ROBUST STABILIZATION OF LINEAR TIME-DELAY SYSTEMS VIA THE OPTIMIZATION OF REAL STABILITY RADII

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Abstract: We present a new numerical procedure for the robust stabilization of linear time-delay systems, based on the position of the eigenvalues in the complex plane. We assume static perturbations on the system matrices and express the robustness of stability in terms of real stability radii. We outline how real stability radii can be computed and maximized as a function of controller parameters. The latter synthesis problem is solved by a quasi-continuous shaping of appropriate frequency response plots.

Keywords: linear time-delay systems, robust stabilization, stability radii

1. INTRODUCTION

We study the robust stabilization of the system

$$\dot{x}(t) = \sum_{i=1}^{l} A_i x(t-\tau_i) + \sum_{i=1}^{l} B_i u(t-\tau_i), \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ are the inputs and $0 \leq \tau_1 \ldots < \tau_l$ are delays in both state and inputs. We use a controller of the form

$$u = Kx(t). \qquad (2)$$

In a previous paper (Michiels *et al.*, 2002), the eigenvalue based *continuous pole placement method* was developed to compute a stabilizing feedback matrix K (if it exists). Because the closed-loop system has infinitely many eigenvalues (Hale and Verduyn Lunel, 1993), the method consists of controlling only the rightmost or unstable eigenvalues, which are moved to the left half plane in a quasi-continuous way by applying small changes to the feedback gain K, and meanwhile monitoring the other eigenvalues with a large real part. The procedure can be applied until the rightmost eigenvalues cannot be moved further to the left using the available controller parameters, i.e. when the function

$$F(K) = \sup\Big\{\Re(\lambda) : \det(\lambda I - \sum_{i=1}^{l}(A_i + B_i K)e^{-\lambda\tau_i}) = 0\Big\} \qquad (3)$$

is minimal. For stabilizable systems, this means that the exponential decay-rate of the closed-loop solutions is maximal. This may however not be the best solution from a robustness point of view, as illustrated in (Michiels and Roose, 2003).

In this work we assume static perturbations on the system matrices and the feedback gain of the stabilized system (1)-(2) and consider the adaptation of the feedback gain in such a way that some robust stability measures are optimized. These measures are expressed by real stability radii (Qiu *et al.*, 1995), defined as the norm of the smallest destabilizing real perturbations. The overall eigenvalue based robust stabilization algorithm then consists of two steps, first computing a stabilizing gain using the continuous pole placement method (Michiels *et al.*, 2002) and, second, optimizing the robustness of the achieved stability by maximizing real stability radii.

In the literature on DDEs, the robust stabilization problem has been widely studied in a Lyapunov context, see for instance (Li and de Souza, 1997*b*; Li and de Souza, 1997*a*; Cao *et al.*, 1998; Niculescu, 1998; Mahmoud, 2000). This approach allows to include easily more general types of perturbations (e.g. time-varying). However, practical stability results are in the form of sufficient conditions They are usually expressed by the feasibility of LMIs or the solvability of AREs, which in general induces a lot of conservatism, due to the choice of the form of the functional and the estimates involved in the derivation of the stability criteria. The results presented in (Michiels and Roose, 2003) can be considered as a way to tighten the gap between sufficient and necessary conditions by an alternative approach, directly related to the position of the eigenvalues in the complex plane. In this reference robustness of stability is expressed in terms of complex stability radii, which are optimized in function of the controller parameters. Some conservatism however remains in the obtained uncertainty bounds because the allowed perturbations are complex matrices. This is removed in this paper by restricting the perturbations to be (realistic) real matrices, at the price of a higher computational cost.

The structure of the paper is as follows. First we rehearse the concept of stability radii in the context of robust stability of time-delay systems, based on (Michiels and Roose, 2003). Then we describe a numerical procedure to compute real stability radii. Finally we outline an algorithm for the optimization of real stability radii and present an example.

2. STABILITY RADII AS ROBUSTNESS MEASURES

We assume that the controlled system (1)-(2) is asymptotically stable and consider the stability of the perturbed system,

$$\dot{x}(t) = \sum_{i=1}^{l} \big((A_i + \delta A_i) + (B_i + \delta B_i)(K + \delta K)\big) x(t - \tau_i), \quad (4)$$

under various classes of perturbations on the system matrices. For sake of generality we assume that the elements of the matrix perturbations belong to a field $\mathbb{F}$ which can be either $\mathbb{R}$ or $\mathbb{C}$.

Stability radii correspond to the size of the smallest perturbations which result in a shift of an eigenvalue to the closed right half plane and, hence, cause instability. For instance, we define the stability radius w.r.t. changes of A_1 in $\mathbb{F}$ (for the Euclidean norm) as

$$r_{\mathbb{F}}^{A_1} \triangleq \inf_{\delta A_1 \in \mathbb{F}^{n\times n}} \Big\{ \sigma_1(\delta A_1) : \exists \omega \in \mathbb{R} \text{ s.t. } \det\Big(j\omega I - (A_1 + \delta A_1)e^{-j\omega\tau_1} - \sum_{i=2}^{l}(A_i + B_i K)e^{-j\omega\tau_i}\Big) = 0 \Big\}. \quad (5)$$

In a similar fashion, one can define $r_{\mathbb{F}}^{B}$, $r_{\mathbb{F}}^{K}$ etc. When $\mathbb{F} = \mathbb{R}$ resp. $\mathbb{F} = \mathbb{C}$ then (5) is called the real, resp. complex stability radius w.r.t. changes of A_1.

For any of the perturbations in (4), the characteristic equation of the perturbed system on the imaginary axis can be written in the form

$$\det(I - M_K(j\omega)\Delta) = 0,$$
$$M_K(j\omega) = X(j\omega)\left(j\omega I - \sum_{i=1}^{l}(A_i + B_i K)e^{-j\omega\tau_i}\right)^{-1} Y(j\omega), \quad (6)$$

where $\Delta \in \mathbb{F}^{p\times q}$ is the perturbation under consideration and X and Y depend on the type of this perturbation [1], e.g. for $\delta A_1 = \Delta$ we have $X = I$, $Y = e^{-j\omega\tau_1} I$ and for $\delta K = \Delta$, we have $X = I$, $Y = \sum_{i=1}^{l} B_i e^{-j\omega\tau_i}$.

When defining the matrix function $\mu_{\mathbb{F}}(.)$ by

$$\mu_{\mathbb{F}}(M) = \begin{cases} 0, \quad \text{when } \det(I - M\Delta) \neq 0,\ \forall \Delta \in \mathbb{F}^{p\times q}, \\ \left[\inf\limits_{\Delta \in \mathbb{F}^{p\times q}} \{\sigma_1(\Delta) : \det(I - M\Delta) = 0\}\right]^{-1}, \\ \qquad\qquad\qquad \text{otherwise} \end{cases}, \quad (7)$$

we can express

$$r_{\mathbb{F}} = \frac{1}{\sup_{j\omega} \mu_{\mathbb{F}}(M_K(j\omega))}. \quad (8)$$

Indeed, notice from (6) and (7) that $\mu_{\mathbb{F}}(M_K(j\omega))$ is the inverse of the size of the smallest perturbation with elements in $\mathbb{F}$, which shifts an eigenvalue to $\lambda = j\omega$, in case such perturbations exist, and is equal to zero otherwise.

As a standard result from robust control theory, we have

$$\mu_{\mathbb{C}}(\cdot) = \sigma_1(\cdot), \quad (9)$$

see e.g. (Zhou *et al.*, 1995). The main result of (Qiu *et al.*, 1995) states that $\mu_{\mathbb{R}}(\cdot)$ can be computed in the following way:

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,\ 1]} f(M, \gamma) := \inf_{\gamma \in (0,1]} \sigma_2\left(\begin{bmatrix} \Re(M) & -\gamma \Im(M) \\ \gamma^{-1}\Im(M) & \Re(M) \end{bmatrix}\right). \quad (10)$$

Furthermore, in the special case rank $\Im(M) = 1$ it satisfies

$$\mu_{\mathbb{R}}(M) = \max\left\{\sigma_1(U_2^T \Re(M)), \sigma_1(\Re(M)V_2)\right\}, \quad (11)$$

where U_2 and V_2 come from any singular value decomposition of $\Im(M)$:

$$\Im(M) = [U_1\ U_2]\begin{bmatrix} \sigma_1(\Im(M)) & 0 \\ 0 & 0 \end{bmatrix}[V_1\ V_2]^T.$$

3. COMPUTATION OF REAL STABILITY RADII

From (8) and (10)-(11) one can determine real stability radii by searching the maximum of the function

$$\omega \in \mathbb{R}^+ \to \mu_{\mathbb{R}}(M_K(j\omega)). \quad (12)$$

[1] This is also the case for *structured* perturbations, e.g. $\delta A_1 = S_1.\Delta.S_2$, and for classes of combined perturbations on several system matrices. In the reduction to the standard form (6) the relation $\det(I + GH) = \det(I + HG)$ is useful.

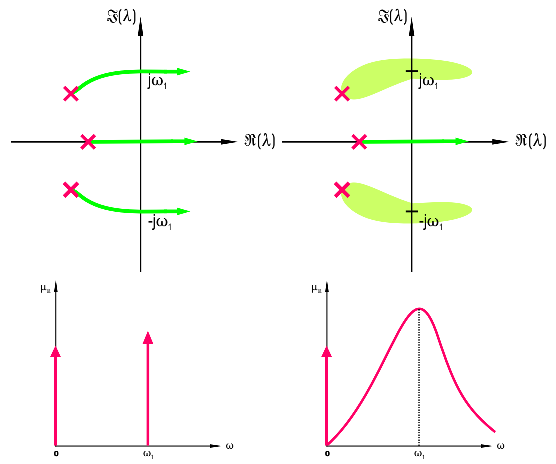

Fig. 1. Typical behavior of eigenvalues as a function of perturbed parameters, and the associated curve $\omega \to \mu_{\mathbb{R}}(\omega)$. (Left) - perturbation on one parameter; (right) - perturbation on multiple parameters.

For notational convenience we will from now on write $\mu_{\mathbb{R}}(\omega)$ instead of $\mu_{\mathbb{R}}(M_K(j\omega))$, when no confusion is possible.

Properties of the function (12) carry over from the results of (Qiu *et al.*, 1995) on finite-dimensional systems. Of major importance from a computational point of view is the fact that it is *not* continuous, yet only upper semi-continuous. More precisely, discontinuities can occur at frequencies where $\Im(M_K(j\omega)) = 0$. At such point we have $\mu_{\mathbb{R}} = \mu_{\mathbb{C}}$, while in general we have $\mu_{\mathbb{R}} \leq \mu_{\mathbb{C}}$ (the set of real perturbations form a subset of the set of complex perturbations). As follows intuitively from the interpretation (7) discontinuities in the curve $\omega \to \mu_{\mathbb{R}}(\omega)$ generically occur in specific cases only. In case of *one* parameter being perturbed, i.e. $\Delta \in \mathbb{R}$, the root locus of eigenvalues as a function of Δ consists of a number of curves. By changing Δ eigenvalues can only cross the imaginary axis at a countable number of frequencies. Therefore, $\mu_{\mathbb{R}}$ only differs from zero at these frequencies, see Fig. 1 (left). In case of multiple parameters being perturbed simultaneously, the situation of Fig. 1 (right) applies. The multiple degrees of freedom allow eigenvalues to cross the imaginary axis at several adjacent frequencies. An exception however holds for to the case $\omega = 0$, because an isolated real eigenvalue cannot leave the real axis under any sufficiently small real perturbation. Indeed, since eigenvalues occur in complex-conjugate pairs, this would require the interaction with another real eigenvalue.

When there is one uncertain parameter, searching the maximum of the function (12) requires to compute the frequencies where a discontinuity occurs, being the zeros of the *scalar* function $\omega \to \Im(M_K(j\omega))$, and to evaluate $\mu_{\mathbb{R}}$ at these frequencies. Following their definition real stability radii can alternatively be computed using a continuation procedure, where the uncertain parameter is changed quasi-continuously and the rightmost eigenvalues are monitored meanwhile until a crossing of the imaginary axis occurs. This can be done using the package DDE-BIFTOOL (Engelborghs *et al.*, 2001).

For multiple uncertain parameters parts of the function (12) need to be computed. The positive ω-axis is screened from $\omega = 0$ on, while computing $\mu_{\mathbb{R}}$ at points ω_i of a frequency grid and storing the maximum over the interval $[0,\ \omega_i]$. Regarding discontinuities only $\omega = 0$ deserves special attention, but to detect a non-generic case we also monitor $\|\Im(M_K(j\omega)\|$ along the ω-axis. The efficiency of the algorithm is improved in the following way:

- The function to be minimized in the right-hand side of (10), $f(M,\cdot)$, is unimodal (Qiu *et al.*, 1995), which makes a *golden search* method appropriate.
- The computation of $r_{\mathbb{R}}$ based on (8) and (10) is in fact a mini-max problem, which is exploited as follows: when for $\omega_2 > \omega_1$ and some $\gamma^* \in (0,\ 1]$, we have
$$f(M_K(j\omega_2),\gamma^*) \leq \mu_{\mathbb{R}}(\omega_1), \qquad (13)$$
then $\mu_{\mathbb{R}}(\omega_2) \leq \mu_{\mathbb{R}}(\omega_1)$. Thus if we have computed $\mu_{\mathbb{R}}(\omega_1)$ and (13) holds, then the knowledge of $\mu_{\mathbb{R}}(\omega_2)$ is not required to find the maximum of the curve $\omega \to \mu_{\mathbb{R}}(\omega)$. Therefore, the optimization in (10) is not needed anymore and, while further screening the ω-axis, γ can be frozen temporarily until $f(M_K(j\omega),\gamma^*) > \mu_{\mathbb{R}}(\omega_1)$.
Note that when M_K (and the perturbations) are *vectors* one can use (11) to directly compute $\mu_{\mathbb{R}}$, instead of (10).
- A frequency grid with adaptive steplength is used.
- The maximum frequency of the grid is adapted during computations. It is easy to show that $\mu_{\mathbb{R}}(\omega) < \xi$ for some $\xi > 0$, if
$$\omega \geq \sum_{i=1}^{l} \|A_i + B_i K\| + \frac{1}{\xi}\|X(j\omega)\|_\infty \|Y(j\omega)\|_\infty. \qquad (14)$$
The maximum frequency of the grid is initially determined from (14) with $\xi = \mu_{\mathbb{R}}(0)$. This bound can be refined by using
$$\xi = \max_{\omega \in [0,\ \bar{\omega}]} \mu_{\mathbb{R}}(\omega)$$
after having screened the frequency axis over $[0,\ \bar{\omega}]$.

Remark 1. The fast iterative methods to calculate stability radii of finite-dimensional systems, as (Streedhar *et al.*, 1996), are not directly applicable in the DDE case. Although the frequencies ω where a singular value of $M_K(j\omega)$ equals a constant value also coincide with the imaginary eigenvalues of a Hamiltonian system, the latter is now infinite-dimensional and described by a functional differential equation with both delayed and advanced terms. For a given dimension n of the system, even arbitrarily many crossing frequencies are possible, which prevents a fast direct calculation, see (Michiels and Roose, 2003) for an illustration.

But despite of the infinite-dimensional nature of DDEs, they describe an evolution in the finite- dimensional space $\mathbb{R}^n$, and the dimensions of $M_K(j\omega)$ are of order n. Hence, when n is small, the computation of parts of (12) is reasonable and it will be hard to find an alternative which performs better than this 'rough' approach. Notice that a small dimension occurs in many applications and that high dimensional systems (e.g. described by PDEs) may be approximated precisely by *low-order models with delays*, see (Niculescu, 2001; Kolmanovskii and Myshkis, 1999).

4. OPTIMIZATION OF REAL STABILITY RADII

Following from (8) maximizing stability radii as a function of the gain K corresponds to the minimization problem:

$$\min_K \sup_{\omega\geq 0} \mu_{\mathbb{R}}(M_K(j\omega)). \tag{15}$$

To solve this highly complex optimization problem (e.g. a non-convex, non-differentiable objective function), we propose an iterative numerical procedure, which applies a local strategy consisting of a quasi-continuous reduction of the objective function by making *small changes* to the feedback gain per iteration step. The latter is important because it assures the asymptotic stability of (1)-(2) during the iteration process when the initial gain in stabilizing, as the robustness of stability is enlarged all the time. Notice at this point that the objective function is defined for all gain values, where (1)-(2) has no eigenvalues on the imaginary axis, and that a large change of the gain could eventually lead to a local minimum of (15), where the corresponding gain is not stabilizing.

The basic algorithm is similar to the algorithm of (Michiels and Roose, 2003) for the optimization of complex stability radii, which was in turn inspired by the continuous pole placement algorithm for the (non-robust) stabilization of DDEs (Michiels *et al.*, 2002):

Algorithm 1. [Minimization $\sup_\omega \mu_{\mathbb{R}}(M_K(j\omega))$ as a function of K]

A. Initialize $m = 1$.
B. Compute $\sup_\omega \mu_{\mathbb{R}}(\omega)$ and the frequencies ω_i, $i = 1, m$ with $\mu_{\mathbb{R}}(\omega_i) = \sup_\omega \mu_{\mathbb{R}}(\omega)$.
C. Compute the sensitivity of the peak values, $\mu_{\mathbb{R}}(M_K(j\omega_i(K)))$, $i = 1, m$ w.r.t. changes in the feedback gain K.
D. Reduce the m peaks $\mu_{\mathbb{R}}(\omega_i)$ by applying small changes to the feedback gain, using the computed sensitivities.
E. Regularly check the presence of other frequencies ω_e with $\mu_{\mathbb{R}}(\omega_e) \approx \sup_\omega \mu_{\mathbb{R}}(\omega)$. If necessary, increase the number of controlled peak values in the $(\omega, \mu_{\mathbb{R}})$-plot, m. Stop when the available controller parameters do not allow to further reduce $\sup_\omega \mu_{\mathbb{R}}(\omega)$. In the other case, go to step B.

We now outline some of the steps in more depth.

Performing step C. can be reduced by the *chain rule for differentiation* to the computation of the sensitivity of singular values of parametrized matrices. A complication is the minimization over γ in expression (8). According to (Qiu *et al.*, 1995) several generic situations characterizing the minimum of $f(M,\cdot)$ occur. When it correspond to a smooth minimum of

$$\sigma_2\left(\begin{bmatrix} \Re(M) & -\gamma\Im(M) \\ \gamma^{-1}\Im(M) & \Re(M) \end{bmatrix}\right) \tag{16}$$

at some value $\gamma^* \in (0,1)$ the multiplicity of this second singular value is generically equal to one. When the minimum is reached for the limit $\gamma \to 0$, one can show that expression (11) holds. When it is reached for $\gamma = 1$, then the multiplicity is generically equal to two, but one shows that $\mu_{\mathbb{R}}(M) = \sigma_1(M)$, the largest singular value of M being isolated. In all of these cases the problem of computing the derivative of a peak value w.r.t. a parameter ultimately relies on taking the derivative of an *isolated singular value* of a matrix, which smoothly depends on a parameter. This is a standard problem, see (Michiels and Roose, 2003) for mathematical expressions. In case $\mu_{\mathbb{R}}(\omega_i)$ corresponds to a non-smooth minimum of (16) at some $\gamma^* \in (0,\, 1)$, typically two intersecting branches of singular values as a function of γ are involved. When this configuration is structurally stable w.r.t. changes of the controller parameter, standard calculus allows to compute the sensitivity of the position of the intersection point. In the other case a bifurcation to the smooth case may occur [2].

When collecting the sensitivities of the peak values w.r.t. the components k_j of the gain K in a matrix S,

$$S_{i,j} = \frac{\partial \mu_{\mathbb{R}}(\omega_i(K))}{\partial k_j}, \tag{17}$$

and with Δh the desired reduction of $\sup_\omega \mu_{\mathbb{R}}(\omega)$ per iteration step, one can compute the necessary change of the feedback gain as

$$\Delta K = -S_m^{\dagger} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \Delta h. \tag{18}$$

For the new feedback gain, a correction has to be made on both ω_i and $\mu_{\mathbb{R}}(\omega_i)$, because (18) is based on linearization. When no reduction of the objective function is achieved the steplength Δh is automatically decreased in our algorithm. Vice versa a reduction leads to a larger steplength in the next iteration.

There are generally two types of optima. One can have a smooth optimum, where for some $i \in \{0, \ldots, m\}$,

$$\frac{\delta\mu(\omega_i(K))}{\delta k_j} = 0,\ \forall j,$$

[2] In our experiments we have so far not encountered this situation

or a nonsmooth optimum, where all peak values can be reduced but there is no common descending direction. For instance, for $m = 2$ this occurs when for some $\alpha > 0$,

$$\frac{\delta\mu(\omega_1(K))}{\delta k_j} = -\alpha\frac{\delta\mu(\omega_2(K))}{\delta k_j}, \ \forall j.$$

Since it is hard to check whether such criteria are (approximately) satisfied, a more practical criterion is implemented in our algorithm: it terminates when the adaptive steplength Δh has become smaller than a threshold.

Remark 2. Discontinuities in the the derivative of the objective function due multiple peaks in the $\mu_{\mathbb{R}}$ plot are dealt with by monitoring and reducing the peaks simultaneously. As alternative a random gradient bundle method (Burke *et al.*, 2002) may be used to find a descending direction of the objective function.

The computationally intensive step of Algorithm 1 is Step B, the computation of a real stability radius. In the similar algorithm of (Michiels and Roose, 2003) for optimizing complex stability radii this step is replaced by the computation of a complex stability radius. This is less expensive, especially when the perturbation is not a vector, as comparing (10) and (9) reveals that no optimization is needed to evaluate $\mu_{\mathbb{C}}$. As a consequence, the overall computational cost can often be reduced when optimizing complex stability radii as a pre-processor for Algorithm 1, i.e. to generate a starting value for the gain K.

5. NUMERICAL RESULTS

We take the system (4) with $l = 2, \tau_1 = 0, \tau_2 = 5$, $A_2 = 0$, $B_1 = 0$ and

$$A_1 = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad (19)$$

which was also considered in (Michiels *et al.*, 2002; Michiels and Roose, 2003). In the first reference the continuous pole placement method was applied, yielding $K = [0.471\ 0.504\ 0.607]$ and $F(K) = -0.15$ in the optimum of (3). We wish to improve the robustness of stability w.r.t. (real) perturbations of A_1. To have a better starting value for K in Algorithm 1, hence to reduce the number of iterations required, we first optimize the complex stability radius $r_{\mathbb{C}}^{A_1}$ using the algorithm of (Michiels and Roose, 2003), see Figure 2. This results in $K \approx [0.649\ 1.05\ \ 0.741]$. Then we apply Algorithm 1. Some iterations are shown in Figure 3, where we display the *full* $\mu_{\mathbb{R}}$ plot for illustrative purposes - as explained in Section 3 it is sufficient to compute only parts of the plot. Notice the discontinuity of the function $\mu_{\mathbb{R}}(\omega)$ at zero. As a result we find that $r_{\mathbb{R}}^{A_1}$ is maximal for $K \approx [0.832\ 1.12\ 0.705]$.

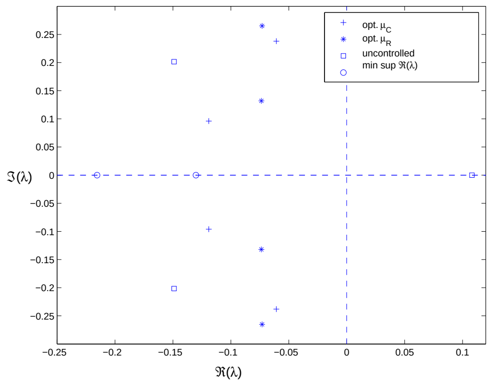

Fig. 4. Rightmost eigenvalues for gain values corresponding to a minimum of F, maxima of $r_{\mathbb{C}}^{A_1}$ and $r_{\mathbb{R}}^{A_1}$, as well as the open loop eigenvalues.

Table 1 compares the results of optimizing $F(K)$, $r_{\mathbb{C}}^{A_1}$ and $r_{\mathbb{R}}^{A_1}$, whereas Figure 4 shown the corresponding rightmost eigenvalue configuration. As expected they reveal the classical trade-off between performance and robustness.

opt.	$r_{\mathbb{C}}^{A_1}$	$r_{\mathbb{R}}^{A_1}$	$\sup \Re(\lambda)$
$F(K)$	0.0130	0.0130	−0.13
$r_{\mathbb{C}}^{A_1}$	0.0343	0.0351	−0.061
$r_{\mathbb{R}}^{A_1}$	0.0296	0.0450	−0.074

Table 1. Comparison of results of different optimization criteria.

6. CONCLUSIONS

Procedures to compute and optimize real stability radii of stable time-delay systems were presented. Combined with previous work on computing stabilizing feedback controllers (Michiels *et al.*, 2002) an overall eigenvalue based solution for the robust stabilization problem is obtained.

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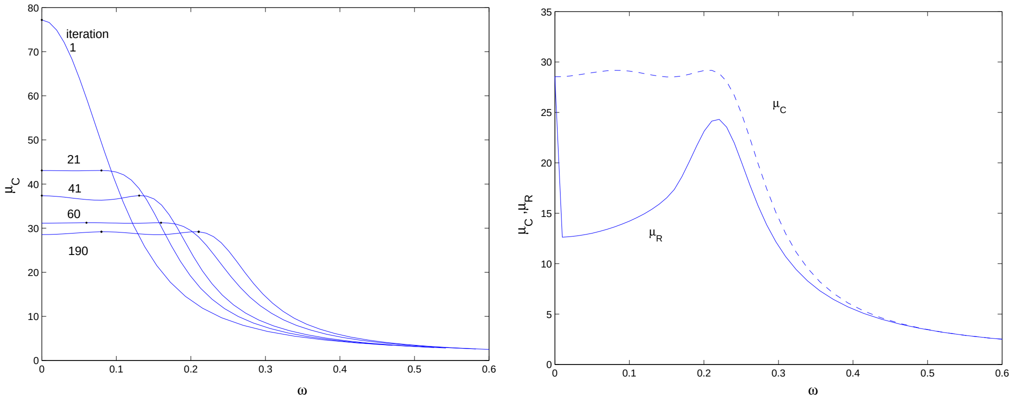

Fig. 2. (left) Iterations of the algorithm of (Michiels & Roose, 2003) to maximize $r_{\mathbb{C}}^{A_1}$. The initial condition is the minimum of (3). (right) Plot of $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ after optimizing $r_{\mathbb{C}}^{A_1}$.

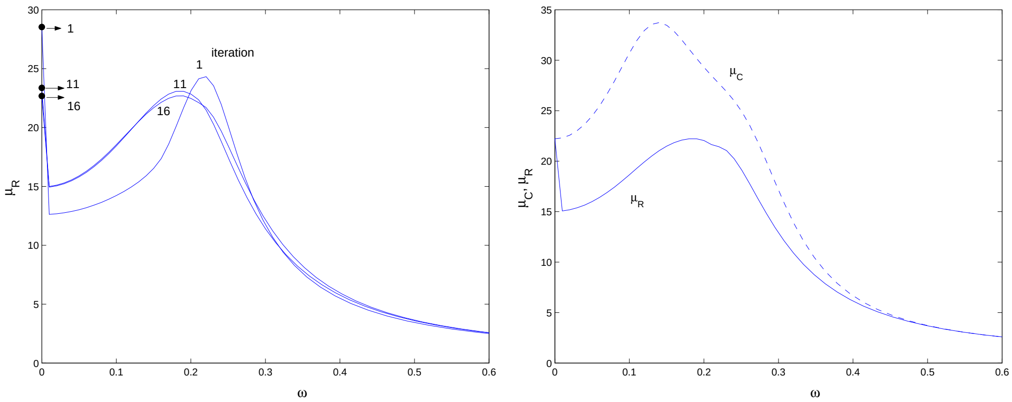

Fig. 3. (left) Iterations of Algorithm 1 to maximize $r_{\mathbb{R}}^{A_1}$. (right) Plot of $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ after optimizing $r_{\mathbb{R}}^{A_1}$.

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