaginary root or not? This is a tough question to answer. Increasing the digits higher than 14 (obviously including 20) this concern is eliminated.

When the more reliable procedure is followed, as mentioned in the description above, **U** and **V** matrices are trivially obtained from their definitions. The generalized eigenvalues of (**U**, **V**) pair that have unitary magnitudes (obtained with 20 digit-precision again) are given in Table 3.

Interestingly on this implementation one can use 9 digit-precision and can still obtain acceptable errors looking at the real parts of the imaginary roots.

Table 3. The tolerances in evaluating z's and the corresponding frequencies.

$1-\lvert z\rvert$	ω_i
$-0.14\ 10^{-17}$	$-0.25\ 10^{-17} \mp 0.8404\,i$
$-0.34\ 10^{-17}$	$0.27\ 10^{-17} \mp 2.1109\,i$
$0.12\ 10^{-16}$	$0.18\ 10^{-16} \mp 2.9123\,i$
$-0.78\ 10^{-17}$	$0.31\ 10^{-16} \mp 3.0351\,i$
$0.4\ 10^{-18}$	$-0.95\ 10^{-18} \mp 15.5032\,i$

Table 4. The tolerance in evaluation z's and the corresponding frequencies

$1-\lvert z\rvert$	ω_i
0	$-0.15\ 10^{-7} \mp 0.8404\,i$
-10^{-8}	$0.66\ 10^{-8} \mp 2.1109\,i$
-10^{-8}	$0.4\ 10^{-7} \mp 2.9123\,i$
$-0.18\ 10^{-7}$	$-0.42\ 10^{-7} \mp 3.0351\,i$
0	$0.27\ 10^{-7} \mp 15.5032\,i$

This table indicates that the generalized eigenvalue computation can be performed with a much smaller number of digits, such as 9, without sacrificing the accuracy of the roots.

(d) <u>Kronecker multiplication and elementary transformation method</u> Following the procedure given in method (d) in Section II, one obtains equation (26), again with 20-digits precision, as

$$\begin{aligned}
&\det \boldsymbol{\lambda}(s) = s^{18} + 0.3\,10^{-9}\, s^{17} + 165.81\, s^{16} \\
&+ 0.99\,10^{-6}\, s^{15} - 18681.53\, s^{14} + 0.0007\, s^{13} \\
&- 248693.48\, s^{12} + 0.16\, s^{11} + 35183.49\, s^{10} \\
&+ 12.97\, s^9 + 0.41\,10^{10}\, s^8 + 370.2\, s^7 \\
&+ 0.94\,10^{11}\, s^6 + 3348.51\, s^5 + 0.71\,10^{12}\, s^4 \\
&+ 7209.8\, s^3 + 0.19\ 10^{13}\, s^2 + 1480\, s + 0.1\,10^{13} = 0
\end{aligned} \tag{41}$$

The roots of Eq(41) which are almost purely imaginary are the expected crossing frequencies as given in Table 5. With this precision level, the accumulated errors in real parts are relatively large (in ¼-digit sense) in magnitudes as can be seen in this table.

As mentioned in the earlier section, in order to improve the accuracy of the imaginary roots of (1), we pursue computing the eigenvalues of matrix **F**, *eig*(**F**). For this computation 10 and 20 digit-precisions are used separately. These imaginary roots, which are formed, are tabulated on Table 6, with the error terms in real parts. They are all acceptably small deviations. Thus we conclude that this method can produce the imaginary roots accurately even with 10-digit operations.

Table 5. Crossing frequencies

ω_i
$0.1\ 10^{-8} \mp 0.8404\,\mathrm{i}$
$0.13\ 10^{-7} + 2.1109\,\mathrm{i}$
$-0.82\ 10^{-7} + 2.9123\,\mathrm{i}$
$0.53\ 10^{-7} + 3.0352\,\mathrm{i}$
$-0.1\ 10^{-9} + 15.5032\,\mathrm{i}$

Table 6. Crossing frequencies

10 digit precision ω_i	20 digit precision ω_i
$0.8847\ 10^{-15} \mp 0.8404\,i$	$0.88\ 10^{-18} \mp 0.8404\,\mathrm{i}$
$0.111\ 10^{-13} \mp 2.1109\,i$	$-0.1\ 10^{-18} + 2.1109\,\mathrm{i}$
$-0.1856\ 10^{-12} \mp 2.9123\,i$	$0.19\ 10^{-16} + 2.9123\,\mathrm{i}$
$0.172\ 10^{-12} \mp 3.0351\,i$	$-0.19\ 10^{-16} + 3.0352\,\mathrm{i}$
$-0.355\ 10^{-14} \mp 15.5032\,i$	$0.66\ 10^{-17} + 15.5032\,\mathrm{i}$

(e) **Rekasius substitution** In this procedure, at least 13-digit of precision is required in order to obtain the crossing frequencies as listed in the beginning of this section. Characteristic equation of (35) turns out to be

$$\begin{aligned} CE(s,T) &= T^3 s^6 + (5.1\,T^3 + 3\,T^2)s^5 \\ &+ (253.2\,T^3 + 17.1\,T^2 + 3\,T)s^4 \\ &+ (-171.4\,T^3 + 162.4\,T^2 + 18.9\,T + 1)s^3 \\ &+ (898.4\,T^2 - 71.2\,T + 6.9)s^2 \\ &+ (137.8\,T + 19.6)s + 23.2 \end{aligned} \tag{42}$$

for which we are looking for $T \in \Re$ to cause imaginary roots. The numerator of the only term of the s^1 row of the respective Routh's array on equation (42) is

$$\begin{aligned} &T^3 (0.4004\,10^7\,T^9 - 0.5418\,10^6\,T^8 \\ &-0.1060\,10^7\,T^7 - 0.7869\,10^5\,T^6 \\ &-0.1501\,10^5\,T^5 + 0.1216\,10^4\,T^4 \\ &+ 0.4011\,10^3\,T^3 - 0.1025\,10^2\,T^2 \\ &+ 0.1197\,T - 0.1120) = 0 \end{aligned} \tag{43}$$

Notice that we are seeking the real roots of equation (43) and 3 of them are at $T = 0$. Remaining ones, i.e., 9 ($= n^2$) of them have to be solved. Out of these 9 roots only five happen to be real and all five satisfy the sign agreement condition in the s^2 row of Routh's array (see earlier note). They concur with the results of (Olgac and Sipahi, 2002) which uses a slightly different procedure to arrive at these T values. Auxiliary equation is formed by the terms of the row s^2 generating the crossing frequencies, ω. The T vs. ω correspondence is shown in Table 7.

Table 7. Rekasius substitution results

$$[T] = \begin{bmatrix} -0.4269 \\ -0.1332 \\ 0.0828 \\ 0.0952 \\ 0.6232 \end{bmatrix} \rightarrow [\omega] = \begin{bmatrix} 15.5032 \\ 0.8404 \\ 3.0351 \\ 2.9123 \\ 2.1109 \end{bmatrix}$$

Note that using T and ω, one can also find the respective delays, τ as per equation (30).

IV. COMPARATIVE COMMENTS AND CONCLUSIONS

The five procedures described above reduce the problem at hand to the solution of

(a) Schur-Cohn: $2n^2$ degree of complex polynomial for purely imaginary roots.
(b) Elimination of transcendental terms: $n2^n$ degree real polynomial for imaginary roots.
(c) Kronecker sum: $2n^2$ degree polynomial for complex roots with $|z|=1$.
(d) Kronecker multiplication: $2n^2$ degree polynomial is solved for imaginary roots.
(e) Rekasius: n^2 degree polynomial for real roots ($T \in \Re$).

From the perspective of developmental steps, the least involved path appears to be in (e). Nevertheless, utilizing the appropriate matrix operations the methods (a), (c) and (d) prove to be equally potent producing the desired results. In fact for the example case method (d) generates the results with the smallest number of significant digits among the five. Method (b) is the one that requires special attention avoiding the false solutions. As the dimension, n, increases method (b) becomes more problematic to process leaving the other 4 methods as very attractive paths to follow.

We wish to mention two attractive features of method (e): First, the degree of the polynomial in question is considerably smaller than all the other four. Second, we seek the real roots (*T*) only, not complex ones. This is a great relief as complex (particularly purely imaginary) roots are very hard to detect when numerical errors creep in. Similar hardship appears when the test of $|z|=1$ is performed.

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APPENDIX

Let $\mathbf{A} = [a_{ij}] \in \Re^{k \times \ell}$, $\mathbf{B} = [b_{ij}] \in \Re^{m \times n}$. Then the Kronecker product of **A** and **B**, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as follows:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & & \vdots \\ a_{n1}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{bmatrix} \in \Re^{km \times \ell n} \tag{A1}$$

If $k = \ell$ and $m = n$, the Kronecker sum of **A** and **B**, denoted by $\mathbf{A} \oplus \mathbf{B}$, is defined by

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_m + \mathbf{I}_k \otimes \mathbf{B} \in \Re^{km \times km} \tag{A2}$$