ORDER REDUCTION & STRUCTURE SIMPLIFICATION FOR NONLINEAR SYSTEMS

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Abstract: This paper considers the problem of passing from a nonlinear simple-structure model to a low-order approximation by preserving simplicity. The approximation problem is often best posed as an ℓ_2 optimization problem. This optimization problem is the core of our order reduction method so-called system matrices optimization. The simplification is formulated as special secondary conditions which can be added to the original optimization problem. In order to find the simplest reduced order model a search process should be performed that an appropriate and effective fitness function and some techniques for shrinking the search space and accelerating the search process is presented.

Keywords: Structure Simplification, Order Reduction,Optimization , Nonlinear Systems, Genetic Algorithm

1. INTRODUCTION

Typical nonlinear dynamical systems are modelled by means of a set of first order coupled differential equations or a set of partial differential equations. The models which are described with partial differential equations can be also solved numerically by first spatially discretizing them by means of finite element, boundary element and similar methods which leads to a set of ordinary differential equations. In the first case the order of the system (number of state variables) depends on the quality of modelling and complexity of the system, but in the second case it depends on the quality of discretization.

Recent advances in hardware and software technology provide this ability to solve very large systems of ordinary differential equations. Nevertheless, typically these calculations need parallel processing which increases the cost of simulation drastically, and as a result, limits simulation applicability considerably. Therefore the complexity of simulation, analysis and controller design of a system depends directly on the complexity of the corresponding system model. In order to face this dilemma, there are two popular methods: order reduction and structure simplification. The idea behind order reduction is to approximate a dynamic system with a model with less number of state variables. Although there are different methods for order reduction of general nonlinear systems such as Proper Orthogonal Decomposition (Volkwein 1999), System Matrices Optimization (Lohmann 1995*b*) and Nonlinear Balancing and Truncation (Scherpen 1994), but they generally result in reduced systems with high number of internal interconnections, i.e. the model structures are complex. The idea behind structure simplification is simplifying the relations and coupling among the state variables. This idea was first introduced by (Buttelmann, M. and Lohmann, B. 2000) and it was developed in later publications.

The problem that we address through this paper is a combination of these two ideas in one algorithm. Starting from the system matrices optimization method, secondary conditions can be formulated to calculate reduced systems with simpler structures. One of the methods to find these secondary conditions is exploiting genetic algorithm in order to perform a global search within the search space. In this paper a modified fitness function is presented, which simplifies the search procedure and enormously reduces the computation effort. Also a method for omitting improper candidates is suggested that accelerates the whole search process by shrinking the search space. It should be also noted here that one of the advantages of the new method in comparison to the old ideas is *simulation free concept*. In this algorithm only the snapshots of the original system are required and no further simulation of the original system or the reduced order system is necessary. Our method includes three major steps are as follows:

(1) Order reduction using system matrices optimization method,
(2) Simplifying the reconstruction matrix ($\mathbf{W_{nc}}$),
(3) Simplifying the reduced order system matrices ($\tilde{\mathbf{E}}_{\mathbf{nc}}$).

They will be elaborated in the proceeding sections.

2. ORDER REDUCTION

The first step is reducing the number of state variables. In order to put this idea into practice we use the system matrices optimization method. This method has first been proposed by (Lohmann 1995*b*) and it can be exploited for nonlinear systems with the following representation:

$$\mathcal{S} : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \tag{1}$$

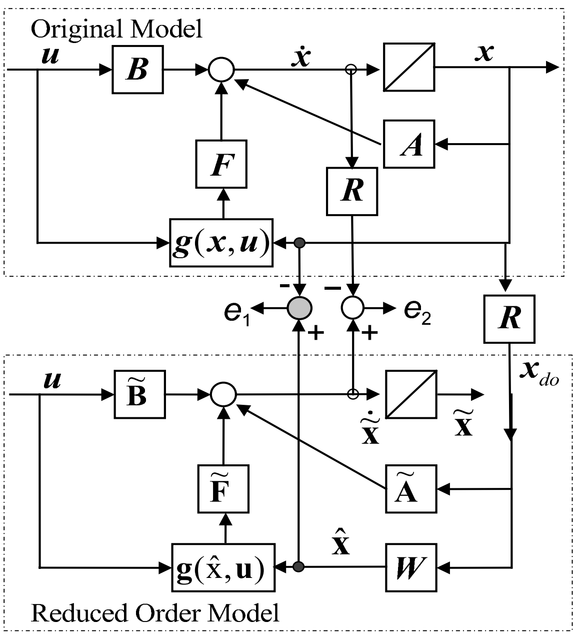

Fig. 1. Step I: system matrices optimization method

where $\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$ comprises the nonlinear part of the differential equations. Using this method we find a system of lower order $\tilde{n}$ which delivers an approximation of the dominant state variables. These dominant state variables are chosen by the designer and are combined in the vector $\mathbf{x_{do}}$ which is related to the original vector $\mathbf{x}$ by $\mathbf{x_{do}} = \mathbf{R}\mathbf{x}$. Based on the given system (1) and the dominant state variables, the system matrices optimization method calculates matrices $\mathbf{E} = [\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{F}}]$ and $\mathbf{W}$ such that they optimally fit the snapshots of the dominant state variables of the original system. In fact this method tries to minimize the errors e_1 and e_2 (shown graphically in Figure (1)) without any additional constraints and for some typical input signals. Assume that matrices χ, $\chi_{\mathbf{do}}$, $\dot{\chi}_{\mathbf{do}}$, $\mathbf{\Psi}$ and $\mathbf{\Gamma}$ are the snapshots of the original system for typical inputs which respectively show the numerical values of state variables, dominant state variables, derivative of dominant state variables, inputs and nonlinear part. For evaluating matrices $\mathbf{E}$ and $\mathbf{W}$, the following optimization problems should be solved (The notation $\|\mathbf{A}\|$ in this paper is the square of the Euclidian (Frobenius) norm of matrix A and is defined as $\sum \sqrt{diag(\mathbf{A^T A})}$):

$$\min_{\mathbf{E}} \| \dot{\chi}_{\mathbf{do}} - \underbrace{[\tilde{A} \quad \tilde{B} \quad \tilde{F}]}_{\mathbf{E}} \underbrace{\begin{bmatrix} \chi_{do} \\ \Psi \\ \Gamma \end{bmatrix}}_{\mathbf{M}} \|,$$

$$\min_{\mathbf{W}} \ \| \chi - \mathbf{W}\chi_{\mathbf{do}} \| \tag{2}$$

which is equivalent to solving $n + \tilde{n}$ independent optimization problems as follows (for the proof refer to (Yousefi, A. *et al.* 2004)):

$$\mathbf{W_{nc}} = \min_{\mathbf{w_i}} \| \mathbf{x_i^T} - \mathbf{w_i^T}\chi_{\mathbf{do}} \|, \ i = 1, 2, \ldots, n \tag{3}$$

$$\tilde{\mathbf{E}}_{\mathbf{nc}} = \min_{\mathbf{e_i^T}} \| \dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}} - \underbrace{[\tilde{A}_i \quad \tilde{B}_i \quad \tilde{F}_i]}_{\mathbf{e_i^T}} \underbrace{\begin{bmatrix} \chi_{do} \\ \Psi \\ \Gamma \end{bmatrix}}_{\mathbf{M}} \|,$$

$$i = 1, 2, \ldots, \tilde{n} \tag{4}$$

where $\dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}}$ is the snapshots of derivative of the i_{th} state variable, $\tilde{\mathbf{A}}_{\mathbf{i}}, \tilde{\mathbf{B}}_{\mathbf{i}}$ and $\tilde{\mathbf{F}}_{\mathbf{i}}$ are the i_{th} row of the reduced order system matrices $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ and $\tilde{\mathbf{F}}$ respectively, $\mathbf{x_i^T}$ is the snapshots of i_{th} state variable and $\mathbf{w_i^T}$ is the i_{th} row of matrix $\mathbf{W}$. The index *nc* means no constraints is applied to the optimization problem. In fact $\tilde{\mathbf{E}}_{\mathbf{nc}}$ and $\mathbf{W_{nc}}$ are the best answers (with respect to accuracy criterion) that can be resulted from system matrices optimization method. The optimal solution can be evaluated using (5).

$$\begin{aligned} \mathbf{e^T_{opt_i}} &= \dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}} \mathbf{M^T}(\mathbf{M}\mathbf{M^T})^{-1} \\ \mathbf{w^T_{opt_i}} &= \mathbf{x_i^T}\chi^{\mathbf{T}}_{\mathbf{do}}(\chi_{\mathbf{do}}\chi^{\mathbf{T}}_{\mathbf{do}})^{-1} \end{aligned} \tag{5}$$

Exploiting the result of (5) the reduced system is completely determined and in addition to dominant state variables the non-dominant state variables are approximated using $\mathbf{W}$. By assuming $\tilde{\mathbf{x}}$ as the approximation of $\mathbf{x_{do}}$, the reduced system is set up as follows:

$$\mathcal{S}_{red}:\begin{cases}\dot{\tilde{\mathbf{x}}}(t)=\tilde{\mathbf{A}}\tilde{\mathbf{x}}(t)+\tilde{\mathbf{B}}\mathbf{u}(t)+\tilde{\mathbf{F}}\mathbf{g}(\mathbf{W}\tilde{\mathbf{x}},\mathbf{u})\\ \mathbf{y}(t)=\tilde{\mathbf{C}}\tilde{\mathbf{x}}(t)\end{cases}\quad (6)$$

Accordingly, the vector $\mathbf{g}$ of the nonlinearities is taken over from the original system (1) into the reduced order system and no additional nonlinearities are introduced. With respect to the fact that typically all the elements of matrices $\mathbf{E_{nc}}$ and $\mathbf{W_{nc}}$ are nonzero, in the proceeding steps of our algorithm we will try to simplify (removing the coupling elements) the reduced order nonlinear system in order to increase the sparsity (higher number of zeros) of system matrices.

2.1 Secondary Conditions

The general linear equality constrained minimization problem can be written as follows:

Find X such that it minimizes $\|\mathbf{A}X-\mathbf{B}\|$ and fulfills the equality $\mathbf{C}X=\mathbf{D}$

where $\mathbf{A}$ is an m-by-n matrix ($m\leq n$) and $\mathbf{C}X=\mathbf{D}$ defines a linear equality constraint. In (Lawson, C.L. and R.J. Hanson 1974, Fletcher 1980) some methods for solving this optimization problems are proposed. The ability to solve optimization problems with constraints can be used to combine some additional features to system matrices optimization method for order reduction and structure simplification. In (Lohmann 1995*b*) this basic idea is used for improving the steady state performance. The application of secondary conditions to structure simplification is elaborated in the next subsection.

2.2 Application of Secondary Conditions to Structure Simplification

Since each non-zero element represents one internal coupling within the system, it is therefore appropriate to not only reduce the system order but also to keep the reduced system simple by aiming at a significant number of zero elements in $\mathbf{E}$ and $\mathbf{W}$. In order to achieve this, we first formulate complexity constraints on the reduced model (6) by the following secondary conditions:

$$\mathbf{e_i^T}\mathbf{h_{e,i}}-\mathbf{l_{e,i}}=\mathbf{0^T},\quad \mathbf{w_i^T}\mathbf{h_{w,i}}-\mathbf{l_{w,i}}=\mathbf{0^T}\quad (7)$$

where $\mathbf{e_i^T}$ is the i_{th} row of matrix $\mathbf{E}$ and $\mathbf{w_i^T}$ is the i_{th} row of matrix $\mathbf{W}$. It is very easy to prove that for instance for forcing a zero at the first element in the second row of matrix $\mathbf{A}$ (a_{12}), we can choose the following secondary conditions for the second row of matrix $\mathbf{E}$ ($\mathbf{e_2^T}$):

$$\mathbf{h_{e,2}}=[1,0,\ldots,0]^T,\ \mathbf{l_{e,2}}=[0]$$

The optimization problems (4) with secondary conditions of type (7) results in optimal solutions (8):

$$\begin{aligned}\mathbf{e^T_{opt_i}}&=\dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}}\mathbf{M^T}(\mathbf{MM^T})^{-1}+\\&+(\mathbf{l_{e,i}}-\dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}}\mathbf{M^T}(\mathbf{MM^T})^{-1}\mathbf{h_{e,i}})\cdot\\&\cdot(\mathbf{h^T_{e,i}}(\mathbf{MM^T})^{-1}\mathbf{h_{e,i}})^{-1}\mathbf{h^T_{e,i}}(\mathbf{MM^T})^{-1}\\ \mathbf{w^T_{opt_i}}&=\mathbf{x_i^T}\chi^{\mathbf{T}}_{\mathbf{do}}(\chi_{\mathbf{do}}\chi^{\mathbf{T}}_{\mathbf{do}})^{-1}+\\&+(\mathbf{l_{w,i}}-\mathbf{x_i^T}\chi^{\mathbf{T}}_{\mathbf{do}}(\chi_{\mathbf{do}}\chi^{\mathbf{T}}_{\mathbf{do}})^{-1}\mathbf{h_{w,i}})\cdot\\&\cdot(\mathbf{h^T_{w,i}}(\chi_{\mathbf{do}}\chi^{\mathbf{T}}_{\mathbf{do}})^{-1}\mathbf{h_{w,i}})^{-1}\mathbf{h^T_{w,i}}(\chi_{\mathbf{do}}\chi^{\mathbf{T}}_{\mathbf{do}})^{-1}\end{aligned}\quad (8)$$

Using secondary conditions presented in (7), we can force any element of system matrices to zero deliberately. But the problem is that we don't know which elements are not significant and can be replaced with zeros. Therefore we should search between the possible options to find the elements that can be substituted with zeros. Often there are a large number of different choices to carry out this task and this number is related directly to the size and complexity of the original system. In fact it is not sometimes possible to check every single option independently and find the best solution, therefore some methods for pioneered searching such as genetic algorithm is demanded. But if the suitable complexity constraints $\mathbf{l}$ and $\mathbf{h}$ are found, the optimization problems (3,4) with secondary conditions of type (7) result in a reduced simplified model.

3. SIMPLIFYING THE RECONSTRUCTION MATRIX $\mathbf{W_{NC}}$

Each row of $\mathbf{W_{nc}}$ shows the optimal estimation of the corresponding state variable in the original system based on the state variables of the reduced order system. In the case that the original state variable be one of the dominant state variables, the corresponding row of $\mathbf{W}$ has very simple structure as follows:

$$\mathbf{w_{i_{do}}}=\begin{pmatrix}0\ \ldots\ 0 & \underbrace{1}_{i_{th}col.} & 0\ \ldots\ 0\end{pmatrix}$$

therefore no further simplification is applicable. But other rows need to be checked for the possibility of simplification. For instance if one of the rows of $\mathbf{W_{nc}}$ looks like the following row vector:

$$\mathbf{w_j}=\begin{pmatrix}4 & 3\times10^{-17} & -7 & 3\times10^{-14} & 10^{-14}\end{pmatrix}$$

our method tries to replace the very small elements with zeros and simultaneously examine precisely the effects of this replacement on the approximation error. In order to carry out this task we define an acceptable error range (accuracy criterion) for each row as follows:

$$\begin{aligned}&\text{Error Range}=[\mathbf{Error_{min_{W_i}}},(1+k)\mathbf{Error_{min_{W_i}}}]\\&\mathbf{Error_{min_{W_i}}}=\|\,\mathbf{x_i^T}-\mathbf{w^T_{nc_i}}\chi_{\mathbf{do}}\|\end{aligned}\quad (9)$$

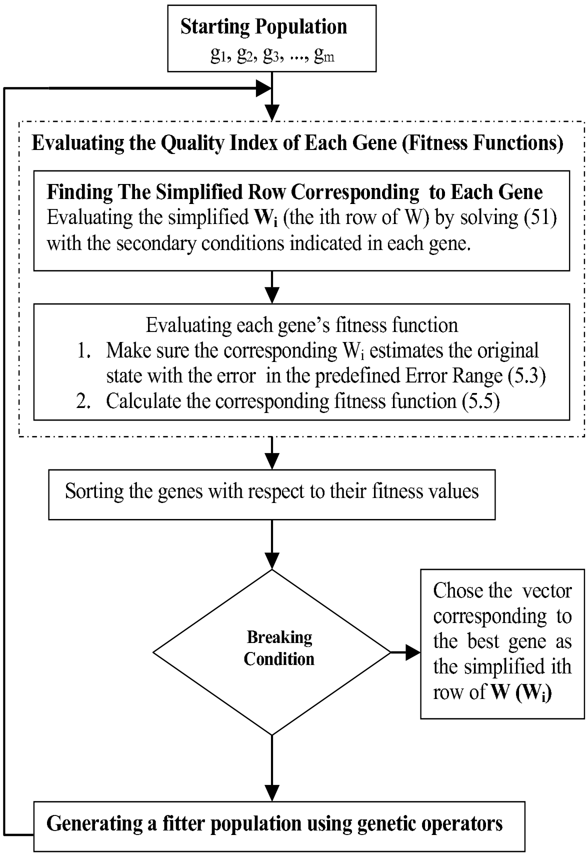

Fig. 2. Step II: structure simplification flow chart using genetic algorithms for each row of $\mathbf{W}$.

where parameter k can be any value greater than zero. The typical value of k is around 0.1 which shows losing the accuracy not less than 10 percent of the optimum answer, of course this value can be changed with regard to the application. Then our algorithm replaces some elements with zeros by adding the corresponding (zero forcing) secondary conditions to (3) and solving it. Then if the added secondary condition results in an approximation error in the *Error Range* (accuracy criterion), it calculates a simplicity cost using the following cost function:

$$F = \textit{number of nonzero elements} \qquad (10)$$

otherwise F will be set to zero. For small matrices it is possible to check all the possible (zero forcing) secondary conditions and compare all the corresponding resulted rows which have an approximation error in the acceptable error range with respect to their simplicity cost. Thus the row with the higher cost function has the highest simplicity and at the same time it doesn't lose much accuracy. But when the number of different options exceeds a limit, our method uses genetic algorithm to carry out a pioneered search among different options. The steps and algorithm of this search is depicted in form of a flowchart in Figure (2).

3.1 How Does Genetic Algorithm Help Structure Simplification?

With respect to the previous section, suitable choices of $\mathbf{l}$ and $\mathbf{h}$ are needed as candidates for the optimal simplified reduced order system. Genetic algorithms can be used to search between different options. In this method each option is presented in form of a bit string (so-called individual) that only consists of ones and zeros and ones show the places that zeros should be inserted in the corresponding row of matrices $\mathbf{E}$ and $\mathbf{W}$. For instance suppose the second row of matrix $\mathbf{W}$ is equal to the following row vector:

$$\mathbf{W_2} = [\, 2.1 \quad 5.10^{-5} \quad 6.10^{-17} \,]$$

The constraint for the genetic algorithm that forces the element w_{23} of of matrix $\mathbf{W}$ to zero, can be presented by the following row vector:

$$\mathbf{g^T_{W_2}} = \underbrace{0}_{1_{st}\text{ col.}} \ \underbrace{0}_{2_{nd}\text{ col.}} \ \underbrace{1}_{3_{rd}\text{ col.}} = 001$$

Consequently every row vector that has the length three and contains only ones and zeros corresponds to a simplified row of matrix $\mathbf{W}$. In this method the starting population is selected randomly and the tournament selection, two point cross over and normal mutation are used as genetic operators (Mitchell 1996) and the genetic algorithm produces new generations with better and better individuals as long as the breaking condition (number of produced generations) is not fulfilled.

4. SIMPLIFYING THE REDUCED ORDER SYSTEM MATRICES ($\tilde{\mathbf{E}}_{\mathbf{NC}}$)

Each row of $\tilde{\mathbf{E}}_{\mathbf{nc}}$ shows the optimal estimation of the corresponding state variable's derivative in the original system. Similar to the previous step, our method tries to replace the very small elements of each row of $\tilde{\mathbf{E}}$ with zeros and simultaneously examine the effects of this replacement on the approximation error. In order to carry out this task an acceptable error range for each row is defined as follows:

$$\text{Error Range} = [\mathbf{Error}_{\mathbf{min}_{\tilde{\mathbf{E}}\mathbf{i}}}, (1+k)\mathbf{Error}_{\mathbf{min}_{\tilde{\mathbf{E}}\mathbf{i}}}]$$

$$\mathbf{Error}_{\mathbf{min}_{\tilde{\mathbf{E}}\mathbf{i}}} = \min_{\mathbf{e_i^T}} \| \dot{\mathbf{x}}^{\mathbf{T}}_{\mathbf{do_i}} - \mathbf{E}^{\mathbf{T}}_{\mathbf{nc_i}} \underbrace{\begin{bmatrix} \chi_{do} \\ \Psi \\ \Gamma \end{bmatrix}}_{\mathbf{M}} \| \qquad (11)$$

At first our algorithm replaces some elements with zeros by adding corresponding (zero forcing) secondary conditions to (4) and solving it. Then it checks the accuracy criterion and calculates the cost function:

$$F = \textit{number of nonzero elements} \qquad (12)$$

Depends on the dimension of the problem sometimes it is possible to check all the possible (zero forcing)

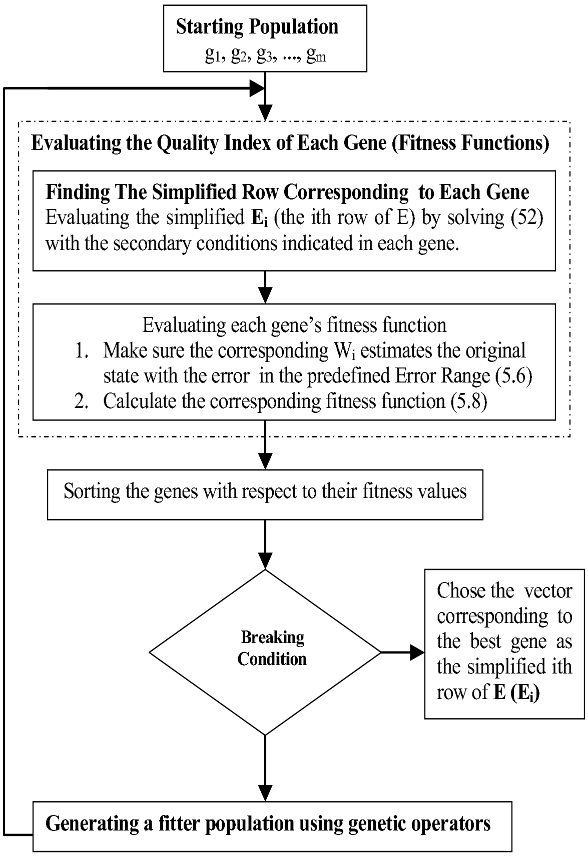

Fig. 3. Step III: structure simplification flow chart using genetic algorithms for each row of $\tilde{\mathbf{E}}$.

secondary conditions and find the simplest $\tilde{\mathbf{E}}$ that fulfills the accuracy criterion. But when the number of different options exceeds a limit, our method uses genetic algorithm, to carry out a pioneered search among different options. The steps and algorithm of this search is depicted as a flow chart in Figure (3).

5. NUMERICAL RESULTS

5.1 Combustion Engine with An Eddy Current Break

One of the systems that was used for implementation of our methods is combustion engine which is shown in figure (4). It is a test bed for a combustion engine linked to an eddy current break with a flexible shaft. The break stator is linked with a spring damper unit to the foundations (more details in (Lohmann 1994)). The system has a 7th order model and is used as the reference model in the fitness function and order reduction. The dimension of system matrices is shown in Table 1. The dominant states of the original model are selected as $x_{do} = [x_1, x_3, x_4]$. Starting the order reduction method without any secondary conditions delivers a model of 3rd order, but with a high complexity. The dimension of reduced order matrices is shown in Table 2. In order to achieve a simplified reduced model, using the old method results in a search space with 4.3×10^{12} elements. But by exploiting the new algorithm in this paper, it breaks into ten smaller search

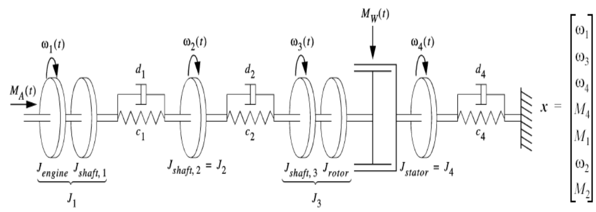

Fig. 4. A 7th order model of a combustion engine with an eddy current break

spaces with maximum 2^7 elements. In Table(3) the general formula for the search space size is presented. As it is shown in Table 3 in this example the search space shrinks to three smaller search spaces of order 2^7 and seven search spaces of order 2^3.

Table 1. The dimensions of system matrices

	A	**B**	**F**
Model of order n	$n.n$	$n.b$	$n.f$
Example 1 (7) Combustion Engine	7×7	7×2	7×2
Example 2 (10) Vehicle Suspention	10×10	10×2	10×5

5.2 Hydropneumatic Vehicle Suspension

Another system that was used for further implementation of the methods is an active hydropneumatic suspension. This device increases comfortableness and safety by significantly reducing the incongruous movements of the car body compared to a traditional passive spring shock-absorber system. The inputs of this system are the in and outflow of oil in the hydropneumatic system which should be regulated by the controller using measurement data from the sensors. Figure(5) shows the mechanical construction of the suspension for a single wheel and the related part of the car body (more details in (Lohmann 1995*a*)). The dimension of system matrices is shown in Table(1). The system has a 10th order model and there exist seven dominant state variables, so the reduced order model of order 7th can be calculated.

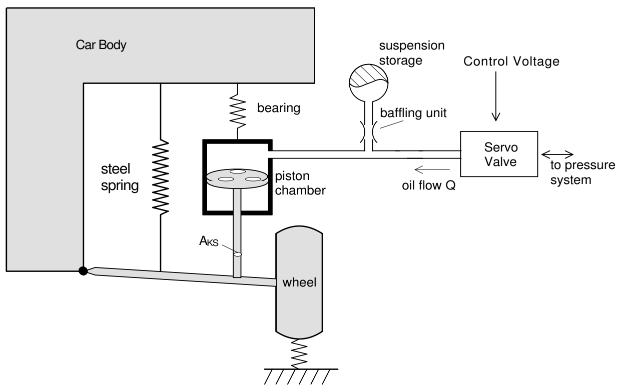

Fig. 5. Construction of hydropneumatic suspension

Table 2. The dimensions of reduced order system matrices

	$\tilde{\mathbf{A}}$	$\tilde{\mathbf{B}}$	$\tilde{\mathbf{F}}$	$\mathbf{W}$
Model of order n to order $\tilde{n}$ $(n \rightarrow \tilde{n})$	$\tilde{n}.\tilde{n}$	$\tilde{n}.b$	$\tilde{n}.f$	$n.\tilde{n}$
Example 1 $(7 \rightarrow 3)$ Combustion Engine	3×3	3×2	3×2	7×3
Example 2 $(10 \rightarrow 7)$ Vehicle Suspention	7×7	7×2	7×5	10×7

The dimension of reduced order matrices is shown in Table 2. By applying order reduction the approximation of the reduced order system is good, but the complexity of the model is very high. In order to achieve a simplified reduced model, using the old method results in a search space with 3.7×10^{50} elements. But by exploiting the new algorithm in this paper, it breaks into ten smaller search spaces with maximum 2^{14} elements. As it is shown in Table 3 in this example the search space shrinks to seven smaller search spaces of order 2^{14} and ten search spaces of order 2^7.

Table 3. The search space dimension in the old and new method

	Search Space Size (old)	Search Space Size (new)
Model of order n to order $\tilde{n}$ $(n \rightarrow \tilde{n})$	$2^{\tilde{n} \times (\tilde{n}+b+f+n)}$	$(\tilde{n}) \times 2^{(\tilde{n}+b+f)}$ $+(n) \times 2^{\tilde{n}}$
Combustion Engine $(7 \rightarrow 3)$	4.3×10^{12}	$(3) \times 2^7$ $+(7) \times 2^3$
Vehicle Suspention $(10 \rightarrow 7)$	3.7×10^{50}	$(7) \times 2^{14}$ $+(10) \times 2^7$

6. CONCLUSION

The reduction scheme presented in this paper delivers models of reduced order AND simple inner structure at the same time. Model structures are coded in binary strings and are optimized using Genetic Algorithms, whereas the reduced model's system matrices (respecting the structure constraints) are calculated by explicit formula (8). Due to the specific optimization criterion (2), the huge search space can be separated into $n + \tilde{n}$ subspaces that are independent of each others. In other words: if the optimum within each subspace can be found (being within the error ranges and with maximum number of zero elements), then the global optimum is found. It is this fact that reduces the computational effort drastically while still delivering good approximation results in practice.

Future work shall focus on the suitable choices of starting populations of the genetic algorithm, gained from the original model for instance, and on the application of the method to even larger and more challenging practical engineering problems.

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